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Magnetohydrodynamic surface waves in a running stream

SUSAMA SAHA AND KRIPASINDHU CHAUDHURI
Department of Mathematics, Jadavpur University, Calcutta 700 032, India.

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Abstract

The three-dimensional problem of surface waves excited in a running stream of finite depth by arbitrary surface disturbances under the presence of magnetic and current fields has been studied. The integral expression of the free surface elevation η has been asymptotically evaluated for large distances by applying the method of stationary phase. The results have been illustrated for rectangular and elliptic areas of disturbance on the free surface. Some features of the wave motion have been discussed.

Key words : Free surface elevation, capillary-gravity waves, running-stream, method of stationary phase.

1. Introduction

The two-dimensional problem of waves generated by an oscillatory pressure acting at the surface of a running stream of finite depth has been investigated by Debnath and Rosenblat[1]. Debnath[2] has also studied the propagation of two-dimensional capillary-gravity waves excited by an oscillating pressure distribution acting at the free surface of a running stream of finite, infinite and shallow depth. The theory of formation of waves on a running stream is also well known[3].

However, the three-dimensional problem of waves in a running stream due to any initial time-independent surface impulse or elevation remains to be solved. In the present paper, we have solved this problem in water of finite depth in the presence of magnetic and current fields. Formal solutions of the problem in the form of infinite integrals are obtained by the applications of double Fourier transforms. These integrals are then asymptotically evaluated for large distances by using the method of stationary phase. Next we consider a class of physically plausible models of the disturbance on areas of elliptic and rectangular shapes on the surface. Some features of the wave motion are then discussed.

2. Formulation of the problem

We take the origin on the undisturbed horizontal free surface which coincides with the xy-plane and the z-axis is drawn vertically upwards.

Let h be the uniform depth of a running stream which is under the influence of a uniform horizontal magnetic field $\vec{B} = (B_x, B_y, o)$, a uniform horizontal current field $\vec{J} = (J_x, J_y, o)$ and the gravity g. It may be mentioned here that the horizontal component of magnetic field is present in the equatorial region of the Earth.

The total body force consists of the vertical component $(J_x B_y - J_y B_x)$ of the Lorentz force $\vec{J} \times \vec{B}$ and the gravity. The magnetic field will damp or boost up the action of gravity according as

$$J_x / J_y > \text{ or } < \frac{B_x}{B_y}.$$

The well known equations of motion[4] of the conducting fluid are

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \vec{\nabla}) \vec{q} = -g\vec{k} + \frac{1}{\rho} \vec{J} \times \vec{B} - \frac{1}{\rho} \vec{\nabla} p, \tag{1}$$

$$\vec{\nabla} \cdot \vec{q} = o, \tag{2}$$

$$\vec{\nabla} \cdot \vec{B} = o, \tag{3}$$

$$\vec{\nabla} \times \vec{B} = \mu \vec{J}, \tag{4}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{5}$$

$$\vec{J} = \sigma (\vec{E} + \vec{q} \times \vec{B}) \tag{6}$$

where $\vec{q} = (u, v, w)$ is the velocity field, ρ the density, μ the magnetic permeability and σ the electrical conductivity of the fluid. ρ, μ and σ are assumed to be constant throughout the region of flow.

Equation (2) implies that the fluid considered here is incompressible.

We take $\quad \vec{\nabla} \times \vec{E} = 0$

Now

$$\vec{\nabla} \times \vec{E} = 0 \Rightarrow \frac{\partial \vec{B}}{\partial t} = 0$$

so that the induced magnetic field remains constant over time.

Again,

$$\vec{\nabla} \times \vec{E} = 0 \Rightarrow \vec{\nabla} \times \vec{E} \gg \frac{\partial \vec{B}}{\partial t}.$$

A dimensional analysis of this condition reveals that

$$E/L \gg B/T \; ; \; i.e., \; qB/L \gg B/T$$

so that

$$T \gg \frac{L}{q} = \frac{Lq}{c^2} \frac{c^2}{q^2},$$

c being the velocity of light.

Since MHD conditions require $T \gg Lq/c^2$, the system considered here is a restricted MHD system.

A wave motion is set up by the action of an initial surface impulse

$$(\rho\Phi)_{\substack{z=0\\t=0}} = F(x, y) \tag{7}$$

together with an initial surface displacement

$$(\eta)_{t=0} = f(x, y). \tag{8}$$

We assume that the motions arising from disturbances created in the uniform stream have a velocity potential $\phi\,(x, y, z\,; t)$

where

$$\Phi\,(x, y, z\,; t) = Ux + \phi\,(x, y, z\,; t) \tag{9}$$

for

$$-\infty < (x, y) < \infty, \; -h < z < \eta, \; t \geqslant 0.$$

Here $\phi\,(x, y, z\,; t)$ is the velocity potential due to the external disturbances only and η is the vertical displacement.

Assuming a small disturbance on the running stream, the equations of motion and the boundary conditions for the subsequent irrotational motion may be written as

$$\nabla^2 \phi = 0, \; -\infty < (x, y) < \infty, \; -h < z < \eta; \tag{10}$$

$$\phi_t + \left(g + \frac{J_y B_x - J_x B_y}{\rho}\right)\eta + U\phi_x = 0 \text{ on } z = 0; \tag{11}$$

$$\eta_t + U\eta_x - \phi_z = 0 \text{ on } z = 0; \tag{12}$$

$$\phi_z = 0 \text{ at } z = -h. \tag{13}$$

Eliminating η between (11) and (12), we get

$$\phi_{tt} + 2U\phi_{xt} + U^2\phi_{xx} + \lambda\phi_z = 0 \tag{14}$$

where

$$\lambda = g + \frac{J_y B_x - J_x B_y}{\rho}. \tag{15}$$

3. Solution

To solve (10), we introduce the double Fourier transform $\bar{\phi}(k_1, k_2, z; t)$ of the velocity potential $\phi(x, y, z; t)$ with respect to x and y, that is

$$\bar{\phi}(k_1, k_2, z; t) = \frac{1}{2\pi}\int\int_{-\infty}^{\infty} \phi(x, y, z; t)\, e^{i(k_1 x + k_2 y)}\, dx dy. \tag{16}$$

By (16), (10) becomes

$$\bar{\phi}_{zz} - k^2\bar{\phi} = 0 \tag{17}$$

where

$$k^2 = k_1^2 + k_2^2. \tag{18}$$

The solution of (17) is

$$\bar{\phi} = R(k_1, k_2, t)\cosh k(z + h). \tag{19}$$

Transformed forms of (14), (11), (13), (7), (8) are

$$\bar{\phi}_{tt} - 2Uik_1\bar{\phi}_t - U^2 k_1^2\bar{\phi} + \lambda\bar{\phi}_z = 0 \text{ on } z = 0, \tag{20}$$

$$\lambda\bar{\eta} + \bar{\phi}_t - Uik_1\bar{\phi} = 0 \qquad \text{on } z = 0 \tag{21}$$

$$\bar{\phi}_z = 0 \qquad \text{on } z = -h \tag{22}$$

$$(\rho\bar{\phi})_{\substack{z=0\\t=0}} = \bar{F}(k_1, k_2), \tag{23}$$

$$(\bar{\eta})_{t=0} = \bar{f}(k_1, k_2). \tag{24}$$

From (19) and (20), we get

$$R_{tt} - 2Uik_1R_t - (U^2k_1^2 - \lambda k \tanh kh) R = 0 \tag{25}$$

Solution of this equation is

$$R = e^{iUk_1t}[A_1 \cos pt + A_2 \sin pt] \tag{26}$$

where

$$p^2 = \lambda k \tanh kh.$$

Using (19), (23), (24), (26), we have

$$A_1 = \frac{\bar{F}(k_1, k_2)}{\rho \cosh kh}, \quad A_2 = -\frac{\lambda \bar{f}(k_1, k_2)}{\rho \cosh kh}.$$

Then,

$$\bar{\phi} = e^{iUk_1t}\left[\frac{\bar{F}}{\rho}\cos pt - \frac{\lambda \bar{f}}{\rho}\sin pt\right]\cosh\{k(z+h)\}\operatorname{sech} kh.$$

Inverting this expression for $\bar{\phi}$ by the Fourier double inversion theorem, we have finally

$$\Phi = (2\pi)^{-1}\iint_{-\infty}^{\infty}[\rho^{-1}\bar{F}(k_1, k_2)\cos pt - p^{-1}\lambda \bar{f}(k_1, k_2)\sin pt]$$
$$\times \cosh\{k(z+h)\}\operatorname{sech} kh\, e^{iUk_1t} \times e^{-i(k_1x+k_2y)}\, dk_1dk_2. \tag{27}$$

By using (21) and (27), the expression for $\bar{\eta}$ is deduced and then it is inverted by Fourier double inversion theorem to get

$$\eta = (2\pi\lambda)^{-1}\iint_{-\infty}^{\infty} e^{iUk_1t}[p\bar{F}\rho^{-1}\sin pt + \bar{f}\lambda\cos pt] \times e^{-i(k_1x+k_2y)}\, dk_1dk_2. \tag{28}$$

Expressions (27) and (28) give the formal solutions of the problem in terms of infinite integrals.

4. Asymptotic value of η

We can write (28) as

$$\eta = \eta_i + \eta_e. \tag{29}$$

where

$$\eta_i = \frac{1}{2\pi\lambda}\iint_{-\infty}^{\infty}\rho^{-1}\bar{F}p e^{i(Uk_1t - k_1x - k_2y)}\sin pt\, dk_1dk_2 \tag{30}$$

and

$$\eta_e = \frac{1}{2\pi} \int\!\!\int_{-\infty}^{\infty} \bar{f} e^{i(Uk_1 t - k_1 x - k_2 y)} \cos pt \, dk_1 dk_2. \tag{31}$$

Here the subscripts 'i' and 'e' refer to 'impulse' and 'elevation' components respectively of η.

Let

$$\left.\begin{aligned} & k_1 = k\cos\psi,\ k_2 = k\sin\psi,\ x = r\cos\theta,\ y = r\sin\theta, \\ & R = \frac{r}{h},\ \mu = kh,\ \bar{h} = h^{-1},\ \tau = \frac{t(\lambda h)^{1/2}}{2r} \\ & \frac{Ut}{r} = v,\ \omega = (\mu \tanh \mu)^{1/2} \end{aligned}\right\} \tag{32}$$

Then

$$\eta_e = (2\pi h^2)^{-1} \int_0^{\infty} \mu \cos(2R\omega\tau)\, d\mu \int_0^{2\pi} \bar{f}(\mu\bar{h}\cos\psi,\ \mu\bar{h}\sin\psi)\, e^{i\mu R A(\psi)}\, d\psi \tag{33}$$

where

$$A(\psi) = v\cos\psi - \cos(\psi - \theta). \tag{34}$$

We now evaluate the ψ-integral and the μ-integral correct to the first term of the asymptotic expansions by the method of stationary phase under the conditions $R \gg 1$ while τ remains fixed so that τ does not tend to zero as $R \to \infty$.

Writing

$$\left.\begin{aligned} A(\psi) &= v\cos\psi - \cos(\psi - \theta), \\ A'(\psi) &= -v\sin\psi + \sin(\psi - \theta), \\ A''(\psi) &= -v\cos\psi + \cos(\psi - \theta). \end{aligned}\right\} \tag{35}$$

The stationary point $\psi = \alpha$ is given by $A'(\psi) = 0$

i.e., $\sin(\alpha - \theta) = v\sin\alpha$.

which gives on simplification

$$\alpha = \frac{\theta}{2} + \tan^{-1}\left(\frac{1+v}{1-v}\tan\frac{\theta}{2}\right). \tag{36}$$

Also

$$A''(\alpha) < 0$$

when

$$v > \frac{\cos(\alpha - \theta)}{\cos\alpha}. \tag{37}$$

For each value of μ with $\mu R \gg 1$ the method of stationary phase[3] gives

$$\eta_s \simeq (4\pi h^2)^{-1} \left[\int_0^\infty \mu \bar{f}\,(\mu \bar{h} \cos a,\ \mu \bar{h} \sin a) \right.$$

$$\times \left(- \frac{2\pi}{\mu R A''(a)} \right)^{1/2} \times \{e^{iRP_1(\mu)} + e^{iRP(\mu)}\}\, d\mu, \tag{38}$$

where

$$P_1(\mu) = 2\omega\tau + \mu A(a) - \frac{\pi}{4R}, \tag{39}$$

$$P(\mu) = 2\omega\tau - \mu A(a) + \frac{\pi}{4R}. \tag{40}$$

Therefore

$$P'(\mu) = \tau \left[\left(\frac{\tanh \mu}{\mu} \right)^{1/2} + \left(\frac{\mu}{\tanh \mu} \right)^{1/2} \operatorname{sech}^2 \mu \right] - 1$$

and

$$P''(\mu) = \tau \left[\left(\frac{\tanh \mu}{\mu} \right)^{1/2} \{\operatorname{cosech} 2\mu - (2\mu)^{-1}\} \right.$$

$$\left. + \left(\frac{\mu}{\tanh \mu} \right)^{1/2} \operatorname{sech}^2 \mu\ \{(2\mu)^{-1} - \coth 2\mu - \tanh \mu\} \right]$$

For $P''(\mu)$ holds $P''(\mu) < 0$ for $0 < \mu < \infty$.

Hence

$$P'(\mu) \to 2\tau - A(a) \text{ as } \mu \to 0^+,$$

$$P'(\mu) \to -A(a) \quad \text{as } \mu \to +\infty.$$

The equation $P'(\mu) = 0$ has a real positive root $\mu = \beta$ (say) when $\tau > \frac{1}{2} A(a)$ where

$$\tau \left[\left(\frac{\tanh \beta}{\beta} \right)^{1/2} + \left(\frac{\beta}{\tanh \beta} \right)^{1/2} \operatorname{sech}^2 \beta \right] = A(a). \tag{41}$$

The stationary phase method applied again to (38) leads to

$$\eta_s \simeq (2h^2 R)^{-1} \left(\frac{\beta}{A''(a) P''(a)} \right)^{1/2} \operatorname{Re} [\bar{f}(\beta \bar{h} \cos a,\ \beta \bar{h} \sin a)$$

$$\times e^{-iR\{2\tau(\beta \tanh \beta)^{1/2} - \beta A(a)\}}] \tag{42}$$

for $\qquad R\beta \gg 1,\ R \gg 1$ and $\tau > \frac{1}{2} A(a)$.

Similarly,

$$\eta_i \simeq \frac{1}{\rho h^2 R \sqrt{\lambda h}} \left(\frac{\beta \tanh \beta}{A''(\alpha) P''(\beta)} \right)^{1/2} Im [\bar{F}(\beta \bar{h} \cos \alpha, \beta \bar{h} \sin \alpha)$$

$$\times e^{-iR\{2\tau(\beta \tanh \beta)^{1/2} - \beta A(\alpha)\}}].$$

Finally, because $\eta = \eta_i + \eta_e$, we obtain

$$\eta \simeq (2Rh^2)^{-1} \left(\frac{\beta}{A''(\alpha) P''(\alpha)} \right)^{1/2} \Big[\text{Re} \{ \bar{f}(\beta \bar{h} \cos \alpha, \beta \bar{h} \sin \alpha)$$

$$\times e^{-iR\{2\tau(\beta \tanh \beta)^{1/2} - \beta A(\alpha)\}} + \frac{2}{\rho} \left(\frac{\tanh \beta}{h\lambda} \right)^{1/2}$$

$$\times Im \{ \bar{F}(\beta \bar{h} \cos \alpha, \beta \bar{h} \sin \alpha) e^{-iR\{2\tau(\beta \tanh \beta)^{1/2} - \beta A(\alpha)\}} \} \Big].$$

5. Illustrative cases

5.1. *Initial impulse or elevation on a rectangular area*

(*a*) Let

$$F(x, y) = \frac{I}{4ab}, \ |x| \leqslant a, \ |y| \leqslant b$$

$$= 0, \ |x| > a, \ |y| > b$$

$$f(x, y) = 0,$$

where I represents the total impulse over the surface.

Then

$$\bar{F}(k_1, k_2) = \frac{I}{2\pi ab} \cdot \frac{\sin k_1 a}{k_1} \cdot \frac{\sin k_2 b}{k_2},$$

$$\eta_i \simeq \frac{I}{2\pi \, ab \, \rho \, R_4^1 \sqrt{h\lambda}} \left(\frac{\beta \tanh \beta}{A''(\alpha) P''(\beta)} \right)^{1/2} \frac{\sin (a\bar{h} \beta \cos \alpha) \sin (b\bar{h} \beta \sin \alpha)}{\beta^2 \cos \alpha \sin \alpha}$$

$$\times \sin [R \{2\tau(\beta \tanh \beta)^{1/2} - \beta A(\alpha)\}]$$

for

$$R \gg 1, \ R\beta \gg 1, \ \alpha \neq \left(0, \frac{\pi}{2}\right).$$

When $\alpha = 0$,

$$\eta_t \simeq \frac{I}{2\pi aRh\sqrt{h\lambda\beta}}\left(\frac{\tanh\beta}{A''(\alpha)P''(\beta)}\right)^{1/2}\sin(\beta a\bar{h})$$

$$\times \sin R\{2\tau(\beta\tanh\beta)^{1/2} - \beta A(\alpha)\},$$

for $R \gg 1,\ R\beta \gg 1$.

When

$$\alpha = \frac{\pi}{2}$$

$$\eta_t \simeq \frac{I}{2\pi bRh\sqrt{h\lambda\beta}}\left(\frac{\tanh\beta}{A''(\alpha)P''(\beta)}\right)^{1/2}\sin(\beta b\bar{h}$$

$$\times \sin R\{2\tau(\beta\tanh\beta)^{1/2} - \beta A(\alpha)\},$$

for $R \gg 1,\ R\beta \gg 1$.

(*b*) Let.

$$f(x, y) = \frac{W}{4ab},\ |x| \leqslant a,\ |y| \leqslant b,$$

$$= 0,\ |x| > a,\ |y| > b.$$

$$F(x, y) = 0,$$

where W is the total volume of elevated fluid.

Then

$$\bar{f}(k_1, k_2) = \frac{W}{2\pi ab}\cdot\frac{\sin k_1 a}{k_1}\cdot\frac{\sin k_2 b}{k_2}$$

and we find

$$\eta_e \simeq \frac{W}{4\pi abR}\left(\frac{\beta}{A''(\alpha)P''(\beta)}\right)^{1/2}\frac{\sin(\beta\bar{h}\,a\cos\alpha)\sin(\beta\bar{h}b\sin\alpha)}{\beta^2\sin\alpha\cos\alpha}$$

$$\times \cos R\{2\tau(\beta\tanh\beta)^{1/2} - \beta A(\alpha)\}$$

for

$$R \gg 1,\ R\beta \gg 1,\ \alpha \neq \left(0, \frac{\pi}{2}\right).$$

When $\alpha = 0$,

$$\eta_e \simeq \frac{W}{4\pi aRh}\left(A''(\alpha)P''(\beta)\cdot\beta\right)^{-1/2}\sin(\beta\bar{h}a)$$

$$\times \cos R\{2\tau(\beta\tanh\beta)^{1/2} - \beta A(\alpha)\}.$$

When $\alpha = \frac{\pi}{2}$,

$$\eta_{\bullet} \simeq \frac{W}{4\pi b\, Rh} \left(A''(\alpha)\, P''(\beta) \cdot \beta\right)^{-1/2} \sin\,(\beta \bar{h} b)$$

$$\times \cos\, R\,\{2\tau\,(\beta\,\tanh\,\beta)^{1/2} - \beta A\,(\alpha)\}.$$

4.2. *Impulse or elevation over elliptic regions*

(*a*) Let

$$F(x, y) = \frac{I\gamma}{\pi ab}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\gamma-1}, \text{ inside } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$= 0 \qquad\qquad , \text{ outside } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$f\,(x, y) = 0,$$

where I again denotes the total impulse on the surface.

We have

$$\bar{F}(k_1, k_2) = \frac{2^{\gamma-1} I_\gamma\, \Gamma(\gamma)}{\pi} \times \frac{J_\gamma\,[k_1^2\, a^2 + k_2^2\, b^2]^{1/2}}{(k_1^2\, a^2 + k_2^2\, b^2)^{\gamma/2}}$$

(see Erdelyi, 1, 1·3 (8) and 1·3 (50)).

For $\gamma = 2$, we have

$$\eta_t \simeq \frac{4I}{\pi\rho R\beta\,\sqrt{h\lambda\beta}} \left(\frac{\tanh\beta}{A''(\alpha)\,P''(\beta)}\right)^{1/2} \frac{J_2\{\beta\bar{h}\,(a^2\cos^2\alpha + b^2\sin^2\alpha)^{1/2}\}}{a^2\cos^2\alpha + b^2\sin^2\alpha}$$

$$\times \sin\, R\,\{2\tau\,(\beta\tanh\beta)^{1/2} - \beta A\,(\alpha)\}$$

for $R\beta \gg 1,\ R \gg 1$.

(*b*) Let

$$f(x, y) = \frac{W\gamma}{\pi ab}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\gamma-1}, \quad \text{inside } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$= 0 \qquad\qquad , \quad \text{outside } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$F(x, y) = 0 \qquad\qquad ,$$

where W again is the total volume of elevated fluid.

Then for $\gamma = 2$,

$$\eta_{\circ} \simeq \frac{2W}{\pi R\beta \sqrt{\beta}\,[A''(\alpha)\,P''(\beta)]^{1/2}} \;\frac{J_2\{\beta\bar{h}\,(a^2\cos^2\alpha + b^2\sin^2\alpha)^{1/2}\}}{a^2\cos^2\alpha + b^2\sin^2\alpha}$$

$$\times \cos R\,\{2\tau\,(\beta\tanh\beta)^{1/2} - \beta A\,(\alpha)\}$$

for

$$R \gg 1,\; R\beta \gg 1.$$

5. Discussion

We find from (53) that the motion consists of two different modes, one of which is due to the impulse only without any initial surface elevation and the other is caused by the initial elevation of the surface without any impulse.

The phase function is

$$R\,\{2\tau\,(\beta\tanh\beta)^{1/2} - \beta A\,(\alpha)\} = t\,(\lambda\beta\bar{h}\tanh\beta)^{1/2} - \beta\bar{h}\gamma\, A\,(\alpha).$$

The motion, therefore, represents a progressive wave of length

$$\frac{2\pi}{\beta\bar{h}\, A\,(\alpha)},$$

period

$$\frac{2\pi}{(\lambda\beta h\tanh\beta)^{1/2}}$$

and phase velocity

$$c = \frac{1}{A\,(\alpha)}\left(\frac{\lambda h\tanh\beta}{\beta}\right)^{1/2}.$$

Further, eqn. (50) implies that the results we find for η apply to waves that we observe when we move with the group velocity

$$V = \frac{c}{2}\cdot\frac{\tanh\beta + \beta\,\mathrm{sech}^2\beta}{\tanh\beta}.$$

The quantity λ accounts for the effects of magnetic and current fields as well as gravity. As a result, the wavelength, period, phase velocity and the group velocity are all influenced by the presence of magnetic and current fields. We further note that the amplitude of the waves decays like r^{-1}.

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