J. Indian Inst. Sci. 63 (B), June 1981, Pp. 131–140

On some eigenvalue problems associated with a differential operator

N. K. CHAKRABORTY AND SUDIP KUMAR ACHARYYA
Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Calcutta 700 019, India.

Received on May 27, 1980 ; Revised on June 22, 1981.

Abstract

The present paper deals with some problems on the variation of the eigenvalues and their application to study the nature of the spectrum associated with the matrix operator

$$M \equiv \begin{pmatrix} -D(p_0 D) + p_1 & r_1 \\ r_1 & -D(q_0 D) + q_1 \end{pmatrix}, \quad D \equiv \frac{d}{dx} \tag{A}$$

with prescribed boundary conditions. By employing, among others, some of the ideas and techniques of E. C. Titchmarsh and those of Chakraborty and Sen Gupta, it is found that under certain conditions, satisfied by the coefficients of the system (A), the spectrum of the system is discrete.

Key words : Spectrum (discrete), differential operator, Hilbert space, Green's matrix, absolutely uniformly continuous, pseudomonotonics, variation of the eigenvalues, meromorphic, Dirichlet (Neumann) problem.

1. Introduction

Chakraborty and Sen Gupta[6] employed the Titchmarsh method[7] involving the variation of the eigenvalues to obtain interalia a criterion for the discreteness of the spectrum associated with the differential system

$$M_1 \equiv \begin{pmatrix} -D^2 + p & q \\ q & -D^2 + r \end{pmatrix}, \quad D \equiv \frac{d}{dx}. \tag{1.1}$$

In a recent paper Sen Gupta[8] generalises certain results of the above paper, for a slightly more generalised system

$$M_1 [f] = -D^2 f + Pf = \lambda Sf, \tag{1.2}$$

where $P = \begin{pmatrix} p & q \\ q & r \end{pmatrix}$ and $S = \begin{pmatrix} s & h \\ h & t \end{pmatrix}$.

Our object in the present paper is to obtain certain results involving the criteria for the discreteness of the spectra for the general system

$$M\phi = \lambda F\phi, \tag{1.3}$$

$$M \equiv \begin{pmatrix} -D(p_0 D) + p_1 & r_1 \\ r_1 & -D(q_0 D) + q_1 \end{pmatrix}$$

where (i) $p_0, q_0 \geq 1$, $p_1, q_1, r_1 \in C^1(I)$, where $I : a \leq x < b$ ($a = 0$, $b = \infty$ being allowed) and p_1, q_1, r_1 are absolutely continuous over any compact subinterval of I.

(ii) $F = (F_{ij}(x))$ is a symmetric 2×2 matrix of real valued continuous functions, with $\det F \geq (\max(p_0, q_0))^2$ on I. Thus, $\det F \geq 1$, on I.

(iii) $\lambda \in C$, the set of all complex numbers and

(iv) $\phi = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathscr{D}$,

the set of all

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in C^2(I),$$

such that $f^T Ff$, $(Ff)^T F(Ff)$, $(Mf)^T F(Mf)$, $(Mf)^T F^{-1}(Mf) \in \mathscr{H}$, the basic Hilbert space $L(a, b)$;

$$f^T = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^T = (f_1, f_2),$$

the transpose of f.

It is well known[1] that (1.3) along with prescribed boundary conditions at the end points gives rise to an eigenvalue problem, both in the finite as well as in the singular case.

The boundary conditions to be considered for our problem are for the finite interval:

$$u(\alpha) = v(\alpha) = 0$$
$$u(\beta) = v(\beta) = 0 \tag{1.4}$$

or

$$u'(\alpha) = v'(\alpha) = 0$$
$$u'(\beta) = v'(\beta) = 0 \tag{1.5}$$

where, $a < \alpha < \beta < b$; $\phi = \begin{pmatrix} u \\ v \end{pmatrix}$, a solution of (1.3).

We thus encounter the Dirichlet or the Neumann problems for the interval (α, β) according as the boundary conditions are given by (1.4) or (1.5).

When the interval is $[0, \infty)$, the corresponding Dirichlet and the Neumann problems are (1.3) with $u(0) = v(0) = 0$ and (1.3) with $u'(0) = v'(0) = 0$ respectively.

2. The Dirichlet integral

The Dirichlet integral associated with the system (1.3) is defined by $D_I(g, h) \equiv D_I(g, h, p) = \int_a^b (G, H, P)\, dt$, $I = (a, b)$, where

$$P = \begin{pmatrix} p_1 & r_1 \\ r_1 & q_1 \end{pmatrix}, \quad G = \begin{pmatrix} g_1 & g_2 \\ g_1' & g_2' \end{pmatrix} = \begin{pmatrix} g \\ g' \end{pmatrix}, \quad g = (g_1, g_2),$$

$$H = \begin{pmatrix} h_1 & h_2 \\ h_1' & h_2' \end{pmatrix} = \begin{pmatrix} h \\ h' \end{pmatrix}, \quad h = (h_1, h_2),$$

$$(G, H, P) = p_0 g_1' h_1' + g_0 g_2' h_2' + p_1 g_1 h_1 + q_1 g_2 h_2 + r_1 g_1 h_2 + r_1 g_2 h_1 ;$$

with corresponding definitions for $D_b(g, h)$, $D_b(g)$, for $I = [0, b]$ and $D(g, h)$, $D(g)$ for $I = [0, \infty)$ (See Chakraborty and Sen Gupta[6]).

If $p_1 > 0$, $q_1 > 0$ and $\det p \geq 0$, $D_b(g)$ is always positive.

If $\lambda_n = \lambda_n(b)$, and $\psi_n(x) \equiv \psi_n(b, x)$, $n = 0, 1, 2, 3, \ldots$, be the eigenvalues and the eigenvectors, normalised in the sense

$$\| \psi_n(x) \|_{0,b} = \int_0^b \psi_n^T F \psi_n \, dt = 1,$$

and also if

$$C_n = \int_0^b \psi_n^T F f \, dt = \int_0^b f^T F \psi_n \, dt$$

be the Fourier coefficient of $f \in C^1(I)$, then if $p_0, q_0 > 0$, $p_1 > cF_{11}$, $\det(P - cF) \geq 0$ on $[0, b]$, the eigenvalues for both the Dirichlet and the Neumann problems, are greater than or equal to c. Other results concerning $D_b(f, g)$ as obtained in § 3 of Chakraborty and Sen Gupta[6], also follow for the present operator.

Let $p_0, q_0 \geq 1$ satisfy $\frac{p_0'}{p_0}, \frac{q_0'}{q_0} = 0(1)$ and $p_0, q_0 = 0(x^c)$, for large x, $0 < c < 1$, or alternatively, $p_0 \psi_n$, $p_0' \psi_n$, $q_0 \psi_n$, $q_0' \psi_n \in L_2[0, \infty]$. Then for the singular case $[0, \infty)$, $D(\psi_m, \psi_n) = \lambda_n \delta_{m,n}$, $\delta_{m,n}$, the Kronecker delta.

We say that $p_1, q_1, r_1, F \in \mathcal{M}$, if the following additional conditions are satisfied:

(i) $|p_1|, |q_1|, |r_1| \leq Q(x)$, $Q(x) \geq \delta > 0$

(ii) $\lim_{x \to \infty} \frac{Q'(x)}{Q^c(x)} < \infty$, $0 < c \leq \frac{3}{2}$, $Q'(x)$ continuous ;

(iii) $\lim_{x \to \infty} \frac{F'_{ij}}{F_{ij}} < \infty$, $i, j = 1, 2$.

(iv) $t(x) \leq F_{ij} \leq S(x)$, $i, j = 1, 2$, $\frac{s(x)}{t(x)}$ tends to a finite nonzero limit as x tends to infinity.

(v) $Q(x)/S(x)$ tends to infinity as x tends to infinity.

(vi) $\int^{\infty} Q(t)^{-1/2}\, dt$ is divergent.

If $f^T Ff, f'^T Ff \in L[0, \infty)$ (with $f(0) = 0$ for the Dirichlet problem and $f'(0) = 0$ for the Neumann problem), then

$$D(f, \psi_n) = \lambda_n C_n \tag{2.1}$$

and if, moreover, $p_1 \geq 0$, $\det P \geq 0$,

$$D(f) \geq \sum_{n=0}^{\infty} \lambda_n C_n^2. \tag{2.2}$$

It may be noted that the condition $f'^T Ff' \in L[0, \infty)$ as required for the derivations of (2.1) and (2.2) may be dispensed with when $p_1, q_1, r_1, F \in \mathcal{M}$.

3. Variation of the eigenvalues

As in Chakraborty and Sen Gupta[6], we say that a sequence of symmetric matrices

$$P_0 \equiv \{P_j\},\ P_j = \begin{pmatrix} p_{1j} & r_{1j} \\ r_{1j} & q_{1j} \end{pmatrix},\ j = 1, 2 \ldots,$$

defined over I is pseudo-monotonic over I, if and only if for $j < k, j, k = 1, 2, \ldots$, $p_{1j} \leq p_{1k}$, $q_{1j} \leq q_{1k}$, $p_{11} > 0$, $\det P_1 \geq 0$, and $\det(P_j - P_k) \geq 0$, for all $x \in I$.

In particular, the matrix $P \equiv \begin{pmatrix} p & r \\ r & q \end{pmatrix}$ is pseudo-monotonic over $[0, \infty)$, if for $j > k, j, k = 0, 1, 2, \ldots$, $p_j \geq p_k$, $q_j \geq q_k$, $\det(P_j - P_k) \geq 0$, where p_s, q_s, P_s are p, q, P at $x_s \in [0, \infty)$.

We denote the class of pseudo-monotonic sequences of matrices P_0 over I, by $PM(I)$.

Then by utilising the Minkowski inequality for two positive definite harmitian Matrices A, B of order n, viz.,

$$|A|^{1/n} + |B|^{1/n} \leq |A + B|^{1/n} \text{ (see Mirski}^{9}\text{, p. 419)} \ldots (A_0)$$

it easily follows that

(i) $\alpha P_0 + \beta Q_0 \in PM(I)$, where α, β are positive scalars and $P_0, Q_0 \in PM(I)$.

Also if $\{P_j\}, \{Q_j\} \in PM(I)$

(ii) $\det(P_j Q_j - P_k Q_k) \geq 0$, $j, k = 1, 2, 3, \ldots$.

The product sequence of the two sequences $\{P_j\}$, $\{Q_j\}$, is denoted by $[\{P_j\}, \{Q_j\}]$. We note that the product sequences o two pseudo-monotonic sequences $\{P_j\}$, $\{Q_j\}$, are not necessarily pseudo-monotonic.

Put $F(r, x) = \begin{pmatrix} F_{11}(r, x) & F_{12}(r, x) \\ F_{12}(r, x) & F_{22}(r, x) \end{pmatrix}$,

and $\gamma(r, x) = \begin{pmatrix} \gamma_{11}(r, x) & \gamma_{12}(r, x) \\ \gamma_{12}(r, x) & \gamma_{22}(r, x) \end{pmatrix}$, where $x \in I$.

Then the following theorems hold. It is assumed that $p_0 q_0 \geq 1$, in all the following theorems of this article. Further, when we consider the interval $[0, \infty)$, we assume that $p_1, q_1, r_1, F \in \mathcal{M}$.

Theorem 3.1 : If $\{P_j\} \in PM(I)$, $I: 0 \leq x < b$, $b = \infty$ allowed, then $\lambda_n \leq \mu_n$, $n = 0, 1, 2, 3, \ldots$, where λ_n and μ_n are the eigenvalues for the Dirichlet (Neumann) problems, with matrices P_j and P_k respectively for P, $j < k$, $j, k = 1, 2, 3, \ldots$.

Theorem 3.2 : Let $p_1 > 0$, $\det P \geq 0$, $\{F(r, x)\} \in PM(I)$, where $I: 0 \leqq x < b$, $b = \infty$ allowed. Then $\lambda_n \geq \mu_n$, $n = 0, 1, 2, 3, \ldots$, when λ_n and μ_n are respectively the eigenvalues for the Dirichlet (Neumann) problems, with $F(x) = F(r, x)$ and $F(s, x)$ respectively, with $r < s$, $r, s = 1, 2, \ldots$.

Let $I_1 \subset I: 0 \leqq x < b$, $b = \infty$ allowed and let

$$P \equiv \begin{pmatrix} p_1 & r_1 \\ r_1 & q_1 \end{pmatrix} = 0 \text{ on } I_1$$
$$= \gamma(x) F(x) \text{ on } I - I_1,$$

where $\gamma(x) = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$

is a real valued, positive definite, symmetric and absolutely continuous matrix defined on $I - I_1$, $\gamma^T \mathcal{D} \subseteq \mathcal{D}$. Then

Theorem 3.3 : If $\mu_n \geq k$, where k is *a* positive constant and the product sequence $[\{F(r, x)\}, \{KE - \gamma(r, x)\}] \in PM(I - I_1)$, then $\lambda_n \geq \mu_n \geq k$, $n = 0, 1, 2, \ldots$, where $_n$, μ_n are the eigenvalues for the Dirichlet (Neumann) problems, with

$$F(x) = F(r, x),\ P(x) = \gamma F \equiv \gamma(r, x) F(r, x) \quad \text{and}$$
$$F(x) = F(s, x),\ P(x) = \gamma(s, x) F(s, x),\ r < s,\ r, s = 1, 2, 3, \ldots$$

respectively; E is the 2×2 unit matrix.

Let the intervals $[0, b]$ and $[0, B]$, $B > b$ be represented respectively by I_b and I_B and let I_1 be an interval, included in I_b. Then we have

Theorem 3.4 : If $p_1 > 0$, and $\det P \geq 0$, and if λ_n, μ_n denote the nth eigenvalues for the Dirichlet (Neumann) problem of the intervals I_b and I_B respectively, then $\lambda_n \geq \mu_n$, $n = 0, 1, 2, \ldots$

Finally, we have

Theorem 3.5 : If $\mu_n \geq k$, where k is a positive constant and the product sequence $[\{kE - \gamma(r, x)\}, \{F(r, x)\}] \in PM(I_B - I_1)$, then $\lambda_n \geq \mu_n \geq k, n = 0, 1, 2, \ldots$, where λ_n and u_n are the eigenvalues for the problem of the intervals I_b, with

$$F(x) = F(r, x),\ P(x) = \gamma(r, x) F(r, x) \text{ and } I_B \text{ with } F(x) = F(s, x),$$

$$P(x) = \gamma(s, x) F(s, x),\ r < s,\ r, s = 1, 2, 3, \ldots, B > b.$$

E, F, γ, I_b, I_B having the same meanings as before.

The result follows by choosing

$$\begin{aligned} f(x) &= \psi_0(x), & 0 \leq x < b \\ &= 0, & b \leq x \leq B, \end{aligned}$$

so that $D_b(f, P(r, x)) - D_B(f, P(s, x))$

$$= \int_{I_b - I_1} f^T \{\gamma(r, x) F(r, x) - \gamma(s, x) F(s, x)\} f\, dx,$$

and then adopting the familiar Titchmarsh analysis[7] (pp. 89-90).

4 Discreteness of the spectra

Let $p_0, q_0 \in C^2(I)$, $I : a < x < b$, satisfy additional conditions

$$\left.\begin{aligned} p_0'^2(x) - 4 p_0(x) p_0''(x) &= A p_0(x) \\ q_0'^2(x) - 4 q_0(x) q_0''(x) &= B q_0(x) \end{aligned}\right\} \tag{4.1}$$

where $A, B \geq 0$.

Let $0 < a < x < X$, and

$$u_1(x) = u_1(x - a) = 0$$

$$v_1(x) = v_1(x - a) = \frac{1}{b_1} \{q_0(a)\}^{5/4} \{q_0(x)\}^{-1/4} \sin\{b_1(\psi(x) - \psi(a))\},$$

$\psi(x) = \int_0^x q_0(z)^{-1/2}\, dz$, and b_1, a positive constant, which depends on B.

Then it easily follows that $U_1 \equiv \{u_1, v_1\}$ satisfies the system $M_0 U_1 = 0$, where

$$M_0 \equiv \begin{pmatrix} -D(p_0 D) - 1 & 0 \\ 0 & -D(q_0 D) - 1 \end{pmatrix},$$

with initial conditions

$$u_1(a) = v_1(a) = 0$$
$$u_1'(a) = 0, \; v_1'(a) = \{q_0(a)\}^{1/2}.$$

Let $\bar{H}(x, y)$ be the matrix,

$$\bar{H}(x, y) = \begin{pmatrix} H_{11}(x, y) & H_{21}(x, y) \\ H_{12}(x, y) & H_{22}(x, y) \end{pmatrix}$$
$$= \begin{pmatrix} p_0^{-1}(y)\, q_0^{-1/2}(y)\, v_1(x-y) & u_1(x-y) \\ u_1(x-y) & q_0^{-1}(y)\, q_0^{-1/2}(y)\, v_1(x-y) \end{pmatrix}$$

And $\quad H(x, y) = \bar{H}(x, y)$, for $a < y < x$ (4.2)

$\quad 0$, otherwise.

Let $G(X, x, y, \lambda)$ be the Green's matrix for the interval $[0, X]$, with elements $G_{ij}(X, x, y, \lambda)$, which satisfy the discontinuity property

$$\frac{\partial}{\partial x} G_{ij}(X, y+0, y, \lambda) - \frac{\partial}{\partial x} G_{ij}(X, y-0, y, \lambda)$$
$$= \begin{cases} p_0^{-1}(y)\, \delta_{ij}, & \text{if } i = 1 \\ q_0^{-1}(y)\, \delta_{ij}, & \text{if } i = 2 \end{cases} \tag{4.3}$$

(see Bhagat[2]).

Then it clearly follows that $H(x, y)$, although not a Green's matrix, has the same discontinuity property (4.3), as the Green's matrix $G(X, x, y, \lambda)$. Further, $H(x, y)$ always exists in $\delta \equiv (a, x) \subset (0, X)$. (4.4)

Let $\quad \Gamma(X, x, y, \lambda) = (\Gamma_{ij}(X, x, y, \lambda)$

where

$$\left.\begin{aligned} \Gamma_{11}(X, x, y, \lambda) &= G_{11}(X, x, y, \lambda) - H_{12}(x, y) \\ \Gamma_{12}(X, x, y, \lambda) &= G_{12}(X, x, y, \lambda) - H_{11}(x, y) \\ \Gamma_{ij}(X, x, y, \lambda) &= G_{ij}(X, x, y, \lambda) - H_{ij}(x, y) \end{aligned}\right\} \tag{4.5}$$

$i = 2, \; j = 1, 2.$

Then $\quad (M - \lambda F)\, \Gamma_1(X, x, y, \lambda) = -F\{F^{-1} K_1(x, y) - \lambda H_1(x, y)\}$, (4.6)

where $\quad \Gamma_i(.) = \{\Gamma_{i1}, \Gamma_{i2}\}, \; i = 1, 2,$

$$K_1(x, y) = \begin{pmatrix} (p_1(x) + 1)\, H_{12}(x, y) + r_1(x)\, H_{11}(x, y) \\ (r_1(x)\, H_{12}(x, y) + (q_1(x) + 1)\, H_{11}(x, y) \end{pmatrix}.$$

and $H_1(x, y) = (H_{12}, H_{11})^T$.

From (4.6),

$$\Gamma_1(X, x, y, \lambda) = \int_0^X G(X, x, z, \lambda)\{K_1(z, y) - \lambda F(z) H_1(z, y)\}\, dz \tag{4.7}$$

Also, by Bhagat[1], (p. 61)

$$\int_0^X \Gamma_1^T(X, x, y, \lambda) F(x) \bar{\Gamma}_1(X, x, y, \lambda)\, dx$$

$$\leqslant \nu^{-2} \int_0^X \chi^T(z, y) F(z) \bar{\chi}(z, y)\, dz \tag{4.8}$$

where $\chi(x, y) = F^{-1}(x) K_1(x, y) - \lambda H_1(x, y), \lambda = \mu + i\nu, \nu \neq 0.$
Since det $F(X) \geq 1$,

$\Gamma_1^T(.) F\bar{\Gamma}_1(.) \geq |\Gamma_{11}(.)|^2$, and hence from (4.8), after some tedious reductions,

$$\int_0^X |\Gamma_{11}(X, x, y, \lambda)|^2\, dx \leq \nu^{-2} K(y, \delta, |\lambda|) \tag{4.9}$$

where $K(.)$ denotes the constant depending on the arguments shown. Similar results hold for the other Γ_{ij}.
From (4.5)

$$\int_0^X |G_{11}(X, x, y, \lambda)|^2\, dx \leq (1 + \nu^{-2}) K(y, \delta, |\lambda|) \tag{4.10}$$

with similar results for the other $G_{ij}(X, x, y, \lambda)$. From results of type (4.7), by making use of the properties of $H_{ij}(x, y)$, the Schwarz inequality, and the relations of type (4.10), it follows that

$$|\Gamma_{ij}(X, x, y, \lambda)| \leq (\nu^{-2} + 1)^{1/2} K(x, y, \delta, |\lambda|) \tag{4.11}$$

where x, y lie in a fixed $\delta_0 \subset \delta$.

We now make use of the formula, easily verifiable by integration by parts, *viz.*,

$$(\xi - x)^2 \phi(x) h(x)$$

$$= \int_0^\xi (\xi - y)^2 (y - x)\left(\frac{d}{dy}\, \phi\, \frac{d}{dy}\right) h(y)\, dy$$

$$-\int_0^\xi (\xi - y)^2 (y - x) \phi''(y) h(y) dy - 2\int_0^\xi (\xi - y)(\xi - 3y + 2x) \phi'(y) h(y)\, dy$$

$$+ \int_0^\xi (2x + 4\xi - 6y) \phi(y)\, h(y)\, dy,$$

and proceed in a manner, as indicated in Chakraborty[6], so as to derive that G_{ij} (X, x, y, λ) tends uniformly to $G_{ij}(x, y, \lambda)$, as X tends to infinity through a suitable sequence and $G(x, y\ \lambda) = (G_{ij}(x, y, \lambda))$ is the Green's matrix in the singular case $[0, \infty)$ with the usual properties.

We now establish the following theroem :

Theorem 4.1 : Let $p_1(x)/F_{11}(x) > a$, be monotone increasing, det $(P - aF) \geq 0$, for all $x \in I : 0 \leqq x < \infty$, where a is a positive constant and p_0, q_0 satisfy the conditions (i) $p_0\ q_0 \geq 1$, (ii) $p_0 \cdot q_0 \in C(I)$ and (iii) the conditions (4.1). Also let the matrix P be pseudo-monotonic over I. Then the spectrum of the given boundary value problem is discrete over (a, β), where $\beta > a$, is arbitrary.

Let the eigenvalues for the problem of the intervals $[0, X]$ and $[0, X']$, $X \leqq X'$, be represented respectively by $\lambda_{nX'}$ and $\lambda_{nX'}$. Then from the given conditions

$$\lambda_{nX'}, \lambda_{nX'} \geq a > 0, \text{ and that } \lambda_{nX'} \geq \lambda_{nX'} \text{ (Theorem 3.4).}$$

Hence for sufficiently large X, the sequence $\{\lambda_{jX}\}$, $j = 0, 1, 2, \ldots h$ of eigenvalues lying in $(a \mid \beta)$ tend to $\{\lambda_j\}$ $j = 0, 1, 2, \ldots h$ (not necessarily all different).

Let $\lambda_m < \lambda_{m-1}$, the Green's matrix $G(X, x, y, \lambda)$ $\lambda = \mu + iv$ for the interval $[0, X]$ for our problem, is regular except at the points $\lambda = \lambda_{nw}$, which are the simple poles of $G(X, x, y, \lambda)$.

Put $-\delta \leqq v \leqq \delta$, $|\delta| \leqq 1$, and $\lambda_m + 2\delta \leqq \mu \leqq \lambda_m - 2\delta$. Then for given x, y, $x \neq y$, it follows from (4.11), that $|G_{ij}(X, x, y, \lambda)| \leqq M\,|v|^{-1}$, M constant.

The theorem now follows by arguments, similar to those of Titchmarsh[7] (p. 149).

It is easily verifiable that the λ_n are actually the eigenvalues.

Theorem 4.2 : Let p_0, q_0 satisfy the conditions of theorem 4.1 and let

$$\frac{p_1}{F_{11}},\ \frac{q_1}{F_{22}} \geq \left|\frac{r_1}{F_{12}}\right|$$

where r_1/F_{12} is monotone increasing. Then the spectrum of our problem is discrete over $(0, \infty)$.

This is an immediate consequence of the theorem 4.1.

Let P and F be related by $P = \gamma F$, where γ is defined as in §3. Then the following theorem giving the discreteness of the spectra holds.

Theorem 4.3 : If, in addition to the conditions of theorem 4.1, $\gamma_{11}(x) > \alpha$ be monotone increasing, $\det(\gamma - \alpha E) \geq 0$, for all $x \in I : 0 \leqq x < \infty$, and F_{ij} and $\gamma_{ij}, i, j = 1, 2$ maintain the same sign in $[0, \infty)$, then the spectrum of our problem is discrete over (α, β), β arbitrary, $\beta > \alpha > 0$.

Finally, we note that Sen Gupta's theorem[8] is only a special case of the general theorem 4.1 obtained above.

Acknowledgements

The authors express their grateful thanks to the referee for his valuable suggestions and comments.

References

1. BHAGAT, B. — Eigenfunction expansions associated with a pair of second order differential equations, Ph.D. thesis, Patna University, 1966.

2. BHAGAT, B. — Eigenfunction expansions associated with a pair of second order differential equations, *Proc. Natn. Inst. Sci. India*, 1969, Part A35, 161–174.

3. BHAGAT, B. — Some problems on a pair of singular second order differential equations, *Proc. Natn. Inst. Sci. India*, 1969, A35, 232–244.

4. CHAKRABORTY, N. K. — Some problems in eigenfunction expansions (I), *Q. J. Math. (Oxford)*, 1965, 16, 135–150.

5. CHAKRABORTY, N. K. — Some problems in eigenfunction expansions (III), *Q. J. Math. (Oxford)*, 1968, 19, 397–415.

6. CHAKRABORTY, N. K. AND SEN GUPTA, P. K. — On the distribution of the eigenvalues of a matrix differential operator, *J. Indian Inst. Sci.*, 1979, 61 (B), 19–42.

7. TITCHMARSH, E. C. — *Eigenfunction expansions associated with second order differential equations, Part II*, Oxford University Press, 1958.

8. SEN GUPTA, PRABIR KUMAR — The spectrum of a matrix differential operator, *J. Indian Inst. Sci.*, 1980, 62 (B), 43–46.

9. MIRSKY, L. — *An introduction to linear algebra*, Oxford, 1955.