J. Indian Inst. Sci., 64 (B), Aug. 1983, Pp. 211–217
© Indian Institute of Science, Printed in India.

Applications of complementary variational principles

M. A. GOPALAN*
Department of Mathematics, National College, Tiruchirapalli 620 001, Tamil Nadu, India.

Received on March 2, 1983 ; Revised on July 2, 1983.

Abstract

Bounds on quantities of physical interest are derived for boundary value problems by applying the canonical theory of complementary variational principles.

Key words: Variational principles. complementary principles.

1. Introduction

Arthurs[1] has shown that one can construct complementary variational principles for the pair of canonical equations

$$Tx = \frac{\partial W}{\partial y}, \quad T^* y = \frac{\partial W}{\partial x}$$

where T and T^* are adjoint linear operators, $W(x, y)$ is a functional which is convex in x and concave in y and $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ are appropriate functional derivatives.

In fluid mechanics it is possible to identify situations where the governing equations are of the canonical form. More specifically, the procedure has been illustrated by formulating extremum principles for boundary value problems occurring in the study of the following situations :

(i) The steady flow of a mixture of two incompressible Newtonian fluids through a pipe of arbitrary cross-section S. Bounds on the flux are derived.

*A part of this paper is supported financially by the UGC, New Delhi, Grant No. P25-3 (13368/83).

(ii) The Reynolds equation for pressure in a full finite journal bearing which is fed axially with an incompressible lubricant. Bounds on one of the load components are obtained without having recourse to finding the series solution or numerical solution of the differential equation. We content ourselves with merely sketching the method.

2. Mixture of two incompressible Newtonian fluids

2.1 *Mathematical formulation*

From the general theory given by Craine[2], the governing equations may be derived as

$$A\nabla^2 u + B\nabla^2 v - \alpha(u - v) = -K\gamma \qquad (2.1)$$

$$C\nabla^2 u + D\nabla^2 v + \alpha(u - v) = -K(1 - \gamma) \qquad (2.2)$$

where ∇^2 is the two-dimensional Laplacian, u and v are the velocity components of the respective constituents in the mixture, the constant K (> 0) is the pressure gradient for the flow, and γ $(0 < \gamma < 1)$ is a composition factor which is a constant and $A = \mu_1 + \frac{1}{2}\lambda_5$, $B = \mu_3 - \frac{1}{2}\lambda_5$, $C = \mu_4 - \frac{1}{2}\lambda_5$, $D = \mu_2 + \frac{1}{2}\lambda_5$. The coefficients $\mu_1, \mu_2, \mu_3, \mu_4, \lambda_5$ and α are constants and satisfy the relations :

$$\mu_1 \geq 0,\ \mu_2 \geq 0,\ 4\mu_1\mu_2 \geq (\mu_3 + \mu_4)^2,\quad \alpha \geq 0,\ \lambda_5 > 0.$$

The fluid is assumed to satisfy the condition that

$$u = v = 0 \text{ on the boundary } M. \qquad (2.3)$$

The mean velocity is defined to be

$$U = \gamma u + (1 - \gamma)v \qquad (2.4)$$

and the volume flow rate as

$$Q = \int_S U\, dS. \qquad (2.5)$$

Eliminating u and v from (2.1) and (2.2) in turn and using (2.4), we have

$$(\nabla^4 - \beta^2\nabla^2)U = k_1 \qquad (2.6)$$

subject to

$$U = 0,\ \nabla^2 U = -l \text{ on } M \qquad (2.7)$$

where

$$k_1 = \alpha K/(AD - BC)$$

$$\beta^2 = \alpha \sum_{i=1}^{4} \mu_i/(AD - BC)$$

$$l = \frac{K}{(AD - BC)}[\gamma^2 D - \gamma(1 - \gamma)(B + C) + (1 - \gamma)^2 A].$$

It has been shown in ref. 3 that U is always positive for an arbitrary-shaped pipe and $(AD - BC) > 0$, $l > 0$.

To put (2.6) in canonical form, we take

$$T = \begin{pmatrix} \nabla^2 \\ \beta \text{ grad} \end{pmatrix}$$

and its adjoint

$$T^* = (\nabla^2 - \beta \text{ div})$$

so that (2.6) takes the form

$$T^*TU = k_1. \tag{2.8}$$

Now we write

$$TU = \phi = \frac{\partial H}{\partial \phi} \tag{2.9}$$

$$T^*\phi = k_1 = \frac{\partial H}{\partial U}. \tag{2.10}$$

A suitable Hamiltonian H is, therefore,

$$H(U, \phi) = \tfrac{1}{2}\phi^t\phi + k_1 U, \tag{2.11}$$

where ϕ^t is transpose of ϕ.

2.2 *Complementary principles*

Consider the functional

$$I(\bar{U}, \phi) = \int_S (H(\bar{U}, \phi) - \phi^t T\bar{U})\, dS$$

$$+ \int_C \left(\phi_1 \frac{\partial \bar{U}}{\partial n} - \bar{U}\frac{\partial \phi_1}{\partial n} + \beta \bar{U}\phi_2 . n\right) dM \tag{2.12}$$

$$= \int_S (H(\bar{U}, \phi) - \bar{U}T^* \phi)\, dS \tag{2.13}$$

where $\phi = (\phi_1, \phi_2)^t$. The exact solution of the problem is denoted by $\bar{U} = U$, $\phi = \phi$. First, choose a trial function $\bar{U}$ satisfying $T\bar{U} = \phi$ with the conditions $\bar{U} = 0$, $\nabla^2 \bar{U} = -l$ on M. Then (2.12) gives

$$G(\bar{U}) = \int_S (k_1\bar{U} - \tfrac{1}{2}[(\nabla^2 \bar{U})^2 + \beta^2 \nabla \bar{U} \cdot \nabla \bar{U}])\, dS - l \int_C \frac{\partial \bar{U}}{\partial n}\, dM. \tag{2.14}$$

Next, choose another trial function ϕ satisfying $T^*\phi = k_1$. Then (2.13) gives

$$J(\phi) = \tfrac{1}{2}\int_S \phi^t\phi\, dS. \tag{2.15}$$

The functionals $G(\bar{U})$ and $J(\phi)$ provide lower and upper bounds to $I(U, \phi)$, that is,

$$G(\bar{U}) \leq G(U) = I(U, \phi) = J(\phi) \leq J(\phi). \tag{2.16}$$

Multiplying (2.6) by U and integrating we have

$$k_1 Q = \int_S [(\nabla^2 U)^2 + \beta^2 \nabla U \, , \nabla U] \, dS + l \int_C \frac{\partial U}{\partial n} dM. \tag{2.17}$$

Using (2.17) we write

$$I(U, \phi) = \frac{k_1}{2} Q - \frac{l}{2} \int_C \frac{\partial U}{\partial n} \, dM. \tag{2.18}$$

Thus (2.16) is written as

$$\frac{2}{k_1} G(\bar{U}) + \frac{l}{k_1} \int_C \frac{\partial \bar{U}}{\partial n} dM \leq Q \leq \frac{2}{k_1} J(\phi) + \frac{l}{k_1} \int_C \frac{\partial \bar{U}}{\partial n} dM. \tag{2.19}$$

As the exact function ϕ satisfies $TU = \phi$, it is convenient to choose $\phi = T\psi$ where ψ is an approximation to the exact function U. Thus the RHS of (2.19) becomes

$$\frac{2}{k_1} J(T\psi) + \frac{l}{k_1} \int_C \frac{\partial \psi}{\partial n} \, dM,$$

where ψ satisfies the constraint $T^* \, T\psi = k_1$. By choosing properly the trial functions, the bounds on Q are obtained to any desired degree of accuracy.

3. Pressure equation in a full finite journal bearing

3.1. *Governing equations*

Consider a journal bearing of length L and radius R rotating with a constant angular velocity ω. Reynolds equation for the pressure p in a full finite bearing, when the viscosity of lubricant μ is assumed to be constant, is[4]

$$\frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial z} \left(h^3 \frac{\partial p}{\partial z} \right) = 6 \mu U \frac{\partial h}{\partial x} \tag{3.1}$$

where x is taken in the direction of rotation. Here h is the film thickness and U is the velocity of the journal. Now let x and z be the coordinates along the circumferential and axial directions of the bearing (fig. 1). The film thickness may be written as

$$h = c_1 (1 + \epsilon \cos \theta) = c_1 H, \quad \epsilon = e/c_1$$

where c_1 = radial bearing clearance,
e = eccentricity, and
ϵ = eccentricity ratio.

Introducing dimensionless quantities $\theta = x/R$, $\eta = z/R$ (3.1) becomes

$$\frac{\partial}{\partial \theta}\left(H^3 \frac{\partial p}{\partial \theta} \right) + \frac{\partial}{\partial \eta}\left(H^3 \frac{\partial p}{\partial \eta} \right) = \alpha \frac{\partial H}{\partial \theta}, \quad \alpha = 6 \mu \omega (R^2/c_1^2). \tag{3.2}$$

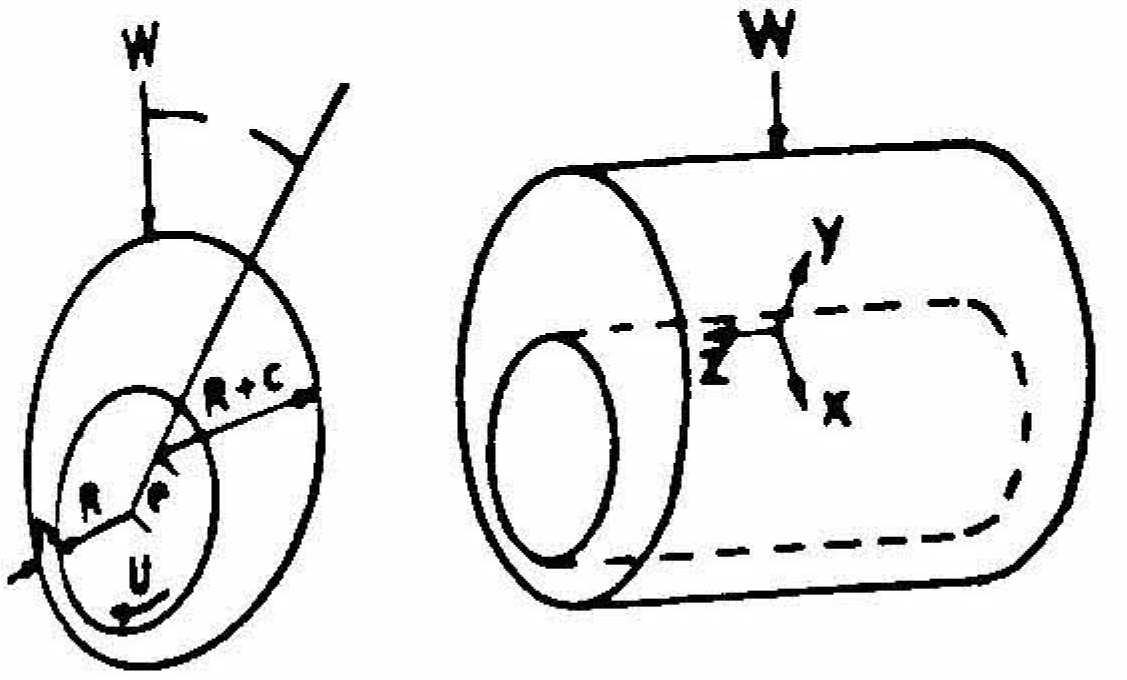

Fig. 1. Journal-bearing configuration.

We apply the boundary conditions that $p = 0$ at $\eta = -\xi\ \xi$, $\xi = L/2R$ and the approximate cavitation condition that $p = 0$ for $0 \leq \theta \leq \pi$, *i.e.*, we will not allow negative pressures in the system. Multiplying both sides of (3.2) by p and integrating over the surface, we get

$$a\epsilon \int_{-\xi}^{\xi}\int_{0}^{\pi} p \sin\theta \; d\theta\, d\eta = \int_{-\xi}^{\xi}\int_{0}^{\pi} H^3 \left[\left(\frac{\partial p}{\partial\theta}\right)^2 + \left(\frac{\partial p}{\partial\eta}\right)^2\right] d\theta\, d\eta. \tag{3.3}$$

To put (3.2) in canonical form, introduce the operators

$$T_1 = H^{3/2}\frac{\partial}{\partial\theta},\ T_2 = H^{3/2}\frac{\partial}{\partial\eta} \tag{3.4}$$

and their adjoints

$$T_1^* = -\frac{\partial}{\partial\theta}(H^{3/2}),\ T_2^* = -\frac{\partial}{\partial\eta}(H^{3/2}) \tag{3.5}$$

where $\iint \psi T_k \phi \, d\theta\, d\eta = \iint (T_k^* \psi)\, \phi \, d\theta\, d\eta$ + Boundary Terms, $k = 1, 2$ for all suitable functions ψ and ϕ. These operators are now used to define the operator T and its adjoint T^* which are such that $Tp = \begin{pmatrix} T_1 p \\ T_2 p \end{pmatrix}$ for all suitable scalar functions p and $T^* u = T_1^* u_1 + T_2^* u_2$ for all suitable vector functions $\bar{u}$ with components u_1 and u_2.

Then (3.2) takes the form

$$T^* T p = -a\,\partial H/\partial\theta. \tag{3.6}$$

Now we write

$$Tp = u = \partial\bar{H}/\partial u,\ T^* u = -a\,\partial H/\partial\theta = \partial\bar{H}/\partial p.$$

A suitable Hamiltonian $\bar{H}$ is, therefore,

$$\bar{H}(u, p) = \tfrac{1}{2} u^t u - ap\,(\partial H/\partial\theta)$$

where u^t = transpose of u.

3.2. *Derivation of bounds*

Consider the functional

$$I(U, P) = \int_{-\xi}^{\xi}\int_{0}^{\pi} (U^t TP - \bar{H}(U, P)\, d\theta\, d\eta - \int_{-\xi}^{\xi} H^3 P (\partial P/\partial\theta) \Big|_0^{\pi} d\eta$$

$$- \int_0^{\pi} H^3 P (\partial P/\partial \eta) \Big|_{-\xi}^{\xi} d\theta \tag{3.7}$$

$$= \int_{-\xi}^{\xi}\int_0^{\pi} [(T^* U)\ P - \bar{H}(U, P)]\, d\theta\, d\eta. \tag{3.8}$$

The exact solution of the problem is denoted by $U = u$, $P = p$. First, choose a trial function P satisfying $TP = U$ with the conditions $P(0, \eta) = P(\pi, \eta) = P(\theta, -\xi) = P(\theta, \xi) = 0$. Then (3.7) gives

$$J(P) = \int_{-\xi}^{\xi}\int_0^{\pi} \left\{ \frac{1}{2} H^3 \left[\left(\frac{\partial P}{\partial \theta}\right)^2 + \left(\frac{\partial P}{\partial \eta}\right)^2 \right] + \alpha P \frac{\partial H}{\partial \theta} \right\} d\theta\, d\eta. \tag{3.9}$$

Next, choose another trial function U satisfying $T^*U = -\alpha(\partial H/\partial\theta)$. Then (3.8) gives

$$G(U) = -\tfrac{1}{2}\int_{-\xi}^{\xi}\int_0^{\pi} U^t U\, d\theta\, d\eta. \tag{3.10}$$

The functionals $J(P)$ and $G(U)$ provide upper and lower bounds to $I(u, p)$, that is,

$$G(U) \le G(u) = I(u, p) = J(p) \le J(P). \tag{3.11}$$

Since

$$I(U, p) = -\tfrac{1}{2}\alpha\epsilon \int_{-\xi}^{\xi}\int_0^{\pi} p \sin\theta\, d\theta\, d\eta$$

by using (3.3), (3.11) is written as

$$-\frac{2}{\alpha\epsilon} R^2 J(P) \le W \sin\phi \le -\frac{2}{\alpha\epsilon} R^2 G(U) \tag{3.12}$$

where

$$W \sin\phi = R^2 \int_{-\xi}^{\xi}\int_0^{\pi} p \sin\theta\, d\theta\, d\eta.$$

Since the exact function u satisfies $Tp = u$, it is convenient to choose $U = T\psi$ where ψ is an approximation to the exact function p. Thus the RHS of (3.12) becomes

$$-\frac{2}{\alpha\epsilon} R^2 G(T\psi)$$

where

$$G(T\psi) = -\frac{1}{2}\int_{-\xi}^{\xi}\int_{0}^{\pi} H^3\left[\left(\frac{\partial\psi}{\partial\theta}\right)^2+\left(\frac{\partial\psi}{\partial\eta}\right)^2\right]d\theta\, d\eta$$

and ψ satisfies the constraint $T^*T\psi = -\alpha(\partial H/\partial\theta)$. It is seen that (3.12) provides upper and lower bounds on the component of the load capacity at right angles to the line of centres of the journal bearing. We note that it is possible to derive bounds on $W \sin\phi$ when the lubricant is fed circumferentially.

References

1. ARTHURS, A. M. *Complementary variational principles*, Clarendon Press, Oxford, 1970.
2. CRAINE, R. E. *Int. J. Engng. Sci.*, 1971, 9, 1177.
3. KNOPS, R. J. AND STRAUGHAN, B. *Q.J.M.A.M.*, 1979, 32, 259.
4. PINKUS, O. AND STERNLICHT, B. *Theory of hydrodynamic lubrication*, McGraw-Hill Book Co., New York, 1961.