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Penny-shaped crack in a nonhomogeneous solid under torsion

P. K. CHAUDHURI AND SUBHANKAR RAY
Department of Applied Mathematics, University College of Science, 92, Acharya Prafulla Chandra Road, Calcutta 700 009, India.

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Abstract

Stress distributions in the neighbourhood of a penny-shaped crack in an elastic medium under torsion and having variable elastic coefficients have been determined and the stress intensity factor and the energy of the crack calculated. Two cases are considered: (a) elastic material is isotropic and nonhomogeneous, and (b) material is transversely isotropic and nonhomogeneous. Numerical results showing the stress distribution in the medium have been presented for nonhomogeneous as well as the associated homogeneous media to assess the effect of nonhomogeneity on stresses. Some of the numerical results have been compared with works of similar nature.

Keywords: Isotropic material, transversely isotropic material, nonhomogeneous medium, penny-shaped crack, stress intensity factor, crack energy.

1. Introduction

The presence of a crack in an elastic medium affects the stress distribution in the medium. When a crack appears in a stressed medium, the associated disturbance may be studied in two separate regions, *viz.*, the near-field region or the local region near the edge of the crack and the far-field region or the region close to the wave front propagating away from the crack. In fracture mechanics, the singular character of the stresses near the periphery of the crack plays an important role in predicting the failure of the solid. In addition to the stresses in the vicinity of the crack, other quantities of physical interest are the stress intensity factor and the energy of the crack. Sack[1] has shown that the presence of a crack in an isotropic solid under uniform torsion alters the free energy of the solid. Bassani and Qu[2] estimated the low-yielding plastic zones surrounding the crack tips of a Griffith crack at the interface of two distinct anisotropic media.

In a recent study, Choudhury and Maity[3] observed that the energy of a crack in torsion in a transversely isotropic material could be less or greater than that in an isotropic medium, depending upon the magnitude of the rigidity modulus of the medium. Following the same line of study, Chaudhuri and Sen[4] investigated the problem in a nonhomogeneous transversely isotropic medium with exponentially varying elastic coefficients. The motivation behind such a consideration is quite natural from the fact that the assumption of homogeneity of the medium does not seem to be very adequate always. Experimental results also confirm[5] the variation of elastic coefficients with position.

Variations of elastic coefficients in more than one direction have also been observed in the literature. Singh[6] solved the Reissner–Sagoci problem in elasticity considering shearing modulus as a function of r and z. Dhaliwal and Singh[7] determined the state of stress in an infinite nonhomogeneous elastic medium containing a Griffith crack under a shear force, the material nonhomogeneity being assumed in quite general forms: $\mu = \mu_0\, p(x)\, q(y)$ and $\mu = \mu_0\, p(r)\, q(z)$. Chaudhuri and Ray[8] discussed the stress distribution in a nonhomogeneous isotropic medium with a penny-shaped flaw considering shear modulus as a function of r and z. Several distinct models are also noticeable in the literature to discuss nonhomogeneity in the elasticity problem. As a matter of fact, the earth crust itself is nonhomogeneous. Hence, the investigation of these problems in a nonhomogeneous medium would be very interesting and realistic as well. As the dependence of elastic parameters with position may be arbitrary, investigators usually think of certain models with specific variations in elastic coefficients keeping in mind that the governing differential equations can be handled effectively with the existing mathematical tools. The applicability of such a rather simplified model can be confirmed only by experimental results. Anyway, these models indicate how elastic behaviours are affected by nonhomogeneity of the medium.

In this paper our objective is to study the elastic behaviours in a nonhomogeneous isotropic and in transversely isotropic media under torsion in the presence of a crack. In both the cases, the governing equations of the mixed boundary value problem have been reduced to Fredholm's integral equations. These equations are to be solved numerically to find the near-field solution for the stresses, surface displacements, stress intensity factor and the energy of the crack. It is also observed that the results of the associated homogeneous cases may easily be recovered from our results by letting the nonhomogeneity parameter zero. Some numerical results have also been presented graphically to get an idea about the effect of nonhomogeneity.

2. Formulation of the problem

Let there be a penny-shaped crack of radius a in an infinite elastic medium of a nonhomogeneous material. We shall suppose that the crack is opened by an 'all-round' torsion in the medium. In reality we may think of a crack on a plane normal to the axis of a cylinder whose radius and height are very large compared to the size of the crack and which is under the action of torsion. We may also think of a crack on a diametrical plane of a very large sphere under torsion. We shall assume the plane of the crack to be $z = 0$ and use cylindrical coordinates (r, θ, z) to specify the position of a point in the medium. Further, we shall assume nonhomogeneity of the medium in the form

$$\mu = \mu_0 (1 + \beta_0 |z|)^\alpha$$

for an isotropic medium, and

$$C_{ii} = C_{ii}^0 (1 + \beta_0 |z|)^\alpha$$

for a transversely isotropic medium, α and β_0 (> 0) being real constants.

From the symmetry of the applied force and also from the symmetry of the material nonhomogeneity with respect to the plane $z = 0$, it would be sufficient to consider the solution of the problem for the half-space $z \geq 0$ only. Since the problem is axisymmetric in nature, the displacement components u_r , u_z vanish everywhere, and the only nonzero displacement component u_θ will be independent of θ.

The equation of equilibrium, which is not automatically satisfied, is

$$\frac{\partial}{\partial r}(\sigma_{r\theta})+\frac{\partial}{\partial z}(\sigma_{\theta z})+\frac{2}{r}\sigma_{r\theta}=0. \tag{1}$$

Let us set

$$\xi = r/a, \quad \eta = z/a, \quad u = u_\theta/a. \tag{2}$$

The boundary conditions for the problem are

$$\left.\begin{aligned} &\sigma_{\theta z}(\xi, 0) = 0, && 0 \leq \xi \leq 1, \\ &u(\xi, 0) = 0, && \xi > 1, \\ &\sigma_{\theta z}(\xi, \eta) \sim S && \text{as } \sqrt{\xi^2+\eta^2} \to \infty, \end{aligned}\right\} \tag{3}$$

S being the applied torque.

Now, as the stress components depend on the medium concerned, and our objective is to study the elastic behaviour in nonhomogeneous isotropic and in transversely isotropic half-spaces, we categorize our discussion as Case A and Case B, respectively.

Case A: Isotropic medium

The nonzero components of stress are

$$\begin{aligned} \sigma_{r\theta}(\xi, \eta) &= \mu\frac{\partial u}{\partial \eta}, \\ \sigma_{r\theta}(\xi, \eta) &= \mu\left(\frac{\partial u}{\partial \xi}-\frac{u}{\xi}\right). \end{aligned} \tag{4}$$

The nonhomogeneity of the medium is given by

$$\mu = \mu_0\,(1+\beta\eta)^\alpha, \quad \eta \geq 0. \tag{5}$$

Using (4), (5) and (2) in (1), the governing differential equation is obtained as

$$\frac{\partial^2 u}{\partial \xi^2}+\frac{1}{\xi}\frac{\partial u}{\partial \xi}-\frac{u}{\xi^2}+\frac{\partial^2 u}{\partial \eta^2}+\frac{\alpha\beta}{1+\beta\eta}\frac{\partial u}{\partial \eta}=0. \tag{6}$$

Thus, the solution of the problem requires the solution of eqn (6), subject to boundary conditions (3).

Case B: *Transversely isotropic medium*

In this case the stress–displacement relations are given by

$$\sigma_{\theta z}(\xi, \eta) = C_{44}\frac{\partial u}{\partial \eta},$$
$$\sigma_{r\theta}(\xi, \eta) = C_{66}\left(\frac{\partial u}{\partial \xi} - \frac{u}{\xi}\right), \tag{7}$$

where C_{44} and C_{66} are elastic coefficients such that

$$C_{ii} = C_{ii}^{0}(1+\beta\eta)^{\alpha}, \quad \eta \geq 0, \tag{8}$$

where C_{ii}^{0} $(i = 4, 6)$ are real constants. The governing differential equation in this case becomes

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial u}{\partial \xi} - \frac{u}{\xi^2} + p^2\left(\frac{\partial^2 u}{\partial \eta^2} + \frac{\alpha\beta}{1+\beta\eta}\frac{\partial u}{\partial \eta}\right) = 0 \tag{9}$$

where

$$p^2 = C_{44}^{0} / C_{66}^{0}.$$

Our problem in this case is to solve eqn (9) subject to boundary conditions (3).

3. Solution of the problem

Case A: We take the solution of eqn (6) in the form

$$u = u(\xi, \eta)$$
$$= \frac{S}{2\mu_0\beta m}\left[(1+\beta\eta)^{2m} - 1\right] + \frac{S}{\mu_0}(1+\beta\eta)^m \int_0^\infty A(\lambda) K_m\{\lambda(1+\beta\eta)/\beta\}\, J_1(\lambda\xi)\, \mathrm{d}\lambda, \tag{10}$$

where

$$m = (1-\alpha)/2. \tag{11}$$

J_1 is a Bessel function of the first kind of order 1 and K_m is the modified Bessel function of the second kind of order m, and $A(\lambda)$ is an arbitrary function of λ.

From eqn (4) using eqns (5) and (10), we have

$$\sigma_{\theta z}(\xi, \eta) = S - S(1+\beta\eta)^{1-m}\int_0^\infty \lambda A(\lambda) K_{1-m}\{\lambda(1+\beta\eta)/\beta\}\, J_1(\lambda\xi)\, \mathrm{d}\lambda \tag{12}$$

and

$$\sigma_{r\theta}(\xi, \eta) = -S(1+\beta\eta)^{1-m}\int_0^\infty \lambda A(\lambda) K_m\{\lambda(1+\beta\eta)/\beta\}\, J_2(\lambda\xi)\, \mathrm{d}\lambda\,. \tag{13}$$

From eqn (12) we find that the third condition of eqn (3) will be satisfied if the Hankel transform exists at all. The first two boundary conditions of eqn (3) will be satisfied if $A(\lambda)$ satisfies the following pair of dual integral equations:

$$\int_0^\infty \lambda A(\lambda) K_{1-m}(\lambda/\beta)\, J_1(\lambda\xi)\, d\lambda = 1, \quad 0 \le \xi \le 1, \tag{14}$$

$$\int_0^\infty A(\lambda) K_m(\lambda/\beta)\, J_1(\lambda\xi)\, d\lambda = 0, \quad \xi > 1. \tag{15}$$

Assuming $B(\lambda) = A(\lambda)\, K_m(\lambda/\beta)$, we get from eqns (14) and (15),

$$\int_0^\infty \lambda B(\lambda)[1+\phi(\lambda)] J_1(\lambda\xi)\, d\lambda = 1, \quad 0 \le \xi \le 1, \tag{16}$$

$$\int_0^\infty B(\lambda) J_1(\lambda\xi)\, d\lambda = 0, \qquad \xi > 1, \tag{17}$$

where

$$\phi(\lambda) = \frac{K_{1-m}(\lambda/\beta)}{K_m(\lambda/\beta)} - 1. \tag{18}$$

For solving the dual integral eqns (16) and (17) we shall follow Sneddon[9]. We assume the solution of the system (16) and (17) in the form

$$B(\lambda) = \sqrt{\frac{2\lambda}{\pi}} \int_0^1 \sqrt{x}\, \theta(x) J_{3/2}(\lambda x)\, dx, \tag{19}$$

where $\theta(x)$ is a certain unknown function to be determined. Introducing eqn (19) into eqn (17) and changing the order of integration and using the formula[10]

$$\int_0^\infty t^{\mu-\nu+1} J_\mu(\alpha t) J_\nu(\beta t)\, dt = \begin{cases} 0, & 0 < \beta < \alpha, \\ \dfrac{2^{\mu-\nu+1}\alpha^\mu(\beta^2-\alpha^2)^{\nu-\mu-1}}{\beta^\nu \Gamma(\nu-\mu)}, & \beta > \alpha > 0, \end{cases} \tag{20}$$

$$(\mathrm{Re}\,\nu > \mathrm{Re}\,\mu > -1)$$

we note that it is identically satisfied. Substituting eqn (19) into eqn (16) and then after some manipulation, we arrive at the following Fredholm's integral equation for the determination of $\theta(x)$:

$$\theta(x) + \frac{1}{\pi}\int_0^1 M(x,t)\,\theta(t)\, dt = \frac{\pi x}{4}, \qquad 0 \le x \le 1, \tag{21}$$

where the kernel $M(x, t)$ is given by

$$M(x,t) = \pi\sqrt{tx}\int_0^\infty \lambda\phi(\lambda) J_{3/2}(\lambda x) J_{3/2}(\lambda t)\, d\lambda. \tag{22}$$

For the evaluation of $\theta(x)$ from the integral eqn (21), the kernel $M(x, t)$, as given by eqn (22), should be convergent. We note that $\phi(\lambda) \to 0$ as $\lambda \to \infty$ and $\phi(\lambda)$ behaves as λ^{-1}

for large λ. Also, $\phi(\lambda)$ satisfies the convergence condition as $\lambda \to 0$, So, $\phi(\lambda)$ satisfies all the necessary conditions for the convergence of $M(x, t)$.

Using the result[11]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-1/2} J_{1/2}(\lambda x)\right] = -\lambda x^{-1/2} J_{3/2}(\lambda x) \tag{23}$$

and applying integration by parts, eqn (19) gives

$$A(\lambda) K_m(\lambda/\beta) = -\sqrt{\frac{2}{\pi\lambda}}[\theta(1) J_{1/2}(\lambda) - \int_0^1 [x\,\theta(x)]'\, x^{-1/2}\, J_{1/2}(\lambda x)\,\mathrm{d}x], \tag{24}$$

where prime denotes differentiation with respect to x. By using eqn (24) and the standard results[11]

$$J_{1/2}(\lambda x) = \sqrt{\frac{2\lambda x}{\pi}}\frac{\sin \lambda x}{\lambda x} \tag{25}$$

and

$$\int_0^\infty \frac{\sin \lambda x}{\lambda} J_1(\lambda\xi)\,\mathrm{d}\lambda = \begin{cases} x/\xi, & x \le \xi, \\ -\dfrac{\sqrt{x^2-\xi^2}-x}{\xi}, & x > \xi, \end{cases}$$

the surface displacement $u(\xi, 0)$ can be expressed from eqn (10) as

$$u(\xi, 0) = \frac{2S}{\pi\mu_0\xi}\Bigg[\theta(1)\left(\sqrt{1-\xi^2}-1\right) + \int_0^\xi [x\,\theta(x)]'\,\mathrm{d}x - \int_\xi^1 [x\,\theta(x)]' \frac{\left(\sqrt{x^2-\xi^2}-x\right)}{x}\,\mathrm{d}x\Bigg], \quad \xi < 1. \tag{26}$$

By using eqns (24) and (25), we get from eqns (12) and (13),

$$\sigma_{\theta z}(\xi, \eta) = S + \frac{2S}{\pi}(1+\beta\eta)^{1-m}\Bigg[\theta(1)\int_0^\infty \frac{K_{1-m}\{\lambda(1+\beta\eta)/\beta\}}{K_m(\lambda/\beta)} J_1(\lambda\xi)\sin\lambda\,\mathrm{d}\lambda - \int_0^1 \frac{1}{x}[x\theta(x)]' \left(\int_0^\infty \frac{K_{1-m}\{\lambda(1+\beta\eta)/\beta\}}{K_m(\lambda/\beta)} J_1(\lambda\xi)\sin\lambda x\,\mathrm{d}\lambda\right)\mathrm{d}x\Bigg], \tag{27}$$

$$\sigma_{r\theta}(\xi, \eta) = \frac{2S}{\pi}(1+\beta\eta)^{1-m}\Bigg[\theta(1)\int_0^\infty \frac{K_m\{\lambda(1+\beta\eta)/\beta\}}{K_m(\lambda/\beta)}\left(\frac{2J_1(\lambda\xi)}{\lambda\xi} - J_0(\lambda\xi)\right)\sin\lambda\,\mathrm{d}\lambda - \int_0^1 \frac{1}{x}[x\,\theta(x)]'\left(\int_0^\infty \frac{K_m\{\lambda(1+\beta\eta)/\beta\}}{K_m(\lambda/\beta)}\left(\frac{2J_1(\lambda\xi)}{\lambda\xi} - J_0(\lambda\xi)\right)\sin\lambda x\,\mathrm{d}\lambda\right)\mathrm{d}x\Bigg]. \tag{28}$$

To find the stress distribution in the vicinity of the crack, we follow the procedure adopted by Sih and Embley[12]. We note that the infinite integrals in the preceding expressions are convergent everywhere in the medium except at the singular points which occupy the crack border. Since the solution near these points is desired, it is necessary to evaluate the unbounded portions of these integrals in the neighbourhood of the singular points. We note that the integrands are finite and continuous for every value of λ (≥ 0); the divergence of the integrals along the crack border, $\xi = 1$, must be due to the behaviour at $\lambda \to \infty$. Hence, the terms that give rise to unbounded stresses correspond to those parts of the integrand that are dominant for large λ; we shall isolate them here.

By carrying out the expansion for large values of λ and retaining the lowest-order terms, we get, from eqns (27) and (28),

$$\sigma_{\theta z}(\xi, \eta) = \frac{2S}{\pi}(1+\beta\eta)^{\alpha/2}\theta(1)\int_0^\infty e^{-\lambda\eta} J_1(\lambda\xi) \sin\lambda \, d\lambda + \cdots, \tag{29}$$

$$\sigma_{r\theta}(\xi, \eta) = \frac{2S}{\pi}(1+\beta\eta)^{\alpha/2}\theta(1)\int_0^\infty e^{-\lambda\eta} J_0(\lambda\xi) \sin\lambda \, d\lambda + \cdots. \tag{30}$$

Using these results,

$$\int_0^\infty e^{-\lambda\eta} J_1(\lambda\xi) \sin\lambda \, d\lambda = \frac{1}{\sqrt{\xi_1 \xi_2}}\left[\frac{1}{\xi}\cos\frac{\Psi_1 - \Psi_2}{2} - \frac{\eta}{\xi}\sin\frac{\Psi_1 - \Psi_2}{2}\right]$$

and

$$\int_0^\infty e^{-\lambda\eta} J_0(\lambda\xi) \sin\lambda \, d\lambda = \frac{1}{\sqrt{\xi_1 \xi_2}}\sin\frac{\Psi_1 - \Psi_2}{2},$$

where $\xi - 1 = \xi_1 \cos\Psi_1 = \xi_2 \cos\Psi_2 - 2$.

By letting $\Psi_2 \to 0$, $\xi_2 \to 2$, and $\eta \to 0$, we get the stress distribution near the crack border as

$$\sigma_{\theta z} = \frac{k_1}{\sqrt{2\xi_1}}\cos\frac{\Psi_1}{2} + \cdots,$$

$$\sigma_{r\theta} = -\frac{k_1}{\sqrt{2\xi_1}}\sin\frac{\Psi_1}{2} + \cdots,$$

where

$$k_1 = \lim_{\xi \to 1+}\sqrt{2(\xi - 1)}\,\sigma_{\theta z} = \frac{2S\theta(1)}{\pi} \tag{31}$$

is the stress intensity factor for torsion.

The energy of the crack is given by

$$W_1 = 2\pi S\int_0^1 \xi u(\xi, 0)\, d\xi$$

$$= \frac{4S^2}{\mu_0}\left[\left(\frac{\pi}{4}-1\right)\theta(1)\right.$$

$$\left. = \int_0^1\left\{\int_0^{\xi}[x\,\theta(x)]'\,\mathrm{d}x - \int_{\xi}^1[x\,\theta(x)]'\left(\frac{\sqrt{x^2-\xi^2}-x}{x}\right)\mathrm{d}\,x\right\}\mathrm{d}\xi\right]. \tag{32}$$

3.1. *Associated homogeneous medium*

In the associated homogeneous medium we have $\alpha = 0$, *i.e.*, $m = 1/2$ from eqn (11). Hence, from eqns (18) and (21), we get $\phi(\lambda) = 0$ and $\theta(x) = \pi x/4$, respectively. So from eqns (26), (27), (31) and (32), we get after some calculations

$$u(\xi, 0) = \frac{S}{2\mu_0}\,\xi\log\frac{1+\sqrt{1-\xi^2}}{\xi}, \qquad 0 \le \xi \le 1,$$

$$\sigma_{\theta z}(\xi, 0) = S\left[\frac{\sqrt{\xi^2-1}}{\xi} + \frac{1}{2\xi\sqrt{\xi^2-1}}\right], \quad \xi > 1,$$

$k_{\mathrm{i}} = S/2$ and $W_{\mathrm{i}} = \pi^2S^2/12\mu_0$,
which are the same as those obtained by Sneddon and Lowengrub[13].

Case B

In this case the solution of eqn (9) may be written as

$$u = u(\xi, \eta) = \frac{S}{2m\beta C_{44}^0}[(1+\beta\eta)^{2m}-1] + \frac{Sp}{C_{44}^0}(1+\beta\eta)^m$$

$$\times\int_0^{\infty}A(\lambda)K_m\{\lambda(1+\beta\eta)/p\beta\}J_1(\lambda\xi)\,\mathrm{d}\lambda, \tag{33}$$

where m is given by eqn (11).

From eqns (7), using eqns (8) and (33), we get

$$\sigma_{\theta z}(\xi, \eta) = S - S(1+\beta\eta)^{1-m}\int_0^{\infty}\lambda\,A(\lambda)K_{1-m}\{\lambda(1+\beta\eta)/p\beta\}J_1(\lambda\xi)\,\mathrm{d}\lambda \tag{34}$$

and

$$\sigma_{r\theta}(\xi, \eta) = -\frac{S}{p}(1+\beta\eta)^{1-m}\int_0^{\infty}\lambda\,A(\lambda)K_m\{\lambda(1+\beta\eta)/p\beta\}J_2(\lambda\xi)\,\mathrm{d}\lambda. \tag{35}$$

As in Case A, here also we have a pair of dual integral equations

$$\int_0^{\infty}\lambda\,A(\lambda)K_{1-m}(\lambda/p\beta)J_1(\lambda\xi)\,\mathrm{d}\lambda = 1, \quad 0 \le \xi \le 1 \tag{36}$$

$$\int_0^{\infty}A(\lambda)K_m(\lambda/p\beta)J_1(\lambda\xi)\,\mathrm{d}\lambda = 0, \quad \xi > 1. \tag{37}$$

Assuming $B(\lambda) = A(\lambda)K_m(\lambda/p\beta)$, we get eqns (16) and (17), where

$$\phi(\lambda) = \frac{K_{1-m}(\lambda/p\beta)}{K_m(\lambda/p\beta)} - 1. \tag{38}$$

Proceeding in the same way as in Case A, ultimately we have

$$A_1(\lambda)K_m(\lambda/p\beta) = -\sqrt{\frac{2}{\pi\lambda}}\left[\theta(1)J_{1/2}(\lambda) - \int_0^1 [x\,\theta(x)]' x^{-1/2} J_{1/2}(\lambda x)\,\mathrm{d}x\right]. \tag{39}$$

From eqn (33), we obtain, by using eqns (39) and (25),

$$u(\xi, 0) = \frac{2Sp}{\pi\xi C_{44}^0}\left[\theta(1)\left(\sqrt{1-\xi^2} - 1\right) + \int_0^\xi [x\,\theta(x)]'\,\mathrm{d}x \right.$$

$$\left. - \int_\xi^1 [x\,\theta(x)]' \frac{(\sqrt{x^2-\xi^2} - x)\,\mathrm{d}x}{x}\right], \quad \xi < 1. \tag{40}$$

We obtain from eqns (34) and (35), using eqns (39) and (25)

$$\sigma_{\theta z}(\xi, \eta) = S + \frac{2S}{\pi}(1+\beta\eta)^{1-m}\left[\theta(1)\int_0^\infty \frac{K_{1-m}\{\lambda(1+\beta\eta)/p\beta\}}{K_m(\lambda/p\beta)}\right.$$

$$\times J_1(\lambda\xi)\sin\lambda\,\mathrm{d}\lambda - \int_0^1 \frac{1}{x}[x\,\theta(x)]' \left\{\int_0^\infty \frac{K_{1-m}\{\lambda(1+\beta\eta)/p\beta\}}{K_m(\lambda/p\beta)}\right.$$

$$\left.\left. \times J_1(\lambda\xi)\sin\lambda x\,\mathrm{d}\lambda\right\} dx\right], \tag{41}$$

$$\sigma_{r\theta}(\xi, \eta) = \frac{2S}{p\pi}(1+\beta\eta)^{1-m}\left[\theta(1)\int_0^\infty \frac{K_m\{\lambda(1+\beta\eta)/p\beta\}}{K_m(\lambda/p\beta)}\right.$$

$$\times\left(\frac{2J_1(\lambda\xi)}{\lambda\xi} - J_0(\lambda\xi)\right)\sin\lambda\,\mathrm{d}\lambda$$

$$- \int_0^1 \frac{1}{x}(x\,\theta(x)]' \left\{\int_0^\infty \frac{K_m\{\lambda(1+\beta\eta)/p\beta\}}{K_m(\lambda/p\beta)}\right.$$

$$\left.\left. \times\left(\frac{2J_1(\lambda\xi)}{\lambda\xi} - J_0(\lambda\xi)\right)\sin\lambda x\,\mathrm{d}\lambda\right\}\mathrm{d}x\right]. \tag{42}$$

Proceeding similarly as in Case A, *viz.*, expanding eqns (41) and (42) for large λ and retaining the lowest-order terms, we get

$$\sigma_{\theta z}(\xi, \eta) = \frac{2S}{\pi}(1+\beta\eta)^{\alpha/2}\theta(1)\int_0^\infty \mathrm{e}^{-\lambda\eta/p} J_1(\lambda\xi)\sin\lambda\,\mathrm{d}\lambda + \cdots, \tag{43}$$

$$\sigma_{r\theta}(\xi,\eta)=-\frac{2S}{p\pi}(1+\beta\eta)^{\alpha/2}\theta(1)\int_0^\infty e^{-\lambda\eta/p}J_0(\lambda\xi)\sin\lambda\,d\lambda+\cdots. \tag{44}$$

The stress distribution near the crack border is

$$\sigma_{\theta z}=\frac{k_t}{\sqrt{2\xi_1}}\cos\frac{\Psi_1}{2}+\cdots,$$

$$\sigma_{r\theta}=-\frac{k_t}{p\sqrt{2\xi_1}}\sin\frac{\Psi_1}{2}+\cdots,$$

where

$$k_t=\lim_{\xi\to 1+}\sqrt{2(\xi-1)}\sigma_{\theta z}=\frac{2S\theta(1)}{\pi} \tag{45}$$

is the stress intensity factor for torsion of transversely isotropic nonhomogeneous medium.

In this case the energy of the crack is given by

$$W_t=\frac{4S^2p}{C_{44}^0}\left[\left(\frac{\pi}{4}-1\right)\theta(1)+\int_0^1\left\{\int_0^\xi[x\,\theta(x)]'\,dx\right.\right.$$

$$\left.\left.-\int_\xi^1[x\,\theta(x)]'\left(\frac{\sqrt{x^2-\xi^2}-x}{x}\right)dx\right\}d\xi\right]. \tag{46}$$

3.2. *Associated homogeneous medium*

In the associated homogeneous medium we have $\alpha=0$, which makes $\phi(\lambda)=0$ and hence $\theta(x)=\pi x/4$. In this case we easily deduce from eqns (40), (41), (45) and (46) that

$$u(\xi,0)=\frac{pS}{2C_{44}^0}\xi\log\frac{1+\sqrt{1-\xi^2}}{\xi},\quad 0\le\xi\le 1,$$

$$\sigma_{\theta z}(\xi,0)=S\left[\frac{\sqrt{\xi^2-1}}{\xi}+\frac{1}{2\xi\sqrt{\xi^2-1}}\right],\qquad \xi>1,$$

$k_t=S/2$ and $W_t=\pi^2S^2p/12C_{44}^0$,
which are the same as those obtained by Choudhury and Maity[3].

4. Numerical results

To get some idea about the magnitude of the stresses and also the effect of nonhomogeneity in the medium, we have computed the values of $\sigma_{\theta z}(\xi,\eta)/S$ for different values of ξ and η and plotted them graphically (Figs 1 and 2). Figure 1 shows the variation of $\sigma_{\theta z}/S$ in Case A for isotropic nonhomogeneous medium, assuming $\alpha=1$,

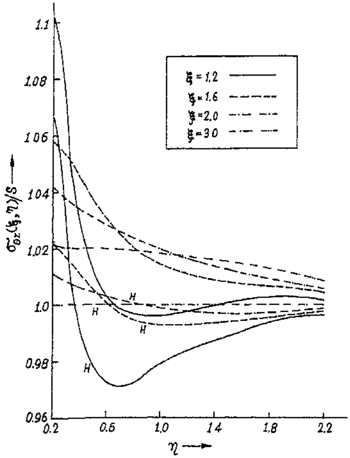

FIG. 1. Variation of $\sigma_{\theta z}(\xi, \eta)/S$ for different values of ξ and η in case A.

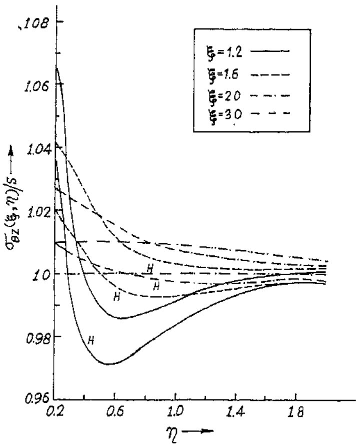

FIG. 2. Variation of $\sigma_{\theta z}(\xi, \eta)/S$ for different values of ξ and η in case B.

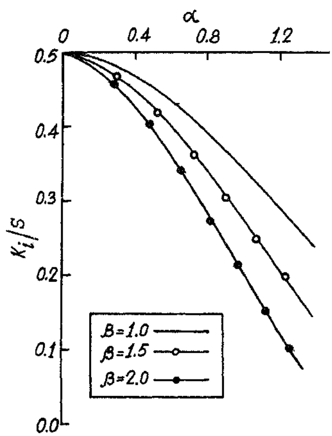

FIG. 3a. Variation of K_t/S with α in case A.

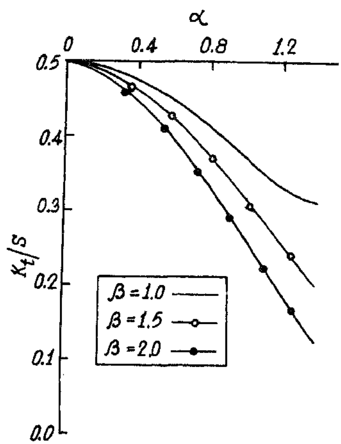

FIG. 3b. Variation of K_t/S with α in case B

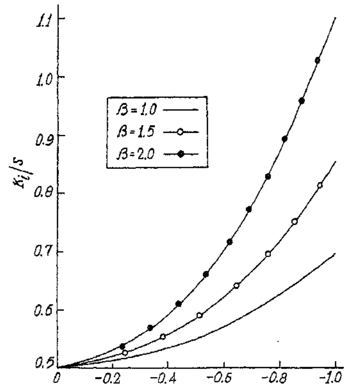

FIG 4a. Variation of K_i/S with α in case A.

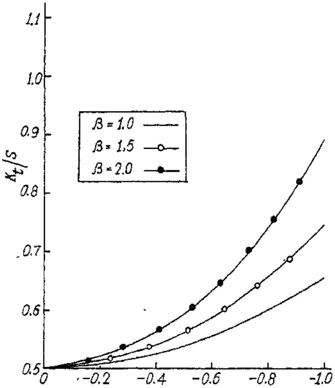

FIG. 4b. Variation of K_t/S with α in case B.

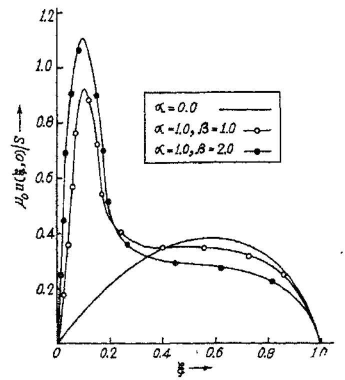

FIG. 5a. Variation of $\mu_0 u(\xi, \eta)/S$ for different values of ξ in Case A.

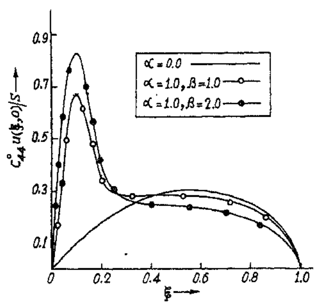

FIG. 5b. Variation of $C_{44}^0 u(\xi, 0)/S$ for different values of ξ in case B.

$\beta = 1$, while Fig. 2 shows the same in Case B for transversely isotropic nonhomogeneous medium, assuming $\alpha = 1$, $\beta = 1$, $C_{44}^0 = 0.40$, $C_{66}^0 = 0.634$. In both the figures the curves marked H represent the variations in the associated homogeneous medium. The effect of nonhomogeneity on stress is quite clear from the graphs.

Figures 3 and 4 show the effects of nonhomogeneity on the stress intensity factor in Cases A and B for different values of the parameters. It is noticed that the stress intensity factor is low for a stiffer material in both the media. Erdogan[14] and Delale and Erdogan[15] have shown that a material may exhibit greater stress intensity factor on the stiffer side in case the material nonhomogeneity is along the plane of the crack. However, in our problem we observe no such anomaly assuming nonhomogeneity along the depth.

As regards crack surface displacements (Fig. 5a, b) we observe significant effect of nonhomogeneity. Similar effects were noticed by Erdogan[14] and Delale and Erdogan[15], although the basic problems and the nature of nonhomogeneity are different.

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