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Short Communication

Stresses and voltage developed in a nonhomogeneous piezoelectric bar due to torsion

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Abstract

Stresses and electric voltage generated in a nonhomogeneous piezoelectric bar acted upon by a torque at one end, the other end being fixed, are considered here. The solution is presented in terms of a series of modified Bessel functions, Macdonald function and Lommel function (in modified form). Results for the homogeneous case are also discussed as a particular case. Numerical results show wide differences in the voltages and the stresses of nonhomogeneous and homogeneous bars.

Keywords: Bar, nonhomogeneous, piezoelectricity, torsion.

1. Introduction

The electrical and mechanical response of laminated piezoelectric bar due to torsion has been investigated by Lee and Moon[1]. We attempt here to present a complete expression of the stress fields and electrical voltage generated in a twisted nonhomogeneous piezoelectric bar.

2. Formulation of the problem and the method of solution

One end of a bar of rectangular section having sides a, b and length l (Figure 1) is fixed and the other end is acted upon by forces reducing to a twisting moment M. The lateral surfaces of the bar are free from external forces.

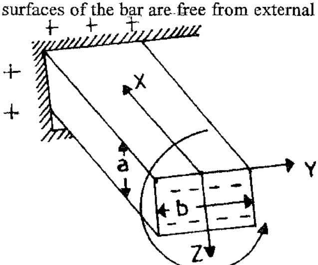

FIG. 1. Polarity of the voltage developed across the bar.

The strain (S_{ij}) and the electric displacement components (D_i) are then expressible in terms of stress (T_{ij}) and electric field components (E_i) as[1, 2]

$$S_{xx} = S_{yy} = S_{zz} = S_{yz} = 0, \tag{1a}$$

$$S_{xy} = s_{66}T_{xy}, \tag{1b}$$

$$S_{xz} = s_{55}T_{xz} - d_{15}E_x, \tag{1c}$$

$$D_x = \varepsilon_{11}E_x - d_{15}T_{xz}, \tag{1d}$$

where s_{ij}, d_{ij} and ε_{ij} represent elastic compliance matrix, piezoelectric strain/charge matrix and permittivity matrix, respectively, and are expressed in terms of power-law function of the sum $(y + y_0)$[2, 3], *i.e.*,

$$s_{ij} = \bar{s}_{ij}\ (y + y_0)^n, \text{ etc.}, \tag{2}$$

where n is an arbitrary real number and is termed as nonhomogeneity parameter, y_0 the distance of a given straight line parallel to the side a from the z axis, $\bar{s}_{ij}$, etc., are material parameters for a homogeneous body.

Under open-circuit condition, following Sirotin and Shaskolskaya[4], one may write from eqn (1c) and (1d)

$$E_x = (d_{15}/\varepsilon_{11})T_{xz}, \tag{3a}$$

$$S_{xz} = s_{55}(1 - k^2)\ T_{xz}, \tag{3b}$$

where $k = d_{15}/(s_{55} \cdot \varepsilon_{11})^{1/2}$ is known as the electromechanical coupling coefficient.

For a continuous medium, T_{xz} and T_{xy} should satisfy the equilibrium equation[2]

$$\frac{\partial T_{xz}}{\partial z} + \frac{\partial T_{xy}}{\partial y} = 0. \tag{4}$$

If the projections of the displacement of a point on the axes of Cartesian coordinates are denoted as u, v, w, eqns (1a), (1b) and (3b) will take the form[2]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0, \tag{5}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = S_{xy}, \tag{6}$$

$$\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = S_{xz}. \tag{7}$$

On integration, eqns (5)–(7) give

$$u = \vartheta\, \phi\, (z, y) + u', \tag{8a}$$

$$v = \vartheta\, xz + v', \tag{8b}$$

$$w = -\vartheta\, xy + w', \tag{8c}$$

where ϑ is the angle of twist per unit length, ϕ is the torsion function, and u', v', w' are rigid-body displacements.

The stress components can be expressed in terms of stress function $\psi(y, z)$ in the following manner, so that eqn (4) is satisfied identically:

$$T_{xz} = \frac{\partial \psi}{\partial y}, \tag{9a}$$

$$T_{xy} = -\frac{\partial \psi}{\partial z}. \tag{9b}$$

From eqns (1b), (3b) and (6)–(8), one gets

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{1}{s_{66}} \frac{\partial}{\partial y}\left[s_{55}(1-k^2) \cdot \frac{\partial \psi}{\partial y} \right] = -\frac{2\vartheta}{s_{66}}. \tag{10}$$

We expand the right-hand side of eqn (10) in a Fourier sine series as

$$-\frac{8\vartheta}{\pi s_{66}} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m} \cdot \sin\frac{m\pi z}{a}. \tag{11}$$

Assuming

$$\psi = \sum_{m=1,3,5,\ldots}^{\infty} Y_m(y_1) \cdot \sin\frac{m\pi z}{a}, \tag{12}$$

so that ψ should vanish beforehand on the two sides $z = 0$ and $z = a$, we get from eqns (10) and (11),

$$Y_m''(y_1) + \frac{n}{y_1} Y_m'(y_1) - \mu^2 \cdot Y_m(y_1) = \frac{\vartheta \cdot P_1(y_1)^{-n}}{m}, \tag{13}$$

where

$$\mu = \frac{m^2\pi^2}{a^2}\left[\bar{s}_{66} / \bar{s}_{55}(1-k^2) \right],$$

$$P_1 = -8b^n/\pi \cdot \bar{s}_{55}(1-k^2)$$

and

$$y_1 = y + y_0.$$

The solution of eqn (13) helps to find the expression of T_{xz} and T_{xy} as

$$T_{xz} = \sum_{m=1,3,5,\ldots}^{\infty} \vartheta \mu y_1^{-N} \left\{ A_m \cdot I_{N+1}(\mu y_1) - B_m \cdot K_{N+1}(\mu y_1) - P_2 \cdot L_{N-1,\,N+1}(\mu y_1) \right\} \sin\frac{m\pi z}{a}, \tag{14a}$$

$$T_{xy} = -\sum_{m=1,3,5,\ldots}^{\infty} \vartheta \cdot \frac{m\pi}{a} \cdot y_1^{-N} \left\{ A_m \cdot I_N(\mu y_1) + B_m \cdot K_N(\mu y_1) + P_2 \cdot L_{N,\,N}(\mu y_1) \right\} \cos\frac{m\pi z}{a}, \tag{14b}$$

where $I_N(\mu y_1)$ and $K_N(\mu y_1)$ are modified Bessel functions and Macdonald function of order N and argument μy_1, respectively, $L_{N,N}(\mu y_1)$ is the Lommel function (modified form) of order N, N and argument μy_1; and $P_2 = P_1/m \cdot \mu^{N+1}$, $N = (1 - n)/2$.

Using the boundary conditions that $\psi = 0$ at $y = \pm b/2$, the arbitrary constants A_m and B_m can be determined. The voltage developed between the fixed and the clamped end of the bar can be found as

$$V = \int_0^1 E_x \mathrm{d}x = \int_0^1 \frac{\bar{d}_{15}}{\bar{\varepsilon}_{11}} T_{xz}\, \mathrm{d}x\,. \tag{15}$$

As $T_{xz} \propto \vartheta$, $V \propto \vartheta$, which tallies with the experimental result of Lee and Moon[1].

For purely elastic cases, the stresses obtained by making $\bar{d}_{15} \rightarrow 0$ and $\bar{\varepsilon}_{11} \rightarrow 0$ tally with those found by Lekhnitskii[2].

3. Numerical results and discussion

Numerical computations have been carried out to obtain stresses and voltage developed in a bar of barium titanate (ceramic). For this ceramic the material constants are[5]

$$\bar{s}_{66} = 2.227 \times 10^{-11}\ \mathrm{m^2/N}, \quad \bar{s}_{55} = 1.8315 \times 10^{-11}\ \mathrm{m^2/N},$$

$$\bar{d}_{15} = 2.6 \times 10^{-12}\ \mathrm{C/N}, \quad \bar{\varepsilon}_{11} = 1596 \times 8.85 \times 10^{-12}\ \mathrm{F/m}.$$

The material parameters are chosen as $y_0 = b$ and $a = 0.033$ m, $b = 0.04$ m and $M = -1 \times 10^{10}$ in SI units (see Table I).

Table I

Nonhomogeneity parameter	*Maximum value of $T_{xz}/\vartheta M$ in SI units*	*Voltage developed ($V/1\vartheta M$) in SI units*
$n = 0$	1.78	3.27×10^{-4}
$n = 1$	2.49	4.58×10^{-4}

It can be noted that the voltage developed for the nonhomogeneous case is greater than that for the homogeneous case.

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