J. Indian Inst. Sci. 63 (B), Oct. 1981, Pp. 181-195


# Asymptotic rates of convergence for the symmetric successive overrelaxation (SSOR) iterative method by means of an associated eigenvalue problem

LALA B. KRISHNA

Department of Mathematics, Patna University, Patna 800 005, India and Kent State University, Kent, Ohio 44242, USA*

Received on February 5, 1981, Revised on October 15, 1981.

**Abstract**

The SSOR iterative method is applied to self-adjoint boundary value problems, such as the Poisson and biharmonic differential equations. We assume the relaxation parameter to be

$$\omega = \frac{2}{1 + Ch^{\sigma}},$$

where $C$ is independent of the mesh spacing $h$ and where, for example, $\sigma = 1$ for the Poisson equation. The asymptotic convergence rates of the SSOR method are then determined from the minimum eigenvalue of an associated non-self-adjoint problem. These results extend the work of Varga for the Successive Overrelaxation (SOR) iterative method.

**Key words:** SSOR iterative method, self-adjoint boundary value problem.

## 1. Introduction

Let $A = [a_{i,j}] \in C^{n,n}$ be a nonsingular complex matrix, and let us seek the solution of the system of linear equations (or at least a good approximation to the solution)

$$\sum_{j=1}^{n} a_{i,j} x_j = b_i, \; 1 \le i \le n, \tag{1.1}$$

which we write in matrix notation as

$$Ax = b \tag{1.2}$$

where $b$ is a given column vector and $x$ is the column vector of unknowns.

* Department of Mathematical Sciences, The University of Akron, Akron, Ohio 44325.

In general, a first degree linear stationary iterative method for approximation to $A^{-1}b$ can be described by

$$x^{(m+1)} = Gx^{(m)} + k, m = 0, 1, 2, \ldots, \quad (1.3)$$

for some iteration matrix $G$ and some vector $k$, where $m$ is the iterative index, and $x^{(0)}$ is an arbitrary guess vector. If $E^{(m)} := x^{(m)} - x$ is the error vector at the end of the $m$-th iteration, then

$$E^{(m)} = GE^{(m-1)} = \ldots = G^m E^{(0)}, m \geq 0. \quad (1.4)$$

We know[2] that the error vectors $E^{(m)}$ of the iterative method tends to zero vector for all $E^{(0)}$ if and only if the spectral radius[2] $\rho(G)$ of the matrix $G$ is less than unity.

The successive overrelaxation (SOR)[2,3] method is given by the equation

$$x^{m+1} = (D - \omega L)^{-1}\{(1 - \omega) D + \omega U\} x^{(m)} + (D - \omega L)^{-1} \omega b, m \geq 0, \quad (1.5)$$

where $\omega$ is the relaxation parameter and $A = D - L - U$, where $D$, $-L$ and $-U$ denote respectively the diagonal, strictly lower and upper triangular parts of $A$. We refer to the matrix

$$\mathcal{L}_\omega := (D - \omega L)^{-1} \{(1 - \omega) D + \omega U\} \quad (1.6)$$

as the successive overrelaxation (SOR) iteration matrix associated with the matrix $A$.

The symmetric successive overrelaxation (SSOR) iterative method is defined by Young[3] as two half iterations. The first half is the same as the SOR method mentioned above, while the second half iteration is the SOR method with the equations taken in reverse order. The SSOR iterative method is defined by the equations

$$x^{(m+1)} = \mathcal{S}_\omega x^{(m)} + \omega (2 - \omega) (D - \omega U)^{-1} (D - \omega L)^{-1} Db, \quad (1.7)$$

where

$$\mathcal{S}_\omega : = I - \omega (2 - \omega) (D - \omega U)^{-1} D (D - \omega L)^{-1} A \quad (1.8)$$

is the SSOR iteration matrix associated with matrix $A$.

## 2. Associated eigenvalue problem

Let $A$ be a $n \times n$ Hermitian positive definite matrix. Consider splitting of $A$ into

$$A = D - L - L^* \quad (2.1)$$

where $D$ and $-L$ denote respectively the diagonal and strictly lower triangular parts of $A$.

Let $\lambda$ be an eigenvalue of the SSOR iteration matrix $\mathcal{S}_\omega$ associated with the matrix $A$ and let $\underline{\varepsilon}$ be a corresponding eigenvector, *i.e.*,

$$\mathcal{S}_\omega \underline{\varepsilon} = \lambda \underline{\varepsilon}. \quad (2.2)$$

Then, from (1.8)

$$\underline{\varepsilon} - \omega(2-\omega)(D-\omega L^*)^{-1} D (D-\omega L)^{-1} A\underline{\varepsilon} = \lambda \underline{\varepsilon}$$

or,

$$(1-\lambda)(D-\omega L) D^{-1} (D-\omega L^*)\, \underline{\epsilon} = \omega(2-\omega) A\underline{\epsilon}$$

or,

$$\omega(1-\omega+\lambda) A\underline{\epsilon} = (1-\lambda)[(1-\omega) D + \omega^2 L D^{-1} L^*]\, \underline{\epsilon}$$

or if $\omega(1-\omega+\lambda) \neq 0$,

$$A\underline{\epsilon} = \frac{(1-\lambda)}{\omega(1-\omega+\lambda)} [(1-\omega) D + \omega^2 L D^{-1} L^*]\, \underline{\epsilon}. \tag{2.3}$$

## 3. Self-adjoint elliptic differential equation in one variable

Consider the self-adjoint elliptic differential equation in one variable

$$-u_{xx} + \sigma u = f, 0 < x < 1 \tag{3.1}$$

subject to the homogeneous boundary conditions

$$u(0) = u(1) = 0, \tag{3.2}$$

where $f(x)$ and $\sigma(x)$ are given real continuous functions on $[0, 1]$ with $\sigma(x) \geq 0$.

Using the standard three-point discrete approximation[2] to (3.1)-(3.2), we get the matrix equation $Au = b$, where the matrix $A$ is given by

$$A = \frac{1}{h^2} \begin{bmatrix} (2+\sigma_1 h^2) & -1 & & \bigcirc \\ -1 & (2+\sigma_2 h^2) & -1 & \\ & \ddots & \ddots & -1 \\ \bigcirc & & -1 & (2+\sigma_n h^2) \end{bmatrix}, \tag{3.3}$$

where

$$h = \frac{1}{n+1} \text{ and } \sigma_i := \sigma(ih)\,; \quad 1 \leq i \leq n.$$

It follows from (2.3) that

$$(A\underline{\epsilon})_i = \frac{(1-\lambda)}{\omega(1-\omega+\lambda)} [(1-\omega)\,(D\underline{\epsilon})_i + \omega^2 (L D^{-1} L^* \underline{\epsilon})_i], \quad 1 \leq i \leq n. \tag{3.4}$$

And using Taylor Series expansion, we get

$$-(\epsilon_{xx})_i + (\sigma\epsilon)_i \doteq \frac{(1-\lambda)}{\omega(1-\omega+\lambda)}\left[(1-\omega)\frac{(2+\sigma_i h^2)}{h^2}\epsilon_i\right.$$

$$\left.+\frac{\omega^2}{h^2}\cdot\frac{1}{(2+\sigma_{i-1}h^2)}\epsilon_i\right],\ i = 2, 3, \ldots, n, \tag{3.5}$$

for sufficiently small $h$. Since $2+\sigma_i h^2 \to 2$ and $2+\sigma_{i-1}h^2 \to 2$ as $h \downarrow 0$, so it follows from (3.5) that

$$-(\epsilon_{xx})_i + (\sigma\epsilon)_i \doteq \frac{(1-\lambda)}{2h^2\omega(1-\omega+\lambda)}(2-\omega)^2\epsilon_i,\ 1 \le i \le n, \tag{3.6}$$

for sufficiently small $h$. Hence, passing to the continuous case, we get

$$-\epsilon_{xx} + \sigma\epsilon \doteq \frac{(1-\lambda)(2-\omega)^2}{2h^2\omega(1-\omega+\lambda)}\epsilon \tag{3.7}$$

for sufficiently small $h$.

Since the above differential equation has been derived from (3.1), we assume that it satisfies the boundary condition (3.2), *i.e.*, $\epsilon(0) = \epsilon(1) = 0$. If we set

$$\alpha := \frac{(1-\lambda)(2-\omega)^2}{2h^2\omega(1-\omega+\lambda)}, \tag{3.8}$$

then from (3.7)

$$-\epsilon_{xx} + \sigma\epsilon \doteq \alpha\epsilon \text{ with } \epsilon(0) = \epsilon(1) = 0.$$

From (3.8), it follows that

$$\frac{d\alpha}{d\lambda} = \frac{-(2-\omega)^2}{2h^2\omega(1-\omega+\lambda)^2} < 0 \text{ for } 0 < \omega < 2.$$

Hence finding the maximum eigenvalue of $\mathcal{S}_\omega \underline{\epsilon} = \lambda\underline{\epsilon}$ is equivalent to finding the minimum eigenvalue of $-\epsilon_{xx} + \sigma\epsilon = \alpha\epsilon$ subject to the boundary conditions $\epsilon(0) = \epsilon(1) = 0$ for sufficiently small $h$, and for all $\omega \in (0, 2)$.

If we set

$$\omega := \frac{2}{1+Ch},$$

where $C$ is a positive real number independent of the step size $h$, then $0 < \omega < 2$. Now, we prove the following proposition :

*Proposition* 1 : Let

$$\omega = \frac{2}{1+Ch},$$

where $C$ is a fixed positive real number independent of the step size $h$. Then for sufficiently small $h$, the optimal $\omega$ for the SSOR method applied to (3.3) is approximately given by $C = \sqrt{\mu}$, where $\mu$ is the minimum eigenvalue of the eigenvalue problem

$$-\epsilon_{xx} + \sigma(x)\,\epsilon = a\epsilon, \quad 0 < x < 1 \tag{3.9}$$

satisfying the homogeneous boundary conditions

$$\epsilon(0) = \epsilon(1) = 0. \tag{3.10}$$

*Proof*: If $\lambda$ is an eigenvalue of the SSOR matrix $\mathcal{S}_\omega$, then from (3.8)

$$a = \frac{(1-\lambda)(2-\omega)^2}{2h^2\,\omega\,(1-\omega+\lambda)} \text{ and } \frac{da}{d\lambda} < 0 \text{ for } 0 < \omega < 2.$$

Hence $\lambda$ will be maximum when $a$ is minimum. Substituting

$$\omega = \frac{2}{1+Ch}$$

and simplifying (3.8), we get

$$\lambda = \frac{C^2 - Cah + a}{C^2 + Cah + a}. \tag{3.11}$$

Note $C^2 + Cah + a \neq 0$, since $C$, $h$ and $a$ are positive.

From (3.11), it follows that

$$\frac{d\lambda}{dC} = \frac{2ha\,(C^2 - a)}{[C^2 + Cah + a]^2}.$$

The critical points of $\lambda$ as a function of $C$ are given by $C = \pm\sqrt{a}$.

The minimum of $\lambda$ as a function of $C$ is attained at $C = \sqrt{a}$, *i.e.*, the optimal $\omega$ for the SSOR method is attained at

$$\omega = \frac{2}{1+\sqrt{\mu}\cdot h},$$

where $\mu$ is the minimum eigenvalue of (3.9).

In particular, if we take $\sigma(x) \equiv 0$, then it is known that the minimum eigenvalue of the problem (3.9)–(3.10) is given by $\pi^2$. Thus we have the following corollary :

*Corollary* 2 :

Let $\quad \omega = \dfrac{2}{1+Ch}$,

where $C$ is a fixed positive real number and $h$ is the mesh size, then, for sufficiently small $h$, the optimal $\omega$ for the SSOR method applied to (3.3) with $\sigma(x) \equiv 0$, is approximately given by $C = \pi$.

The above results are obtained on the assumption that the boundary conditions (3.10) of the eigenvalue problem (3.9) is the same as that of the original differential equation (3.1).

Using the power method, we found in the author's dissertation[4] that the spectral radius of the SSOR iterative matrix $\mathcal{S}_\omega$ with different mesh size $h$ and $C$ agrees with the result of Corollary 2.

## 4. Biharmonic differential equation in one variable

Consider the following biharmonic equation in one variable

$$\Delta^2 u = f;\ 0 < x < 1 \tag{4.1}$$

with boundary conditions

$$u(0) = u(1) = u^{(2)}(0) = u^{(2)}(1) = 0. \tag{4.2}$$

Using a uniform mesh

$$h = \frac{1}{n+1},$$

a standard difference approximation applied to (4.1) gives

$$h^4 u^4(ih) \doteq 6u_i - 4(u_{i-1} + u_{i+1}) + (u_{i-2} + u_{i+2}),\ 1 \le i \le n \tag{4.3}$$

while (4.2) is approximated by

$$u_0 = u_{n+1} = 0;\quad 2u_0 - u_{-1} - u_1 = 0;\quad 2u_{n+1} - u_n - u_{n+2} = 0, \tag{4.4}$$

and these relations further give

$$u_1 = -u_{-1};\quad u_n = -u_{n+2}. \tag{4.4'}$$

This then results in the matrix equation

$$Au = f, \tag{4.5}$$

where

$$A = \begin{bmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & \bigcirc & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & 1 & -4 & 6 & -4 & 1 & \\ \bigcirc & & 1 & -4 & 6 & -4 & \\ & & & 1 & -4 & 5 & \end{bmatrix}, \quad f = h^4 \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_n \end{bmatrix}, \tag{4.6}$$

$A$ being an $n \times x$ real symmetric and positive definite matrix and $u_i := u(x_i)$, $f_i := f(x_i)$ and $x_i := ih$.

From (2.3), it follows that

$$(A\epsilon)_i = \frac{(1-\lambda)}{\omega(1-\omega+\lambda)}\left[6(1-\omega)\,\epsilon_i + \frac{\omega^2}{6}(-4\epsilon_{i-1} + 17\epsilon_i - 4\epsilon_{i+1})\right]. \quad (4.7)$$

From Taylor Series expansion, we have

$$\begin{aligned} &-4\epsilon_{i-1} + 17\epsilon_i - 4\epsilon_{i+1} \quad (4.8)\\ &= -4h^2(\epsilon^{(2)}(x))_i - \frac{1}{3}h^4(\epsilon^{(4)}(x))_i + 0(h^6) + 9\epsilon_i. \end{aligned}$$

And so for sufficiently small $h$, it follows from (4.7) and (4.8) that

$$h^4(\epsilon^{(4)}(x))_i \doteqdot \frac{(1-\lambda)}{\omega(1-\omega+\lambda)}\left[6(1-\omega)\,\epsilon_i + \frac{\omega^2}{6}\left\{-4h^2(\epsilon^{(2)}(x))_i - \frac{1}{3}h^4(\epsilon^{(4)}(x))_i + 9\epsilon_i\right\}\right],\quad 1 \le i \le n,$$

since $(A\underline{\epsilon})_i = h^4(\epsilon^{(4)}(x))_i + 0(h^6)$. Hence passing to the continuous case, we get the following differential equation:

$$h^4\epsilon^{(4)}(x) \doteqdot \frac{(1-\lambda)}{\omega(1-\omega+\lambda)}\left[6(1-\omega)\,\epsilon + \frac{\omega^2}{6}\left\{-4h^2\epsilon^{(2)}(x) - \frac{1}{3}h^4\epsilon^{(4)}(x) + 9\epsilon\right\}\right].$$

Or,

$$\epsilon^{(4)}(x) \doteqdot \frac{3(1-\lambda)[9(2-\omega)^2\epsilon - 4h^2\omega^2\epsilon^{(2)}(x)]}{h^4\omega[(18-17\omega) + \lambda(18-\omega)]}.$$

If

$$\omega = \frac{2}{1+Ch}$$

where $C > 0$, then on simplification, we get

$$\epsilon^{(4)}(x) \doteqdot \frac{27(1-\lambda)C^2\left[\epsilon - \frac{4}{9C^2}\epsilon^{(2)}(x)\right]}{h^2[(9Ch-8) + \lambda(9Ch+8)]}. \quad (4.9)$$

Set

$$a := \frac{27(1-\lambda)C^2}{h^2[9Ch-8) + \lambda(9Ch+8)]}. \quad (4.10)$$

Then from (4.9), we have

$$\epsilon^{(4)}(x) \doteq a\left[\epsilon - \frac{4}{9C^2} \cdot \epsilon^{(2)}(x)\right]. \tag{4.11}$$

*Lemma* 3 : The minimum eigenvalue of the biharmonic differential equation

$$\epsilon^{(4)}(x) = a\left[\epsilon - \frac{4}{9C^2}\, \epsilon^{(2)}(x)\right], \; 0 < x < 1 \tag{4.12}$$

with boundary conditions $\epsilon(0) = \epsilon(1) = \epsilon^{(2)}(0) = \epsilon^{(2)}(1) = 0$, where $C > 0$, is given by

$$\frac{9C^2\pi^4}{9C^2 + 4\pi^2}.$$

*Proof*: Clearly $\epsilon(x) = \sin m\pi x$, where $m$ is a positive integer, is an eigenvector of (4.12). So from (4.12), it follows that

$$\pi^4 m^4 \sin \pi m x = a\left[\sin \pi m x + \frac{4}{9C^2}\pi^2 m^2 \sin \pi m x\right].$$

Or,

$$a = \frac{\pi^4 m^4}{1 + \dfrac{4}{9C^2}\pi^2 m^2} = \frac{9C^2\pi^4 m^4}{9C^2 + 4\pi^2 m^2}. \tag{4.13}$$

Since

$$\frac{da}{dm} = \frac{324C^4\pi^4 m^3 + 72\pi^6 m^5 C^2}{(9C^2 + 4\pi^2 m^2)^2} > 0 \text{ for } C,\ m > 0.$$

Hence the minimum eigenvalue of eigenvalue problem (4.11) will be attained when $m$ is minimum. In particular, if $m$ is a positive integer then the minimum will occur when $m = 1$, and the minimum eigenvalue will be given by

$$\frac{9C^2\pi^4}{9C^2 + 4\pi^2}.$$

*Proposition* 4 : Let $A$ be the $n \times n$ matrix (4.6) obtained using the standard difference approximation to the biharmonic differential equation (4.1)–(4.2). Then the optimal $\omega$ for the spectral radius of the SSOR matrix $\mathcal{S}$, is approximately attained at

$$\omega = \frac{2}{1 + Ch}, \quad C > 0, \text{ where } C = \frac{2\pi}{3}\sqrt{1 + \tfrac{2}{3}h^2\pi^2}$$

and the step size $h$ is sufficiently small, and we assume that the boundary conditions of the associated eigenvalue problem (4.12) satisfy (4.2).

*Proof*: From (4.10), it follows that

$$\frac{d\alpha}{d\lambda} = \frac{-486C^3}{h\left[(9Ch-8)+\lambda(9Ch+8)\right]^2} < 0 \text{ for } C > 0,$$

*i.e.*, $\alpha$ is a decreasing function of $\lambda$. So the maximum value of $\lambda$ is attained at minimum value of $\alpha$. From lemma 3, the minimum value of $\alpha$ is given by

$$\frac{9C^2\pi^4}{9C^2+4\pi^2}.$$

Substituting this in (4.10) and on simplification, we get

$$\lambda = \frac{27C^2 - 9Ch^3\pi^4 + 8h^2\pi^4 + 12\pi^2}{27C^2 + 9Ch^3\pi^4 + 8h^2\pi^4 + 12\pi^2} \tag{4.14}$$

Elementary calculations show that the minimum of the maximum eigenvalue $\lambda$ as function of $C$ occurs at

$$C = \tfrac{2}{3}\pi\sqrt{1+\tfrac{2}{3}h^2\pi^2}$$

and the optimal $\omega$ is given by

$$\omega = \frac{2}{1+\left(\tfrac{2}{3}\pi\sqrt{1+\tfrac{2}{3}h^2\pi^2}\right)\cdot h}.$$

*Remark*: For sufficient small $h$, the otpimal $\omega$ will occur at $C = \tfrac{2}{3}\pi$.

We used the power method to determine the spectral radius of the SSOR iterative matrix $\mathcal{S}_\omega$ for different values of $C$ and $h$. The results given in Table 2 of the author's dissertation[4] agree with Proposition 4.

## 5. Self-adjoint elliptic differential equation in two variables

Consider the second-order self-adjoint elliptic partial differential equation in two variables given by

$$-(P(x,y)\,u_x)_x - (P(x,y)\,u_y)_y + \sigma(x,y)\,u(x,y) = f(x,y),\ (x,y) \in R \tag{5.1}$$

defined in an open, bounded and connected set $R$ in the plane. For the boundary condition, we assume that

$$\alpha(x,y)\,u + \beta(x,y)\,\frac{\partial u}{\partial n} = \gamma(x,y),\ (x,y) \in \Gamma, \tag{5.2}$$

and $\Gamma$, the boundary of $R$, is assumed to be sufficiently smooth. We also assume that

$$\begin{cases} P(x,y) > 0 \\ \sigma(x,y) > 0 \end{cases},\ (x,y) \in \bar{R}. \tag{5.3}$$

Using a finite difference approximation with uniform mesh size $h$, we get a system of linear equations[2]

$$Au = k + \underline{\tau}(u)$$

where

$$(Au)_{i,j} = D_{i,j}\, u_{i,j} - L_{i,j}\, u_{i-1,j} - R_{i,j}\, u_{i+1,j} - T_{i,j}\, u_{i,j+1} - B_{i,j}\, u_{i,j-1};$$

$$\begin{cases} L_{i,j} = P_{i-\frac{1}{2},j}, \; R_{i,j} = P_{i+\frac{1}{2},j}, \\ T_{i,j} = P_{i,j+\frac{1}{2}}, \; B_{i,j} = P_{i,j-\frac{1}{2}}, \\ D_{i,j} = L_{i,j} + R_{i,j} + T_{i,j} + B_{i,j} + \sigma_{i,j}\, h^2 \end{cases} \tag{5.4}$$

and

$$P_{i,j} := P(x_i, y_j),\ \sigma_{i,j} := \sigma(x_i, y_j).$$

Using Taylor Series expansions, we compute $(A\underline{\epsilon})_{i,j}$ and $(D\underline{\epsilon})_{i,j}$ as follows :

$$(A\underline{\epsilon})_{i,j} = L_{i,j}(\epsilon_{i,j} - \epsilon_{i-1,j}) + R_{i,j}(\epsilon_{i,j} - \epsilon_{i+1,j}) + T_{i,j}(\epsilon_{i,j} - \epsilon_{i,j+1}) + B_{i,j}(\epsilon_{i,j} - \epsilon_{i,j-1}) + \sigma_{i,j}\, h^2\, \epsilon_{i,j};$$

$$L_{i,j} = P_{i-\frac{1}{2},j} = P_{i,j} - \tfrac{1}{2}h\,(P_x)_{i,j} + (\tfrac{1}{2}h)^2\,(P_x)_{i,j} + 0\,(h^3),$$

$$R_{i,j} = P_{i-\frac{1}{2},j} = P_{i,j} + \tfrac{1}{2}h\,(P_x)_{i,j} + (\tfrac{1}{2}h)^2\,(P_{xx})_{i,j} + 0\,(h^3).$$

$$T_{i,j} = P_{i,j+\frac{1}{2}} = P_{i,j} + \tfrac{1}{2}h\,(P_y)_{i,j} + (\tfrac{1}{2}h)^2\,(P_{yy})_{i,j} + 0\,(h^3).$$

$$B_{i,j} = P_{i,j-\frac{1}{2}} = P_{i,j} - \tfrac{1}{2}h\,(P_y)_{i,j} + (\tfrac{1}{2}h)^2\,(P_{yy})_{i,j} + 0\,(h^3). \tag{5.5}$$

A simple calculation (using Taylor Series expansion and neglecting terms of order $h^3$) yields

$$(A\underline{\epsilon})_{i,j} \doteqdot -h^2\,[(P\epsilon_x)_x + (P\epsilon_y)_y]_{i,j} + (\sigma\epsilon)_{i,j}\, h^2. \tag{5.6}$$

And

$$(D\underline{\epsilon})_{i,j} = [L_{i,j} + R_{i,j} + T_{i,j} + B_{i,j} + \sigma_{i,j}\, h^2)\, \epsilon_{i,j}$$

$$= [4P_{i,j} + \tfrac{1}{2}h^2\,(P_{xx} + P_{yy})_{i,j} + h^2\sigma_{i,j}]\, \epsilon_{i,j} + 0\,(h^3).$$

Or,

$$(D\underline{\epsilon})_{i,j} \doteqdot 4P_{i,j}\, \epsilon_{i,j} \text{ for sufficiently small } h. \tag{5.7}$$

And

$$(D^{-1})_{i,j} \doteqdot \frac{1}{4P_{i,j}} \text{ for sufficiently small } h. \tag{5.8}$$

So for sufficiently small $h$, we have

$$(LD^{-1} L^* \underline{\epsilon})_{i,j} \doteqdot \left(\frac{L_{i,j}\, T_{i-1,j}}{4P_{i-1,j}}\right) \epsilon_{i-1,j+1} + \left(\frac{L_{i,j}\, R_{i-1,j}}{4P_{i-1,j}}\right.$$

$$+ \left.\frac{B_{i,j}\, T_{i,j-1}}{4P_{i,j-1}}\right) \epsilon_{i,j} + \left(\frac{B_{i,j}\, R_{i,j-1}}{4P_{i,j-1}}\right) \epsilon_{i+1,j-1}$$

$$= \frac{1}{4P_{i-1,j}\, P_{i,j-1}} [L_{i,j}\, T_{i-1,j}\, P_{i,j-1}\, \epsilon_{i-1,j+1} + L_{i,j}\, R_{i-1,j}\, P_{i,j-1}\, \epsilon_{i,j}$$

$$+ B_{i,j}\, T_{i,j-1}\, P_{i-1,j}\, \epsilon_{i,j} + B_{i,j}\, R_{i,j-1}\, P_{i-1,j}\, \epsilon_{i+1,j-1}] .$$

By straightforward computation (using Taylor Series expansions), we get

$$(LD^{-1} L^* \underline{\epsilon})_{i,j} = \frac{1}{4\,(P^2 - hPP_x - hPP_y + h^2 P_x P_y + h^2 PP_{xx} + h^2 PP_{yy})_{i,j}}$$

$$\times \{((4P^3 - 4hP^2 P_x - 4hP^2 P_y + 2h^2 P^2 P_{xx} + \tfrac{8}{2}h^2 PP_x P_y + 2h^2 P^2 P_{yy}$$

$$- h^2 P^2 P_{xy} + \tfrac{1}{4} h^2 PP_x^2 + \tfrac{1}{4} h^2 PP_y^2)_{i,j}\, \epsilon_{i,j} + h^2 (P_{i,j})^2 (P_x \epsilon_x)_{i,j}$$

$$- h^2 (P_{i,j})^2 (P_y \epsilon_x)_{i,j} - h^2 (P_{i,j})^2 (P_x \epsilon_y)_{i,j} + h^2 (P_{i,j})^2 (P_y \epsilon_y)_{i,j}$$

$$+ h^2 (P_{i,j})^2 (\epsilon_{xx} - 2\epsilon_{xy} + \epsilon_{yy})_{i,j}\} + 0\,(h^3). \tag{5.9}$$

For sufficiently small $h$, we drop terms in $h^2$ in (5.9) and so

$$(LD^{-1} L^* \underline{\epsilon})_{i,j} \doteqdot P_{i,j}\, \epsilon_{i,j}. \tag{5.10}$$

Hence for sufficiently small $h$, it follows from (2.3), (5.6), (5.7) and (5.10) that

$$- h^2 ((P\epsilon_x)_x)_{i,j} + ((P\epsilon_y)_y)_{i,j} + (\sigma\epsilon)_{i,j}\, h^2$$

$$= \frac{(1-\lambda)}{\omega\,(1-\omega+\lambda)} [4\,(1-\omega)\, P_{i,j}\, \epsilon_{i,j} + \omega^2 P_{i,j}\, \epsilon_{i,j}],\ 1 \leqslant i \leqslant n. \tag{5.11}$$

Since (5.11) has been derived from the elliptic equation (5.1), we assume that it satisfies the boundary condition of (5.1). So passing to the continuous case, we get from (5.11) that

$$- (P\epsilon_x)_x - (P\epsilon_y)_y + \sigma\epsilon \doteqdot \frac{(1-\lambda)}{h^2\, \omega\,(1-\omega+\lambda)} (\omega - 2)^2 P\epsilon \tag{5.12}$$

with boundary condition (5.2), *i.e.*,

$$\alpha(x, y)\,\epsilon + \beta(x, y)\frac{\partial \epsilon}{\partial n} = \gamma(x, y),\ (x, y) \in \Gamma.$$

Set

$$a := \frac{(1-\lambda)(2-\omega)^2}{h^2\,\omega(1-\omega+\lambda)}, \tag{5.13}$$

then

$$da/d\lambda < 0 \text{ for } 0 < \omega < 2.$$

Hence maximum eigenvalue of SSOR matrix $\mathcal{S}_\omega$ associated with (5.1)-(5.2) is related to the minimum eigenvalue of the associated eigenvalue problem

$$-(P\epsilon_x)_x - (P\epsilon_y)_y + \sigma\epsilon = aP\epsilon. \tag{5.14}$$

subject to the boundary condition (5.2) by the relation (5.13) for $\omega$ in the interval $0 < \omega < 2$.

*Special case* : Now, we consider the Laplace equation with Dirichlet boundary conditions in which case $P \equiv 1$ and $\sigma \equiv 0$. Then for sufficiently small $h$, it follows from (2.3), (5.6), (5.7) and (5.9) that

$$-h^2(\epsilon_{xx} + \epsilon_{yy}) \doteqdot \frac{(1-\lambda)}{\omega(1-\omega+\lambda)}\left[(2-\omega)^2\epsilon + \frac{\omega^2 h^2}{4}(\epsilon_{xx} + \epsilon_{yy} - 2\epsilon_{xy})\right].$$

On simplification, we get

$$-(\epsilon_{xx} + \epsilon_{yy}) \doteqdot \frac{2(1-\lambda)}{h^2\,\omega(2-\omega)(1-\lambda)}\left[(2-\omega)^2\epsilon - \frac{\omega^2 h^2}{4}(\epsilon_{xx} + \epsilon_{yy} + 2\epsilon_{xy})\right]. \tag{5.15}$$

Let

$$\omega = \frac{2}{1+Ch},$$

where $C > 0$, then from (5.15) we get

$$-(\epsilon_{xx} + \epsilon_{yy}) \doteqdot \frac{(1-\lambda)}{2Ch(1+\lambda)}[4C^2\epsilon - (\epsilon_{xx} + \epsilon_{yy} + 2\epsilon_{xy})].$$

Set

$$a := \frac{(1-\lambda)}{2Ch(1+\lambda)}, \tag{5.16}$$

then

$$-(\epsilon_{xx} + \epsilon_{yy}) \doteqdot a(4C^2\epsilon - (\epsilon_{xx} + \epsilon_{yy} + 2\epsilon_{xy})]. \tag{5.17}$$

It is easy to verify that the eigenvalue problem (5.17) with Dirichlet boundary conditions has all positive eigenvalues. From (5.16), $d\alpha/d\lambda < 0$ for $C > 0$, *i.e.*, $\alpha$ is a decreasing function of $\lambda$. Hence, the maximum eigenvalue $\lambda$ of the SSOR matrix $\mathcal{S}_\omega$ derived from Laplace equation and the minimum eigenvalue $\alpha$ of (5.17) with Dirichlet boundary conditions are related by (5.16).

In the author's dissertation[4], the power method has been used to determine the maximum eigenvalue of $\mathcal{S}_\omega$. There it is also shown that the optimal $\omega$ for the SSOR matrix $\mathcal{S}_\omega$ derived from Laplace equation with Dirichlet boundary conditions are attained at $C = 1/h$, i.e., $Ch = 1$ which shows that the optimal $\omega$ is unity.

## 6. Biharmonic differential equation in two variables

Consider the biharmonic partial differential equation

$$\nabla^4 u(x, y) = f(x, y), \quad (x, y) \in R, \tag{6.1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \nabla^4 = \nabla^2 \nabla^2 = \frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

and where $R$ is a simply connected region, for simplicity.

The boundary conditions for $u(x, y)$ are assumed to take one of the following forms on each portion of $\Gamma$ the boundary of $R$ :

$$\begin{aligned} &\frac{\partial u(x, y)}{\partial x} = g_1(x, y), \quad \frac{\partial u(x, y)}{\partial y} = g_2(x, y), \quad (x, y) \in \Gamma, \\ &u(x, y) = g_3(x, y), \quad \frac{\partial u(x, y)}{\partial n} = g_4(x, y), \quad (x, y) \in \Gamma, \end{aligned} \tag{6.2}$$

or

$$\frac{\partial u(x, y)}{\partial n} = 0, \quad \frac{\partial^2 \nabla^2 u(x, y)}{\partial n} = 0, \quad (x, y) \in \Gamma, \tag{6.3}$$

where the functions $g_i(x, y)$ are given and $n$ refers to the outward pointing normal. Using a standard finite difference approximations with uniform mesh size $h$, we get the well-known[6] 13-point biharmonic star :

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{array} \tag{6.4}$$

From (6.4) and (2.3), using Taylor Series expansions and neglecting terms of order $h^6$, we get

$$h^4 (\nabla^4 \underline{\epsilon})_{i,j} \doteq \frac{(1-\lambda)}{\omega(1-\omega+\lambda)} \Big[ 20(1-\omega)\, \epsilon_{i,j} + \omega^2 \Big\{ -\frac{2}{5}\, \epsilon_{i+1,j-2}$$

$$+ \frac{1}{20}\, \epsilon_{i+3,j-1} + \frac{1}{10}\, \epsilon_{i-1,j+1} - \frac{6}{5}\, \epsilon_{i,j-1} + \frac{17}{5}\, \epsilon_{i+1,j-1} - \frac{6}{5}\, \epsilon_{i+2,j-1}$$

$$+ \frac{1}{10}\, \epsilon_{i+3,j-1} + \frac{1}{5}\, \epsilon_{i-2,j} - 2\epsilon_{i-1,j} + \frac{69}{10}\, \epsilon_{i,j}$$

$$- 2\epsilon_{i+1,j} + \frac{1}{5}\, \epsilon_{i+2,j} + \frac{1}{10}\, \epsilon_{i-3,j+1} - \frac{6}{5}\, \epsilon_{i-2,j+1}$$

$$+ \frac{17}{5}\, \epsilon_{i-1,j+1} - \frac{6}{5}\, \epsilon_{i,j+1} + \frac{1}{10}\, \epsilon_{i+1,j+1} + \frac{1}{20}\, \epsilon_{i-2,j+2} - \frac{2}{5}\, \epsilon_{i-1,j+2} \Big\} \Big]$$

$$= \frac{(1-\lambda)}{20(1-\omega+\lambda)} [150(2-\omega)^2 \underline{\epsilon} - h^2 \omega^2 (54\, \underline{\epsilon}_{xx} + 36\epsilon_{xy} + 6\epsilon_{yy})$$

$$+ h^4 \omega^4 (-15\epsilon_{xxxx} + 36\epsilon_{xxxy} - 18\epsilon_{xxyy} + 12\epsilon_{xyyy} - 11\epsilon_{yyyy})]_{i,j}.$$

Let

$$\omega = \frac{2}{1+Ch} \text{ where}$$

$C > 0$, then substituting this value of $\omega$ in the above discrete differential equation, we get

$$(\nabla^4 \epsilon)_{i,j} \doteq \frac{(1-\lambda)}{15h^2 (Ch - 1 + \lambda(Ch+1))} [150\, C^2 \epsilon - 6(9\epsilon_{xx}$$

$$+ 6\epsilon_{xy} + \epsilon_{yy}) + h^2(-15\epsilon_{xxxx} + 36\epsilon_{xxxy}$$

$$- 18\epsilon_{xxyy} + 12\epsilon_{xyyy} - 11\epsilon_{yyyy})]_{i,j}.$$

Dropping $h^2$ terms and passing to the continuous problem, we get

$$\nabla^4 \epsilon \doteq \frac{(1-\lambda)}{15h^2(Ch - 1 + \lambda(Ch+1))} [150\, C^2 \epsilon - 6(9\epsilon_{xx} + 6\epsilon_{xy} + \epsilon_{yy})] \tag{6.5}$$

for sufficiently small $h$. Since (6.5) has been derived from biharmonic equation (6.1), we assume that it satisfies the boundary conditions (6.2) or (6.3) (as the case may be) of (6.1). Now, if we set

$$\alpha = \frac{(1-\lambda)}{15h^2(Ch - 1 + \lambda(Ch+1))}, \tag{6.6}$$

then we have the following eigenvalue problem:

$$\nabla^4 \epsilon \doteq \alpha [150\, C^2 - 6(9\epsilon_{xx} + 6\epsilon_{xy} + \epsilon_{yy})] \tag{6.7}$$

with the boundary conditions (6.2) or (6.3). For $C > 0$, we have $d\alpha/d\lambda < 0$, *i.e.*, $\alpha$ is a decreasing function of $\lambda$. So the maximum eigenvalue of SSOR matrix $\mathcal{S}_\omega$ obtained from (6.1) and the minimum eigenvalue of the eigenvalue problem (6.7) with the same boundary conditions are related by (6.6).

**Acknowledgement**

The author would like to express his thanks to Professor Richard S. Varga for the support and able guidance which he has given throughout the development of this research work.

**References**

1. VARGA, RICHARD S. Poslednie Rezul'taty v Lineinoi Algebre i ee Prilozenijah (Russian) *Vycislitel nye Metody Lineinoi Algebry* # 3,1978, pp. 5–15.
2. VARGA, RICHARD S. *Matrix iterative analysis*, Prentice-Hall, New Jersey, 1962.
3. YOUNG, DAVID *Iterctive solution of large linear system*, Academic Press, 1971.
4. KRISHNA, LALA B. Asymptotic rates of convergence for the symmetric successive overrelaxation method, Ph.D. Dissertation, Kent State University, Ohio, U.S.A., 1979, Ch. 6, pp. 59–84.
5. GRIFFIN, D. S. AND VARGA, RICHARD S. Numerical solution of plane elasticity problems, *J. Soc. Indust. Appl. Math.* 1963, 11, 1046–1062.