JOURNAL OF
THE
INDIAN INSTITUTE OF SCIENCE

VOLUME 49 OCTOBER 1967 NUMBER 4

STABILITY OF EQUATORIAL DISTURBANCES IN AN IDEALIZED DIPOLE MAGNETIC FIELD EMBEDDED IN A NONUNIFORM ATMOSPHERE

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(Received : January 23, 1967)

ABSTRACT

The propagation of the small amplitude disturbances in an inhomogeneous atmosphere embedded in an idealized dipole field of a uniformly magnetized body is considered. The gravitational forces are included. It is found that due to inhomogeneity of either the density or magnetic intensity the disturbances generally tend to grow excepting in a special case. This conforms to the general belief on the formation of shocks in atmospheres. But in the special case when the local Alfven speed is constant and when the phase velocity lies between magneto-acoustic mode and magneto-acoustic mode modified by Lorentz force, the disturbances die out.

INTRODUCTION

1. The hydromagnetic stability of disturbances in the equatorial plane of an idealized dipole field is of importance both in the context of terrestrial and solar fields. For example, in the former case, the disturbances may be due to the passage of equatorial artificial satellites, while in the latter case they may be due to the ejection of the material from equatorial plane. Following Hamlin et al[1] and Avrett[2], we consider the body to be uniformly

magnetized slab and of width a and extending to infinity in the x- and z-directions. The idealized field lines in the undisturbed state are taken parallel to the body and are given by $\mathbf{H}_0 = (Ha^3/r^3, 0, 0)$ where H denotes the intensity at the surface. We have taken the y-axis normal to the surface of the body and r denotes the distance measured along this axis. The z-axis is taken perpendicular to the field lines in the plane of the body We note that this magnetic field configuration is realized in the equatorial plane of a dipole field. The density ρ_0 of the atmosphere in the unperturbed state is taken to vary as $\rho_0 = Ra^n/r^n$, where R is the surface density. Further, the atmosphere is subjected to the gravitational attraction of the body given by $\mathbf{g} = (0, Ga^2/r^2, 0)$, G being the magnitude of the acceleration due to gravity at the surface. We shall further suppose that initially the medium is at rest and is in a steady but nonhomogeneous state, the pressure gradient being balanced by gravitational and Lorentzian forces. Moreover, the medium is taken to be an ideal conductor. Consequently, the basic equations governing the system are

$$\partial \rho/\partial t + \operatorname{div}(\rho \mathbf{v}) = 0, \qquad [1.1]$$

$$\rho\left[\partial \mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = \rho \mathbf{g} + (\mu/c)\,\mathbf{J}\times\mathbf{H} - \nabla p, \qquad [1,2]$$

$$\left.\begin{aligned} &\nabla\cdot\mathbf{B} = 0,\\ &\nabla\cdot\mathbf{D} = 0,\\ &\operatorname{curl}\mathbf{E} = -(1/c)(\partial \mathbf{B}/\partial t),\\ &\operatorname{curl}\mathbf{H} = (4\pi/c)\,\mathbf{J} + (1/c)(\partial \mathbf{D}/\partial t),\\ &\mathbf{B} = \mu\,\mathbf{H},\\ &\mathbf{D} = \epsilon\,\mathbf{E}, \end{aligned}\right\} \qquad [1.3]$$

and

$$\mathbf{E} + (1/c)\,\mathbf{v}\times\mathbf{B} = 0. \qquad [1.4]$$

We shall denote the unperturbed quantities with the suffix 0 and the infinitesimal perturbations in all the physical variables by the sufiix 1. Further, we shall assume that all the perturbed quantities vary with time as $e^{i\omega t}$. Nondimensionalizing the distances with a and the magnetic field with H and eliminating the perturbations in density and electromagnetic fields, we obtain the following equation satisfied by the induced velocity :

$$\begin{aligned} \mathbf{v}_1/\eta^n = {} & (G/a\omega^2)[\operatorname{div}(\mathbf{v}_1/\eta^n)]\mathbf{G}_0 + [\mu H^2/(4\pi\, iRa\,\omega\, V)][3\,\mathbf{J}_0\times\mathbf{h}_1 + (\operatorname{curl}\mathbf{h}_1)\times\mathbf{H}_0] \\ & + (\mu H^2/4\pi R)(\mu\epsilon/c^2)(\mathbf{v}_1\times\mathbf{H}_0)\times\mathbf{H}_0 - (S^2/a^2\omega^2)\operatorname{grad}\operatorname{div}(\mathbf{v}_1/\eta^n) \end{aligned} \qquad [1.5]$$

where

$$\mathbf{h}_1 = (V/ia\omega)\operatorname{curl}(\mathbf{v}_1\times\mathbf{H}_0), \qquad [1.6]$$

V and S being typical speed and sound speed in the medium. The vectors $\mathbf{G}_0$, $\mathbf{J}_0$ and $\mathbf{H}_0$ are given by

$$\left.\begin{aligned} \mathbf{G}_0 &= [0,\ (1/\eta^2),\ 0], \\ \mathbf{J}_0 &= [0,\ 0,\ (1/\eta^4)], \\ \mathbf{H}_0 &= [(1/\eta^3),\ 0,\ 0], \end{aligned}\right\} \qquad [1.7]$$

with $\eta = r/a$ the nondimensional equatorial distance. In order to simplify the discussion we shall introduce the following nondimensional parameters.

$\alpha = (a\omega/V)$, the frequency number,

$\beta = [c/\{\sqrt{(\mu\epsilon)}\cdot V\}]$, the electromagnetic number,

$F = (Ga/V^2)$, the Froude number,

$\gamma = \mu H^2/(4\pi R V^2)$, the Alfven number,

and $\delta = (S/V)$, the pressure number. [1.8]

With the help of these, the equations (1.5) and (1.6) reduce to

$$(\mathbf{v}_1/\eta^n) = (F/\alpha^2)[\operatorname{div}(\mathbf{v}_1/\eta^n)]\,\mathbf{G}_0 + (\gamma/i\alpha)\,[3\,\mathbf{J}_0 \times \mathbf{h}_1 + (\operatorname{curl}\mathbf{h}_1) \times \mathbf{H}_0]$$
$$+ (\gamma/\beta^2)(\mathbf{v}_1 \times \mathbf{H}_0) \times \mathbf{H}_0 - (\delta^2/\alpha^2)\operatorname{grad}\operatorname{div}(\mathbf{v}_1/\eta^n), \qquad [1.9]$$

and
$$\mathbf{h}_1 = (1/i\alpha)\operatorname{curl}(\mathbf{v}_1 \times \mathbf{H}_0) \qquad [1.10]$$

In section 2 we shall consider the case when all the perturbed quantities depend on the normal distance η only and in section 3 when they vary along the direction of the initial field according to $e^{ik\xi}$ in addition to varying transverse to the unperturbed magnetic field. These two cases are of particular importance.

2. When the disturbances vary only along the normal, the equations (1.9) and (1.10) reduce to a single equation for the nondimensional equatorial velocity component v:

$$[\gamma/\eta^6 + \delta^2/\eta^n]\,(d^2 v/d\eta^2) - [9\gamma/\eta^7 + 2n\,\delta^2/\eta^{n+1} + F/\eta^{n+2}]\,(dv/d\eta) + [\gamma\alpha^2/\beta^2\eta^6$$
$$+ 21\gamma/\eta^8 + \alpha^2/\eta^n + \{n(n+1)\,\delta^2/\eta^{n+2}\} + n\,F/\eta^{n+3}]v = 0. \qquad [2.1]$$

In the domain of consideration $1 \leq \eta < \infty$ the above differential equation has only one irregular singular point at $\eta = \infty$ and hence the behaviour of the solutions is completely determined by the asymptotic behaviour of the solutions. The irregular singular point at infinity is of the normal type (Forsyth[3], Tricomi[4]) and hence admits asymptotic solutions of the type $\eta^\rho e^{\theta\eta}$ where ρ is a real constant and θ a, real or purely imaginary constant.

Consequently, the sign of ρ and the nature of θ completely give the information regarding the stability of the perturbations. Further, in order to interpret the results we note the following points of importance:

(i) The successive terms in the coefficient of the middle term of (2.1) arise respectively due to the inhomogeneity of the initial magnetic field, density variation and due to gravitational force.

(ii) For $n < 6$, the significant contribution is from the density gradient and in this case the local Alfven speed (which is proportional to η^{n-6}) tends to zero at large distances and the main process is the pressure build up.

(iii) For $n > 6$, the field gradient is the predominant term and the local Alfven speed tends to infinity as η tends infinity.

(iv) When $n = 6$, the contributions of density and field gradients are of the same order and the magneto-acoustic mode comes into play.

The following are the asymptotic forms for various cases:

(*a*) $n < 5$.

$$v(\eta) \sim \eta^n [A \cos(\alpha\eta/\delta) + B \sin(\alpha\eta/\delta)]; \qquad [2.2]$$

(*b*) $n = 5$.

$$v(\eta) \sim \eta^5 [A \cos\{(\alpha\eta/\delta) + \langle\gamma\alpha(\beta^2 - \delta^2)/2\beta^2\delta^3\rangle \log \eta\} + B \sin\{(\alpha\eta/\delta) + \langle\gamma\alpha(\beta^2 - \delta^2)/2\beta^2\delta^3\rangle \log \eta\}]; \qquad [2.3]$$

(*c*) $n = 6$.

$$v(\eta) \sim \eta^{\rho}[A \cos \lambda\eta + B \sin \lambda\eta]; \qquad [2.4]$$

where $\rho = [(9\gamma + 12\delta^2)/2(\gamma + \delta^2)] > 1$,

$\lambda^2 = [\alpha^2(\beta^2 + \gamma)/\beta^2(\delta^2 + \gamma)]$;

(*d*) $n = 7$.

$$v(\eta) \sim \eta^{9/2} \langle A \cos\{\alpha\eta/\beta + [\alpha(\beta^2 - \delta^2)/2\gamma\beta] \log \quad\} + B \sin\{\alpha\eta/\beta + [\alpha(\beta^2 - \delta^2)/2\gamma\beta] \log \eta\}\rangle; \qquad [2.5]$$

(*e*) $n > 7$.

$$v(\eta) \sim \eta^{9/2} [A \cos(\alpha\eta/\beta) + B \sin(\alpha\eta/\beta)]. \qquad [2.6]$$

From this we conclude that the unbalanced pressure gradient or field gradient always predominates leading to overstable oscillation. In the particular case when $n = 6 \pm 1$ these oscillations are aperiodic also. Hence we conclude that always there would be build up of the perturbations. This conforms with the general belief regarding the formation of shocks in an inhomogeneous atmosphere in such cases which always has a well-defined propagation mechanism.

3. If, in addition to equatorial variation, one has the variation along the unperturbed magnetic field, an additional induced motion in this direction occurs. Hence the system behaves as a diamagnetic material due to the coupling between transverse and longitudinal modes with a possibility of stable modes under certain conditions of applied frequency and wave number. Proceeding as before, the equation satisfied by the equatorial component of the velocity is

$$\left[\frac{\gamma}{\eta^6} + \frac{\alpha^2 \delta^2}{(\alpha^2 - k^2 \delta^2)\,\eta^n}\right] \frac{d^2 v}{d\eta^2} - \frac{1}{\alpha^2 - k^2 \delta^2}\left[\frac{3\gamma(3\alpha^2 - 2k^2\delta^2)}{\eta^7} + \frac{2n\alpha^2\delta^2}{\eta^{n+1}} + \frac{\alpha^2 F}{\eta^{n+2}}\right]\frac{dv}{d\eta}$$

$$+ \left[\frac{\alpha^2}{\eta^n} + \frac{n(n+1)\alpha^2\delta^2}{(\alpha^2 - k^2\delta^2)\,\eta^{n+2}} + \frac{nF\alpha^2}{(\alpha^2 - k^2\delta^2)\,\eta^{n+3}} + \frac{\gamma(\alpha^2 - k^2\beta^2)}{\beta^2\eta^6} + \frac{21\gamma\alpha^2}{(\alpha^2 - k^2\delta^2)\,\eta^8}\right.$$

$$\left. + \frac{3\gamma k^2 F}{(\alpha^2 - k^2\delta^2)\,\eta^9}\right] v = 0. \qquad [3.1]$$

Here also the middle term represents the gradient effects. But the factor $\alpha^2 - k^2\delta^2$ arising due to the coupling of transverse and longitudinal velocity components determine the sign and hence there is no unbalanced retriving force as in the previous case. Hence there is a possibility of stability and growing and decaying modes. We shall summarize the results as follows:

(*a*) $n < 5$.

$$v(\eta) \sim \eta^n [A\cos\lambda\eta + B\sin\lambda\eta] \text{ if } V_w^2 - S^2 > 0,$$

and

$$\sim \eta^n [A e^{\mu\eta} + B e^{-\mu\eta}] \text{ if } V_w^2 - S^2 < 0, \qquad [3.2]$$

where

$$\lambda^2 = (k^2/\delta^2)(V_w^2 - S^2) = -\mu^2,$$

and $V_w = (\omega/akV)$, the nondimensional phase velocity along the field lines. As expected, if V_w is supersonic, very little coupling takes place and hence unbalanced retriving forces induce overstable oscillations. On the other hand, when it is subsonic, there is a growing mode and a decaying mode mainly due to pressure build up.

(*b*) $n = 5$

As in the previous case, apart from an aperiodic phase shift, the results are the same as in the case (*a*). In the supersonic case,

$$v(\eta) \sim \eta^5 [A \cos\{\lambda\eta + (b_1/2\lambda) \log \eta\} + B \sin\{\lambda\eta + (b_1/2\lambda) \log \eta\}],$$

and in the subsonic case

$$v(\eta) \sim \eta^5 [A\, e^{\mu\eta}\, \eta^{-b_1/2\mu} + B\, e^{-\mu\eta}\, \eta^{b_1/2\mu}], \qquad [3.3]$$

where $$b_1 = (\gamma k^4/\alpha^2 \beta^2 \delta^2)(V_w^2 - S^2)[V_w^2 - c_0^2(1 + 1/k^2)].$$

It is interesting to note that whenever the phase velocity lies in the range $(S, c_0(1+1/k^2)^{1/2})$ there is a phase lag and a phase shift otherwise

(*c*) $n = 7$

In the case $n > 6$, the deciding factor is the electromagnetic induction and hence the speed of the light c_0 plays the role of S. If the phase velocity is greater than c_0, then

$$v(\eta) \sim \eta^p [A \cos\{\lambda_1 \eta + (b_2/2\lambda_1) \log \eta\} + B \sin\{\lambda_1 \eta + (b_2/2\lambda_1) \log \eta\}],$$

and if it is less than c_0, then

$$v(\eta) \sim \eta^p [A\, e^{+\mu_1\eta}\, \eta^{-b_2/2\mu_1} + B\, e^{-\mu_1\eta}\, \eta^{b_2/2\mu_1}], \qquad [3.4]$$

where $$\lambda_1^2 = (k^2/\beta^2)(V_w^2 - c_0^2) = -\mu_1^2,$$

and $$b_2 = \frac{\alpha^2(\alpha^2 - \delta^2 - k^2\delta^2)}{\gamma(\alpha^2 - k^2\delta^2)}.$$

(*d*) $n > 7$.

Here also, when $V_w > c_0$

$$v(\eta) \sim \eta^p [A \cos \lambda\eta + B \sin \lambda\eta],$$

and if $V_w < c_0$, then

$$v(\eta) \sim \eta^p [A\, e^{\mu\eta} + B\, e^{-\mu\eta}]. \qquad [3.5]$$

Thus, when $n > 6$, as in the previous case, both types of modes are available. But none of these are stable modes. Either they are overstable or growing and decaying modes.

(*e*) $n = 6$.

By far, the case, when the local Alfven speed remains constant throughout the domain (i.e. when $n = 6$), is the most interesting and the most complicated. Instead of giving the detailed expressions, we shall summarize the results.

Four characteristic velocities appear in this case due to intense coupling between hydromagnetic, acoustic, and transverse and longitudinal modes. They are respectively

$$S, \frac{V_A}{[2+(3/2)(V_A^2/S^2)]^{1/2}}, \frac{V_A}{[1+(V_A^2/c_0^2)]^{1/2}}, \text{ and } \frac{V_A}{[1+(V_A^2/S^2)]^{1/2}},$$

namely, the sound speed, the magneto-acoustic mode modified by Lorentz force term, the magneto-electromagnetic mode, and pure magneto-acoustic mode.

Whenever the phase velocity lies between modified magneto-acoustic velocity and the magneto-acoustic velocity, the perturbations are stable and behave as

$$v(\eta) \sim (1/\eta^{\rho_1})[A\cos\lambda\eta + B\sin\lambda\eta], \qquad [3.6]$$

where

$$\rho_1 = \frac{12\,S^2 + 9\,V_A^2}{S^2 + V_A^2}\left[\frac{V_w^2 - \{V_A^2/[2+(3/2)(V_A^2/S^2)]\}}{V_w^2 - \{V_A^2/[1+(V_A^2/S^2)]\}}\right] > 0.$$

Whenever the phase velocity lies between acoustic velocity and the magneto-electromagnetic velocity, $v(\eta)$ is given by

$$v(\eta) \sim \eta^{\rho}[A\,e^{\kappa\eta} + B\,e^{-\kappa\eta}], \qquad [3.7]$$

and we have both growing and decaying modes.

In the rest of the cases, the asymptotic form shows overstable modes of the type

$$v(\eta) \sim \eta^{\rho}[A\cos\lambda\eta + B\sin\lambda\eta]. \qquad [3.8]$$

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