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Short Communication

Complementary variational principles for convection of heat

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Abstract

The upper and lower bounds on the mean temperature for the steady convection of heat under constant temperature gradient are derived from the canonical theory of complementary variational principles.

Key words : Variational principles, complementary principles, convection of heat.

1. Introduction

Arthurs[1] has shown that one can construct complementary variational principles for the pair of canonical equations

$$Tx = \frac{\partial W}{\partial y}, \qquad T^* y = \frac{\partial W}{\partial x} \tag{1.1, 1.2}$$

where T and T^* are adjoint linear operators, $W(x, y)$ is a functional which is convex in X and concave in Y and

$$\frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}$$

are appropriate functional derivatives. In fluid mechanics it is possible to identify situations where the governing equations are of the canonical form.

The aim of the present paper is to develop complementary variational principles for the steady heat convection equation in which the heat input from the boundary to the fluid is constant in the flow direction and subject to the condition that the temperature $T = 0$ on the boundary C. These extremum principles lead to upper and lower bounds on the mean temperature.

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2. Mathematical formulation

Consider a steady laminar flow of a liquid in a long straight pipe of arbitrary cross-section under the influence of constant pressure gradient along the axis (OX) of the pipe. It is assumed that the flow velocity $u(y, z)$ is entirely parallel with the axis and is independent of distance X along the pipe. As a consequence, the continuity equation

$$\frac{\partial u}{\partial x} = 0$$

is satisfied identically. Viscous dissipation is neglected and the heat input from the boundary to the fluid is constant in the flow direction. The governing equation of temperature distribution is

$$\text{divgrad } T = -\frac{a}{k} u(y, z), \tag{2.1}$$

$$T = O \text{ on } C, \tag{2.2}$$

where

k = the coefficient of thermal diffusivity

$(-a)$ = the constant temperature gradient in the flow direction and divgrad $= (\partial^2/\partial y^2 + \partial^2/\partial z^2)$.

3. Complementary variational principles

Equation (2.1) can be expressed in the form

$$\text{grad } g = \phi, \; g = o \text{ on } C, \tag{3.1}$$

$$-\text{div } \phi = +\frac{a}{k} u, \tag{3.2}$$

where ϕ is a vector having components in the y and z directions. Consider the functional

$$I(g, \phi) = \int_s \left(\frac{1}{2} \phi \cdot \phi + \frac{a}{k} ug - \phi \cdot \text{grad } g \right) ds + \int_c g \phi \cdot \hat{n} \; dc \tag{3.3}$$

$$= \int_s \left(\frac{1}{2} \phi \cdot \phi + \frac{a}{k} ug + g \text{ div } \phi \right) ds \tag{3.4}$$

where s is the cross-section and $\hat{n}$ is the outward unit normal to C. The extremals of this functional with $g = o$ on c lead to (3.1) and (3.2). The exact solution is denoted by $\phi = \phi^*$, $g = T$. The complementary principles are constructed from (3.1)-(3.4) as follows :

First choose a trial function g satisfying (3.1). Then (3.3) gives

$$G(g) = -\int_s \left(-\frac{a}{k} ug + \frac{1}{2} \text{grad } g. \text{ grad } g \right) ds \tag{3.5}$$

Next choose another trial function ϕ satisfying (3.2).

Then (3.4) gives

$$J(\phi) = \tfrac{1}{2} \int_s \phi \cdot \phi \, ds. \tag{3.6}$$

The functionals $G(g)$ and $J(\phi)$ provide lower and upper bounds to the functional $I(T, \phi^*)$, that is,

$$G(g) \leq I(T, \phi^*) \leq J(\phi). \tag{3.7}$$

Now

$$I(T, \phi^*) = -\int_s \left(-\frac{a}{k} uT + \frac{1}{2} \text{grad } T. \text{ grad } T \right) ds.$$

Using (2.1) we obtain after some calculation

$$I(T, \phi^*) = +\frac{1}{2} \frac{a}{k} \left(\int_s u ds \right) T_m \tag{3.8}$$

where

$T_m = \int_s uT ds / \int_s u ds$ is the mean temperature.

From (3.7) and (3.8) we get

$$\frac{2k}{a} \left(\int_s u ds \right)^{-1} G(g) \leq + T_m \leq \frac{2k}{a} \left(\int_s u ds \right)^{-1} J(\phi). \tag{3.9}$$

4. Application

Circle : $y^2 + z^2 = c^2$.

In parametric co-ordinates $y = r \cos \theta$, $z = r \sin \theta$, $o \leq r \leq c$, $o \leq \theta \leq 2\pi$, the boundary C corresponds to $r = c$. Also, it is known that

$$u = 2U_{av} \left(1 - \frac{r^2}{c^2} \right),$$

where U_{av} is the average velocity.

Choose g as

$$g = -\frac{a}{k} \alpha \left(1 - \frac{r^2}{c^2} \right), \tag{4.1}$$

where α is a parameter and the boundary condition is satisfied.
Substituting in (3.5) we get

$$G(g) = -2\pi \frac{a^2}{k^2} \left(U_{av} \frac{c^2}{3} \alpha + \frac{1}{2} \alpha^2 \right). \tag{4.2}$$

The extemals of (4.2) are got by setting $\frac{\partial G}{\partial \alpha} = 0$. This gives

$$\alpha = -\frac{1}{3} c^2 U_{av} \text{ and } \frac{\partial^2 G}{\partial \alpha^2} < 0$$

Therefore

$$G(g) = \frac{\pi}{9} \frac{a^2}{k^2} c^4 U_{av}^2 \tag{4.3}$$

Choose ϕ as

$$\phi = -\frac{aU_{av}}{k} \left\{ y\left(1 + \beta - \frac{y^2 + z^2}{2c^2}\right)\hat{j} + z\left(1 - \beta - \frac{y^2 + z^2}{2c^2}\right)\hat{k} \right\}$$

where β is a parameter and (3.2) is satisfied. Substituting in (3.6) we have

$$J(\phi) = \frac{11}{96} \frac{\pi a^2}{k^2} U_{av}^2 c^4 + \frac{\pi a^2}{4} \frac{U_{av}^2}{k^2} c^4 \beta^2. \tag{4.4}$$

The extremals of (4.4) are got by setting

$$\frac{\partial J}{\partial \beta} = 0.$$

This gives

$$\beta = 0 \text{ and } \frac{\partial^2 J}{\partial \beta^2} > 0.$$

Therefore,

$$J(\phi)_{\cdot_1} = \frac{11}{96} \frac{\pi a^2}{k^2} U_{av}^2 c^4. \tag{4.5}$$

Thus, we have

$$\frac{2}{9} \frac{ac^2 U_{av}}{k} \leq T_m \leq \frac{11}{48} \frac{ac^2 U_{av}}{k}.$$

If, however, we take

$$g = -\frac{a}{k} \alpha \left(3 - \frac{4r^2}{c^2} + \frac{r^4}{c^4}\right), \tag{4.6}$$

from (3.5) one obtains

$$G(g) = -\frac{\pi a^2}{k^2} \left(\frac{11}{6} U_{av} \alpha c^2 + \frac{22}{3} \alpha^2\right). \tag{4.7}$$

The extremals of (4.7) are obtained by setting

$$\frac{\partial G}{\partial \alpha} = 0$$

which leads to

$$\alpha = -\frac{1}{8} C^2 U_{av} \text{ and } \frac{\partial^2 G}{\partial \alpha^2} < 0.$$

Hence,

$$G(g) = \frac{11}{96} \frac{\pi a^2}{k^2} c^4 U_{av}^2 \tag{4.8}$$

Thus, we have

$$\frac{11}{48} \frac{ac^2 U_{av}}{k} \leq T_m \leq \frac{11}{48} \frac{ac^2 U_{av}}{k}. \tag{4.9}$$

We see that the lower and upper bounds coincide.

In conclusion, it is noted that the bounds on the mean temperature can be obtained for other simple geometries.

References

1. Arthurs, A. M. *Complementary variational principles*, Clarendon Press, Oxford, 1970.