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A representation of generalized Meijer–Laplace transformable generalized functions

S. P. MALGONDE AND R. K. SAXENA
Department of Mathematics, Shivaji University, Kolhapur 416 004, India.

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Abstract

In this paper, the spaces $H_{a,b}(I)$ and its dual space $H'_{a,b}(I)$ are given. An extension of the generalized Meijer–Laplace transformation to a certain space of generalized functions (distributions) is given and a structure formula for a class of generalized Meijer–Laplace transformable generalized functions is obtained which shows that every element of the dual space of $H_{a,b}(I)$ is the linear combination of the finite order distributional derivative of continuous functions.

Key words : Generalized functions, generalized Meijer–Laplace transformation.

1. Introduction

Gelfand and Shilov[1], Koh[2] and Pandey[3] have investigated the representation of different kinds of generalized functions. The aim of the present paper is to find a representation formula for the generalized Meijer–Laplace transformable generalized functions in $H'_{a,b}(I)$ space.

The conventional generalized Meijer–Laplace transformation[4] $F(s)$ of a suitably restricted function $f(t)$ is given by

$$F(s) = \int_0^\infty H^{m+1,0}_{m,m+1}\left[st \,\middle|\, \begin{matrix}(\eta_m + a_m, A_m)\\ (\eta_m, B_m), (\rho, \beta_{m+1})\end{matrix}\right] f(t)\, dt \tag{1.1}$$

where $H^{m,n}_{p,q}[z]$ is Fox's H-function[5] which is defined as Mellin–Barnes type integral and studied in detail by Braaksma[6].

The generalized Meijer–Laplace transformation (1.1) has recently been extended to a class of generalized functions by Malgonde and Saxena[7]. The complex inversion formula for (1.1) is shown to be valid for the space of generalized functions $H'_{a,b}(I)$ where a and b are restricted in some way[7].

The notation and terminology will follow ref. 8. Unless otherwise stated, t and x will be understood to be real variables in $I = (0, \infty)$.

2. The testing function space $H_{a,b}(I)$

Let $C = \min_{1 \leqslant j \leqslant m} \operatorname{Re} \{\eta_j/B_j, \rho/B_{m+1}\}$, where $\eta_j\,(j = 1, \ldots, m)$ are complex numbers and $B_j\;(j = 1, \ldots, m)$ are positive numbers. Let a be a fixed real number satisfying $a \leqslant c + 1$ and $b > 0$. $H_{a,b}(I)$ is defined as the collection of all c^{∞}-functions $\phi(t)$ on $I = (0, \infty)$ such that

$$\gamma_k(\phi) \triangleq \gamma_{a,b,k}(\phi) \triangleq \sup_{0<t<\infty} |\, e^{-bt}\, t^{1-a+k} D_t^k \phi(t) \,| < \infty \tag{2.1}$$

for each $k = 0, 1, 2, \ldots$ and $D_t = \dfrac{d}{dt}$.

The topology of $H_{a,b}(I)$ is generated[9] by the semi-norms $\{\gamma_k\}_{k=0}^{\infty}$. A sequence $\{\phi_n\}$ converges to a function ϕ in the topology of $H_{a,b}(I)$ if and only if

$$e^{-bt}\, t^{1-a+k} \frac{d^k}{dt^k} \phi_n(t) \to e^{-bt}\, t^{1-a+k} \frac{d^k}{dt^k} \phi(t)$$

as $n \to \infty$, uniformly in t, for each $k = 0, 1, 2, \ldots$

It turns out that $H_{a,b}(I)$ is countably multinormed space. $H_{a,b}(I)$ is complete and therefore Frechet space[9]. The function

$$H^{m+1,0}_{m,m+1}\left[st \,\middle|\, \begin{matrix} (\eta_m + a_m, A_m) \\ (\eta_m, B_m), (\rho, B_{m+1}) \end{matrix} \right] \quad (0 < t < \infty)$$

for fixed s such that $\operatorname{Re} s > 0$ as a function of t belongs to $H_{a,b}(I)$. Indeed by the analyticity of $H^{m+1,0}_{m,m+1}[z]$ for $z \neq 0$, it follows that $H^{m+1,0}_{m,m+1}[z]$ is smooth on $0 < t < \infty$.

By simple computation we have

$$\gamma_k\left[H^{m+1,0}_{m,m+1}\left[st \,\middle|\, \begin{matrix} (\eta_m + a_m, A_m) \\ (\eta_m, B_m), (\rho, B_{m+1}) \end{matrix} \right]\right]$$

$$= \sup_{0<t<\infty} \left| e^{-bt}\, t^{1-a+k} D_t^k H^{m+1,0}_{m,m+1}\left[st \,\middle|\, \begin{matrix} (\eta_m + a_m, A_m) \\ (\eta_m, B_m), (\rho, B_{m+1}) \end{matrix} \right]\right|$$

$$= \sup_{0<t<\infty} \left| e^{-bt}\, t^{1-a} H^{m+1,1}_{m+1,m+2}\left[st \,\middle|\, \begin{matrix} (0, 1), (\eta_m + a_m, A_m) \\ (\eta_m, B_m), (\rho, B_{m+1}), (k, 1) \end{matrix} \right]\right| < \infty$$

for fixed s such that Re$s > 0$ and for each $k = 0, 1, 2, \ldots$, by using the asymptotic behaviour of H-function[5]. Again if $\{\phi_n\}_{n=1}^{\infty}$ converges in $H_{a,b}(I)$ to zero, then for every non-negative integer k, $\{D_t^k \phi_n(t)\}_{n=1}^{\infty}$ converges to the zero function uniformly on every compact subset of $I = (0, \infty)$ as $n \to \infty$. Thus we have shown that $H_{a,b}(I)$ is a testing function space on I since the three conditions[8] are satisfied. The space D_I is contained in $H_{a,b}(I)$ and the topology of D_I is stronger than that induced on it by $H_{a,b}(I)$. Hence the restriction of any $f \in H'_{a,b}(I)$ to D_I is in D'_I.

The dual space $H'_{a,b}(I)$ contains all distributions of compact support on $I = (0, \infty)$. Also regular distribution f corresponding to any locally integrable function $f(x)$ defined over $I = (0, \infty)$ such that $\int_0^{\infty} |f(x)\, e^{bx}\, x^{a-k-1}|\, dx < \infty$ is a member of $H'_{a,b}(I)$.

3. The generalized Meijer-Laplace transformation

For $f \in H'_{a,b}(I)$, we define the generalized Meijer-Laplace transformation of f as a function $F(s)$ obtained by applying f on the kernel

$$H^{m+1,0}_{m,m+1}\left[st \,\middle|\, \begin{matrix}(\eta_m + a_m, A_m) \\ (\eta_m, B_m), (\rho, B_{m+1})\end{matrix}\right]; \text{ i.e.}$$

$$F(s) \triangleq \left\langle f(t),\ H^{m+1,0}_{m,m+1}\left[st \,\middle|\, \begin{matrix}(\eta_m + a_m, A_m) \\ (\eta_m, B_m), (\rho, B_{m+1})\end{matrix}\right]\right\rangle \quad \text{for } s \in \Omega_f \tag{3.1}$$

$$\Omega_f = \{s : \text{Re}s > 0\}.$$

4. The space $\bar{H}_{a,b}(I)$

For fixed real values of a and $b > 0$, a smooth and complex-valued function $\phi(t)$ defined on $I = (0, \infty)$ is said to be in the space $\bar{H}_{a,b}(I)$ if it satisfies the following order properties:

$$\begin{aligned} D_t^k \phi(t) &= 0(1), \quad \text{as } t \to 0 \\ &= 0(1), \quad \text{as } t \to \infty \end{aligned} \tag{4.1}$$

for each $k = 0, 1, 2, \ldots$

Lemma 1: $\bar{H}_{a,b}(I)$ is a linear subspace of $H_{a,b}(I)$ where $b > 0$.

Proof: The proof is simple and hence omitted.

Lemma 2: For $\phi \in \bar{H}_{a,b}(I)$ and $f \in H'_{a,b}(I)$ there exist a positive constant C and a non-negative integer q such that

$$|\langle f, \phi \rangle| \leqslant C \max_{1 \leqslant k \leqslant q+1} \int_0^{\infty} |e^{-bx}\, x^{k-a}\, D_x^k \phi(x)|\, dx \tag{4.2}$$

Proof : For each $f \in H'_{a,b}(I)$ and in view of boundedness property of generalized functions, we have a positive constant C_1 and a non-negative integer q such that

$$|\langle f, \phi \rangle| \leqslant C_1 \cdot \max_{0 \leqslant k \leqslant q} \gamma_k(\phi)$$

$$\leqslant C_1 \cdot \max_{0 \leqslant k \leqslant q} \sup_{0 < x < \infty} |e^{-bx} x^{1-a+k} D_x^k \phi(x)|$$

$$\leqslant C_1 \cdot \max_{0 \leqslant k \leqslant q} \sup_{0 < x < \infty} |e^{-bx} x^{1-a} x^k D_x^k \phi(x)| \qquad \text{(I)}$$

Further from elementary calculus we have

$$e^{-bx} x^{1-a} x^k D_x^k \phi(x) = \int_0^x D_t [e^{-bt} t^{1-a} t^k D_t^k \phi(t)] dt \qquad \text{(II)}$$

Combining (I) and (II) we obtain

$$|\langle f, \phi \rangle| \leqslant C_1 \cdot \max_{0 \leqslant k \leqslant q} \sup_{0 < x < \infty} \left| \int_0^x D_t [e^{-bt} t^{1-a} t^k D_t^k \phi(t)] dt \right|$$

$$\leqslant C_1 \cdot \max_{0 \leqslant k \leqslant q} \sup_{0 < x < \infty} \left| \int_0^x D_t [e^{-bt} t^{1-a+k}] D_t^k \phi(t) dt + \int_0^x [e^{-bt} t^{1-a+k} D_t^{k+1} \phi(t)] dt \right|$$

$$\leqslant C_1 \max_{0 \leqslant k \leqslant q} \sup_{0 < x < \infty} \left[\left| \int_0^x D_t [e^{-bt} t^{1-a+k}] . \; D_t^k \phi(t) \, dt \right| + \left| \int_0^x e^{-bt} t^{1-a+k} D_t^{k+1} \phi(t) dt \right| \right].$$

Now consider

$$\left| \int_0^x D_t [e^{-bt} t^{1-a+k}] \; D_t^k \phi(t) dt \right|$$

$$= \left| e^{-bx} x^{1-a+k} D_x^k \phi(x) - \int_0^x e^{-bt} t^{1-a+k} D_t^{k+1} \phi(t) dt \right|$$

$$\leqslant |e^{-bx} x^{1-a+k} D_x^k \phi(x)| + \left| \int_0^x e^{-bt} t^{1-a+k} D_t^{k+1} \; \phi(t) dt \right|.$$

Therefore

$$|\langle f, \phi \rangle| \leqslant C_1 \cdot \max_{0 \leqslant k \leqslant q} \sup_{0 < x < \infty} \left[|e^{-bx} \; x^{1-a+k} \; D_x^k \phi(x)| + \left| \int_0^x e^{-bt} t^{1-a+k} \; D_t^{k+1} \phi(t) dt \right| + \left| \int_0^x e^{-bt} \; t^{1-a+k} \; D_t^{k+1} \phi(t) dt \right| \right]$$

$$\leqslant C_1 \cdot \max_{0 \leqslant k \leqslant q} \sup_{0 < x < \infty} \left[|e^{-bx} \; x^{1-a+k} \; D_x^k \phi(x)| + 2 \left| \int_0^x e^{-bt} \; t^{1-a+k} \; D_t^{k+1} \; \phi(t) dt \right| \right]$$

$$\leqslant C_1 \cdot \max_{0 \leqslant k \leqslant q} \sup_{0 < x < \infty} |e^{-bx} \; x^{1-a+k} \; D_x^k \phi(x)|$$

$$+ 2 \sup_{0<t<\infty} \int_0^\infty | e^{-bt} t^{1-a+k} D_t^{k+1} \phi(t) | dt.$$

But for $\phi \in \bar{H}_{a,b}(I) \subset H_{a,b}(I)$ we have

$$\sup_{0<x<\infty} | e^{-bx} x^{1-a+k} D_x^k \phi(x) | < \infty \; (= M, \text{ say})$$

$$\therefore |\langle f, \phi \rangle| \leqslant C_1 . \max_{0 \leqslant k \leqslant q} [M + 2 \int_0^\infty | e^{-bx} x^{1-a+k} D_x^{k+1} \phi(x) | dx]$$

$$\leqslant C_1 . \max_{0 \leqslant k \leqslant q} [(p+2) \int_0^\infty | e^{-bx} x^{1-a+k} D_x^{k+1} \phi(x) | dx],$$

where

$$M = p \int_0^\infty | e^{-bx} x^{1-+k} D_x^{k+1} \phi(x) | dx$$

$$\leqslant C . \max_{0 \leqslant k \leqslant q} \int_0^\infty | e^{-bx} x^{1-a+k} D_x^{k+1} \phi(x) | dx$$

$$\leqslant C \max_{1 \leqslant k \leqslant q+1} \int_0^\infty | e^{-bx} x^{k-a} D_x^k \phi(x) | dx.$$

$$\therefore |\langle f, \phi \rangle| \leqslant C . \max_{1 \leqslant k \leqslant q+1} \int_0^\infty | e^{-bx} x^{k-a} D_x^k \phi(x) | dx$$

which proves Lemma 2.

Now we are in a position to state and prove the main representation formula.

Theorem : Let $f \in H'_{a,b}(I)$ and $\phi \in \bar{H}_{a,b}(I)$, then there exist N-bounded measurable functions $g_k(x)$, $1 \leqslant k \leqslant q+1$, defined on $(0, \infty)$ such that

$$\langle f, \phi \rangle = \sum_{k=1}^{q+1} \langle g_k(x), e^{-bx} x^{k-a} D_x^k \phi(x) \rangle. \tag{4.3}$$

Proof : On account of Lemma 2, we have

$$|\langle f, \phi \rangle| \leqslant C . \max_{1 \leqslant k \leqslant q+1} \int_0^\infty | e^{-bx} x^{k-a} D_x^k \phi(x) | dx$$

$$\leqslant C . \max_{1 \leqslant k \leqslant q+1} \| e^{-bx} x^{k-a} D_x^k \phi(x) \|_{L_1(0,\infty)} \tag{4.4}$$

where $L_1(0, \infty)$ is the space of all equivalence classes of Lebesgue integrable functions on $(0, \infty)$ whose topology is defined through the norm

$$\| \psi(x) \|_{L_1(0,\infty)} = \int_0^\infty | \psi(x) | dx < \infty, \quad \psi \in L_1(0, \infty). \tag{4.5}$$

The result (4.4) defines a linear one-to-one and into mapping

$$M : \bar{H}_{a,b}(I) \to L_1(0, \infty)$$

as

$$\phi \to e^{-bx} x^{k-a} D_x^k \phi(x), \quad 1 \leq k \leq q+1.$$

Since $\bar{H}_{a,b}(I)$ is a linear subspace of $L_1(0, \infty)$, (4.4) further states that f is continuous linear functional[9] on $\bar{H}_{a,b}(I)$ in the topology induced on it by $L_1(0, \infty)$. Hence by Hahn-Banach theorem[10], f can be extended as a continuous linear functional in the whole of $L_1(0, \infty)$. But the conjugate of $L_1(0, \infty)$ is $L_\infty(0, \infty)$. Therefore on account of Riesz-representation theorem[10] there exist N-bounded measurable functions $g_k(x) \in L_\infty(0, \infty)$, $1 \leq k \leq q+1$ such that

$$\langle f, \phi \rangle = \sum_{k=1}^{q+1} \langle g_k(x), e^{-bx} x^{k-a} D_x^k \phi(x) \rangle \tag{4.6}$$

This completes the proof.

Corollary: Let $f \in H'_{a,b}(I)$ and $\phi \in D_I$, the space of smooth functions with compact support on $I = (0, \infty)$. Then there exist N-bounded measurable functions $g_k(x) \in L_\infty(0, \infty)$, $1 \leqslant k \leqslant q+1$ such that

$$\langle f, \phi \rangle = \langle \sum_{k=1}^{q+1} (-1)^k D^{k+1} \int_0^x e^{-bt} t^{k-a} g_k(t)\, dt, \phi(x) \rangle \tag{4.7}$$

Proof: Let $\phi \in D_I$. Then in view of the Theorem, there exist N-bounded measurable functions $g_k(x) \in L_\infty(0, \infty)$, $1 \leq k \leq q+1$ satisfying the relation

$$\langle f, \phi \rangle = \langle \sum_{k=1}^{q+1} \langle D \int_0^x e^{-bt} t^{k-a} g_k(t)\, dt, D_x^k \phi \rangle.$$

Again, since $D_t^k \phi(t) \in D_I$ and the regular distribution corresponding to the integral appearing in (4.7) belongs to D'_I, the relation (4.7) follows immediately using the rules of distributional differentiation[11].

Note: The result (4.7) can be put in Koh's form, *viz*.,

"If $f \in H'_{a,b}$. Then f is equal to a finite sum

$$\sum_{j=0}^{q} C_j \left(\frac{d}{dx}\right)^j [e^{-bx} x^{a-1} P_j(x) F_j(x)]$$

where the $F_j(x)$ are continuous on $(0, \infty)$ and the $P_j(x)$ are polynomials of degree q."

In a view of the general nature of the kernel involved in (1.1), we have been able to extend the representation formulae of the conventional integral transformations like Meijer–Laplace transformation[4] and the other generalizations of Laplace transformations given by Varma and Meijer to generalized functions.

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References

1. GELFAND, I. M. AND SHILOV, G. E. — *Generalized functions*, Vol. II, Academic Press, New York, 1968.

2. KOH, E. L. — A representation of Hankel transformable generalized functions, *SIAM J. Math. Anal.*, 1970, 1, 33–36.

3. PANDEY, J. N. — A representation theorem for a class of convolution transformable generalized functions, *SIAM J. Math. Anal.*, 1971, 3 (2), 286–289.

4. BHISE, V. M. — Inversion formula for a generalized Laplace integral, *J. Vk. Univ.*, 1959, 3 (3), 57–63.

5. FOX, C. — The *G*- and *H*-functions as symmetrical Fourier kernel, *Trans. Am. Math. Soc.*, 1961, 98, 395–429.

6. BRAAKSMA, B. L. J. — Asymptotic expansion and analytic continuation for the class of Barnes integral, *Comp. Math.*, 1964, 15, 239–341.

7. MALGONDE, S. P. AND SAXENA, R. K. — Generalized Meijer–Laplace transformation of generalized functions, Accepted for publication in *Proc. Natn. Acad. Sci., India A.*

8. TREVES, F. — *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967.

9. ZEMANIAN, A. H. — *Generalized integral transformations*, Interscience, New York, 1968.

10. RIESZ, F. AND NAGY, B. S. — *Functional analysis*, Blackie and Son Ltd., London, 1956.

11. ZEMANIAN, A. H. — *Distribution theory and transform analysis*, McGraw-Hill Company, New York, 1965.