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Fault detection by adaptive nonlinear filtering

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Abstract

In a linear system perturbed by Gaussian noise, the state can be estimated from the observations by using Kalman filter. However, if a fault develops in the system at any random time, the Kalman filter will not be able to track the fault and large errors will develop in the state estimate. Consequently, the innovations process will no longer be white. If the random time of occurrence is considered as a state then the system of state equations become nonlinear. In this paper, the Fujisaki, Kallianpur and Kunita nonlinear filtering results have been applied to obtain a representation for the state estimate given the observations. The non-white nature of the innovations process has been modelled as an autoregressive process and an adaptive scheme has been proposed to improve the filter performance.

Key words: Fault detection, nonlinear filter, adaptive filter.

1. Introduction

In a linear system perturbed by Gaussian noise the state can be estimated from the observations by using the Kalman filter. However, if a fault develops in the system at any random time, the Kalman filter will not be able to track the fault and large errors will develop in the state estimate. To limit these large errors the Kalman filter has to be reparametized for which we should have the estimates of the time and amount of fault. Thus, the information from the observations has to be used in both tracking the states and for fault detection. For a certain class of fault detection problems the nonlinear filtering theory developed by Fujisaki *et al*[1] can be used. The white noise nature of the innovations process under optimal conditions is utilized for an adaptive scheme to improve on the Kalman filter performance.

2. Statement of the problem

The problem will be formulated as a scalar, the generalization to a vector case being straightforward.

The state process $\{X_t\ t \in T\}$ is given by an Ito stochastic differential equation

$$dX_t = f_t X_t\, dt + g_t\, dW_t\ \ t \in T,\ X_0 \tag{1}$$

where f_t and g_t are functions of time (not random processes) satisfying the conditions

$$\int_{t \in T} |f_t|\, dt < \infty \qquad \int_{t \in T} g_t^2\, dt < \infty \tag{2}$$

and $\{W_t,\ t \in T\}$ is a Brownian motion process with parameter σ_w^2. X_0 is an arbitary initial condition random variable independent of W_t and T is the time interval $(0, \infty)$.

The observation process $\{Y_t,\ t \in T\}$ is given by another Ito stochastic differential equation

$$dY_t = h_t X_t dt + dV_t\ \ t \in T,\ Y_0 = 0 \tag{3}$$

where h_t is again a function of time satisfying the condition

$$\int_{t \in T} h_t^2\, dt < \infty \tag{4}$$

and $\{V_t,\ t \in T\}$ is another Brownian motion process with parameter σ_v^2, independent of both $\{W_t,\ t \in T\}$ and X_0.

Both the state process $\{X_t,\ t \in T\}$ and the observation process $\{Y_t,\ t \in T\}$ are measurable with respect to the σ-algebra $\mathfrak{B}_t$ defined by

$$\mathfrak{B}_t \supset \sigma\{X_0, X_s, Y_s, Z_s, s \leqslant t,\ t \in T\} \tag{5}$$

Further let F_t be the σ-algebra generated by the observations, *viz.*,

$$F_t = \sigma(Y_s,\ s \leqslant t,\ t \in T) \tag{6}$$

In a general fault problem concerning eqns. (1) and (3), a fault occurs at some random time τ such that

(i) the state noise parameter g_t changes,

(ii) the state parameter f_t changes,

(iii) there is an additive bias in the observation equation,

(iv) there is an additive bias in the state equation,

(v) the observation gain parameter h_t changes,

(vi) there is an increase in the state noise W_t,

(vii) there is an increase in the observation noise V_t.

Among these different fault problems we shall be concerned with (i). Davis[2] has dealt with case (ii) and Chien[3] has treated case (iii).

3. Review of nonlinear filtering results

We shall first state the Doleans–Dade–Meyer extension to the Ito rule. The proof is given in (4).

Let $\{X_t, F_t, t \in T\}$ be a general semi martingale (a sum of a martingale and a process of bounded variation). Let $\psi(t, x, y)$ be a continuous function having continuous partial derivatives

$$\frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial x}, \frac{\partial^2 \psi}{\partial x^2}, \frac{\partial \psi}{\partial y}, \frac{\partial^2 \psi}{\partial y^2}.$$

The scalar process $Z_t = \psi(t, X_t, Y_t)$ admits of a stochastic differential equation given by

$$\begin{aligned} dZ_t = {} & \psi(t, X_t, Y_t) - \psi(t, X_{t-}, Y_{t-}) + \frac{\partial \psi(t, X_t, Y_t)}{\partial t} dt \\ & + \frac{\partial \psi(t, X_{t-}, Y_{t-})}{\partial x} dX_t + \frac{\partial \psi(t, X_{t-}, Y_{t-})}{\partial y} dY_t \\ & + \frac{1}{2} \frac{\partial^2 \psi(t, X_{t-}, Y_{t-})}{\partial x^2} d\langle M_X^c, M_X^c \rangle t \\ & + \frac{1}{2} \frac{\partial^2 \psi(t, X_{t-}, Y_{t-})}{\partial y^2} d\langle M_Y, M_Y^c \rangle t \\ & + \frac{\partial^2 \psi(t, X_{t-}, Y_{t-})}{\partial x \partial y} d\langle M_X^c, M_Y^c \rangle t \\ & - \frac{\partial \psi(t, X_{t-}, Y_{t-})}{\partial x} \Delta X_t \\ & - \frac{\partial \psi(t, X_{t-}, Y_{t-})}{\partial y} \Delta Y_t \end{aligned} \tag{7}$$

where M_X^c, M_Y^c are the continuous F_t-martingales associated with the semi-martingales X_t, Y_t, $\langle M_X^c, M_X^c \rangle_t$ is the quadratic variance process associated with the martingale M_X^c, $\langle M_X^c, M_Y^c \rangle t$ is the quadratic covariance process associated with the martingale M_X^c, M_Y^c, ΔX_t, ΔY_t are the amounts of jump of X_t, Y_t at t and X_{t-}, Y_{t-} are the left hand limits at t of X_t, Y_t.

The special case of the above rule relevent to our situation is when $\psi(t, X_t, Y_t) = X_t Y_t$ where X_t and Y_t are semi-martingales. In this case eqn. (7) can be rewritten as

$$d(X_t Y_t) = X_t Y_t - X_{t-} Y_{t-} + Y_{t-} dX_t + X_{t-} dY_t + d\langle M_X^c, M_Y^c \rangle_t + \Delta X_t \Delta Y_t \tag{8}$$

For details see ref. 4.

We shall now state the nonlinear filtering result[1] for the state process given by eqn. (1) and the observation process given by eqn. (3). For details see ref. 4.

Let $\{\Omega, F, P\}$ be a complete probability space and let $\{X_t, t \in T\}$ and $\{Y_t, t \in T\}$ satisfy eqns. (1) and (3) respectively. Then the filtered estimate $\hat{X}_t$ satisfies the stochastic differential equation

$$d\hat{X}_t = f_t\hat{X}_t dt + \left[E^{F^t}(\tilde{X}_t^2 h_t + \frac{d}{dt}\langle M, V\rangle_t\right]\frac{1}{\sigma_v^2}d\nu_t,\ t \in T \tag{9}$$

where $\tilde{X}_t = \hat{X}_t - X_t$ error in the estimate

$$dM_t = g_t\, dW_t$$

and ν_t the innovations process is given by

$$d\nu_t = dY_t - h_t\hat{X}_t\, dt \quad t \in T \tag{10}$$

Equation (9) is written for the case when W is not independent of V. Since W_t and V_t are assumed independent in eqns. (1) and (3), eqn. (9) can be rewritten as

$$d\hat{X}_t = f_t\hat{X}_t\, dt + \frac{1}{\sigma_v^2}E^{F^t}X_t^2\, h_t \cdot d\nu_t,\ t \in T \tag{11}$$

Equation (9) or (10) is only a representation for the nonlinear filtering problem. We can see from the equations that the first conditional moment $\hat{X}_t$ depends upon the second conditional moment $\hat{X_t^2}$. Similarly, the equation for the second conditional moment will depend upon the third conditional moment and so on. Therefore, closed form solutions, in general, are not possible. Suitable approximations have to be made so that closed form solutions can be obtained.

4. Fault detection with change in state noise parameter, g

The state process is given by the stochastic differential equation

$$dX_t = fX_t\, dt + g\, dW_t,\ t \in T,\ X_0 \tag{12}$$

and the observation process is given by

$$dY_t = hX_t dt + dV_t,\ t \in T,\ Y_0 = 0 \tag{13}$$

where the parameters f, g, h are constants and the Brownian motion parameters are given by σ_w^2 and σ_v^2. We shall also assume as in eqns. (1) and (3), W_t, V_t and X_0 are independent. The σ-algebras $\mathcal{B}_t$ and F_t are defined as before.

In many inertial navigation systems the sttae noise parameter g changes suddenly due to some failure in the system. A fault, therefore, occurs at some random time, τ, so that the state noise parameter, g, changes from g to $g + b$. The random time τ is independent of X_0, W_t, and V_t and $\{\tau \leq t\}$ is measurable with respect to $\mathcal{B}_t$.

As a consequence of the fault, the state process (12) changes to

$$dX_t = fX_t dt + (g + b)\, dW_t \quad t > \tau, \; [t, \tau \in T] \tag{14}$$

Under normal operating conditions the state estimate will be given by the Kalman filter equations. The sudden change in the state noise parameter will induce large errors since the Kalman filter cannot track sudden changes. In order to reparametize the Kalman filter equations, we should know the estimates of the time of fault, τ, and the magnitude of the fault, b.

In order to characterize the fault, we shall introduce a $\mathcal{B}_t$- measurable indicator function, Z_t defined by

$$Z_t = \begin{bmatrix} 1 & t \geq \tau \\ 0 & t < \tau \end{bmatrix} \tag{15}$$

Using eqn. (15), eqns. (12) and (14) can be combined to yield

$$dX_t = fX_t\, dt + (g + b\, Z_t)\, dW \quad t \in T, \; X_0 \tag{16}$$

We have to determine the stochastic differential equation satisfied by the new state variable Z_t. We shall assume that the process of fault occurrence is a Poisson process N_t, with parameter λ independent of W_t and V_t. Hence the process $N_t - \lambda t$ is a Poisson martingale. Z_t is a Poisson process stopped at the first fault occurrence τ. Since a stopped martingale is also a martingale, we have $Z_t - \lambda(t \wedge \tau)$ is $\mathcal{B}_t$-martingale, where $(t \wedge \tau)$ represents $(\min\, (t, \tau)$. Or,

$$Z_t - \lambda(t \wedge \tau) = M_t \quad t \in T \tag{17}$$

where M_t is the discontinuous $\mathcal{B}_t$-martingale associated with the stopped Poisson process Z_t. However, the quantity $(t \wedge \tau)$ can be represented by

$$t \wedge \tau = \int_0^t (1 - Z_s)\, ds \quad t \in T \tag{18}$$

as shown below.

Case (i) $t < \tau$.

Since s is also less than τ, we have $Z_s = 0$ and hence

$$t \wedge \tau = \int_0^t ds = t$$

Case (ii) $t \geq \tau$

Here $\int_0^t (1 - Z_s)\, ds$ can be split as

$$\int_0^t (1 - Z_s)\, dt = \int_0^\tau (1 - Z_s)\, ds + \int_\tau^t (1 - Z_s)\, ds$$

In the first integral in the right hand side $s < \tau$ and $Z_s = 0$ and in the second integral $s > \tau$ and $Z_s = 1$. Hence

$$t \wedge \tau = \int_0^{\tau} ds + 0 = \tau.$$

Substituting eqn. (18) into eqn. (17) the stochastic differential equation for Z_t is given by

$$dZ_t = \lambda (1 - Z_t) dt + dM_t \quad t \in T, \ Z_0 = 0 \tag{19}$$

Due to the occurrence of fault at a random time τ, the state processes are given by eqns. (16) and (19) which are renumbered below.

$$dX_t = fX_t \, dt + (g + bZ_t) \, dW_t, \ t \in T, \ X_0 \tag{20 a}$$

$$dZ_t = \lambda (1 - Z_t) \, dt + dM_t \qquad t \in T, \ Z_0 = 0 \tag{20 b}$$

With the observation process given by eqn. (13)

$$dY_t = hX_t \, dt + dV_t \quad t \in T \tag{16}$$

Using the nonlinear filtering formula eqn. (9) with $(g + bZ_t) \, dt = dM_t$ we can write the filter equations for the estimates $\hat{X}_t$ and $\hat{Z}_t$.

$$d\hat{X}_t = f\hat{X}_t \, dt + \frac{h}{\sigma_0^2} \hat{P}_{Xt} dv_t \quad t < \hat{\tau}, \ X_0 \tag{21a}$$

$$d\hat{Z}_t = \lambda (1 - \hat{Z}_t) \, dt + \frac{h}{\sigma_0^2} \hat{P}_{XZt} \, dv_t \quad t < \hat{\tau}, \ Z_0 = 0 \tag{21 b}$$

where we have defined

$$\hat{P}_{Xt} = E^{F^t} \tilde{X}_t^2$$

$$\hat{P}_{XZt} = E^{F^t} (\tilde{X}_t \tilde{Z}_t)$$

In deriving eqns. 21 we have also used the fact that W_t, V_t, and τ are independent and M_t is a discontinuous martingale and hence the quadratic covariance processes $\langle W, V \rangle_t$ and $\langle M, V \rangle_t$ are zero.

It is interesting to note that the filter eqn. (21 a) is the same for the state process given by (12). Thus, large errors in the estimate manifest itself after the occurrence of the fault.

The unknown quantities in eqns. (21) are the second conditional moments $\hat{P}_{Xt}$ and $\hat{P}_{XZt}$. To determine them we need the stochastic differential equation for the propagation of the error, $\tilde{X}_t$, Subtracting eqn. 20 (*a*) we obtain

$$d\tilde{X}_t = f\tilde{X}_t dt + \frac{h}{\sigma_v^2} \hat{P}_{Xt}\, dv_t - (g + bZ_t)\, dW_t \tag{22}$$

The innovations process v_t (eqn. 10) can also be written as

$$dv_t = h\,(X_t - \hat{X}_t)\, dt + dV_t = -h\tilde{X}_t dt + dV_t \tag{23}$$

which when substituted into eqn. (22) yields

$$d\tilde{X}_t = f\tilde{X}_t dt - \frac{h^2}{\sigma_v^2} \tilde{X}_t \hat{P}_{Xt} dV_t + \frac{h}{\sigma_v^2} \hat{P}_{Xt} dV_t - (g + bZ_t)\, dW_t \tag{24}$$

Using Ito–Doleans–Dade–Meyer rule (eqn. 8) for $d\tilde{X}_t^2$ yields

$$d\tilde{X}_t^2 = \left[\, 2f\tilde{X}_t^2 + - \frac{2h^2}{\sigma_v^2} \tilde{X}_t^2 \hat{P}_{Xt} + \frac{h^2}{\sigma_v^2} \hat{P}_{Xt}^2 + (g + bZ_t)^2 \sigma_w^2 \right] dt$$

$$+ \frac{2h}{\sigma_v^2} \tilde{X}_t \hat{P}_{Xt} dV_t - 2\tilde{X}_t (g + bZ_t)\, dW_t \tag{25}$$

Applying the nonlinear filtering eqn. (9) and using the relation $Z_t^2 = Z_t$, we obtain

$$d\hat{P}_{Xt} = \left\{ 2f\hat{P}_{Xt} - \frac{h^2}{\sigma_v^2} \hat{P}_{Xt}^2 + \sigma_w^2 \left[g^2 + (2gb + b^2)\, \hat{Z}_t\right] \right\} dt$$

$$+ \frac{h}{\sigma_v^2} s_{X^3t} dv_t,\ t < \hat{\tau},\ \hat{P}_{X0} \tag{26 a}$$

where we have defined

$$s_{X^3t} = E^{F_t} \tilde{X}_t^3.$$

In an exactly analogous manner we can write the filter equation for $d\hat{P}_{XZt}$.

$$d\hat{P}_{XZt} = \left[(f - \lambda)\, \hat{P}_{XZt} - \frac{h^2}{\sigma_v^2} \hat{P}_{Xt} \hat{P}_{XZt} \right] dt + \frac{h}{\sigma_v^2} s_{X^2Zt} dv_t t < \hat{\tau},\ P_{XZ0} \tag{26 b}$$

where we have defined

$$s_{X^2Zt} = E^{F^t} X_t^2 Z_t.$$

We can again find filter equations of s_{X^3t} and s_{X^2Zt} thus yielding an expanding set of equations as mentioned earlier. To form a closed set of equations corresponding to a sub-optimal filter we shall use eqns. 21 and 26 as the filter equations and use an adaptive algorithm, as described in the next section, to derive the terms s_{X^3t} and s_{X^2Zt} to zero. We can substitute an apriori estimate for *b* in eqn. 26 (*a*).

From eqns. 21 (*b*) and 26 (*b*) we are in a position to estimate the time of fault. By definition,

$$\hat{Z}_t = E^{F_t} Z_t = P(t \geqslant \tau \mid F_t) \tag{27}$$

Hence Z_t is a probability function conditioned on the observations.

We now define a threshold function $\gamma \in [0, 1]$, the value of which can be set by some criterion of performance linked to the probabilities of false and missed alarms. Having set the threshold value γ, the estimated time of failure, τ, is obtained from

$$\hat{\tau} = \inf_t \{\hat{Z}_t \geqslant \gamma\} \tag{28}$$

After the occurrence of the fault the state equations can be written as

$$dX_t = fX_t dt + (g + b)\, dW_t \quad t \geqslant \tau,\ X_\tau \tag{29 a}$$

$$db_t = 0 \qquad\qquad t \geqslant \tau,\ b \tag{29 b}$$

where we have introduced a new state b and omitted the state Z_t.

Using the nonlinear filtering result (eqn. 9) and the Ito–Doleans–Dade–Meyer rule (eqn. 8) we can write the filter equations.

$$d\hat{X}_t = f\hat{X}_t\, dt + \frac{h}{\sigma_v^2} \hat{P}_{Xt} dv_t \quad t \geqslant \hat{\tau}\ \hat{X}_{\hat{\tau}} \tag{30 a}$$

$$d\hat{b}_t = \frac{h}{\sigma_v^2} \hat{P}_{Xbt} dv_t \qquad t \geqslant \hat{\tau}\ \hat{b} \tag{30 b}$$

where the conditional error covariances

$\hat{P}_{Xt} = E^{F_t} \tilde{X}_t^2$, $\hat{P}_{Xbt} = E^{F_t} \tilde{X}_t\, \tilde{b}_t$ are given by

$$d\hat{P}_{Xt} = \left[2f\hat{P}_{Xt} - \frac{h^2}{\sigma_v^2} \hat{P}_{Xt}^2 + \sigma_w^2 (g + \hat{b})^2 + \hat{P}_{bt} \right] dt$$

$$+ \frac{h}{\sigma_v^2} s_{X^3 t}\, dv_t \quad t \geqslant \hat{\tau},\ \hat{P}_{X\hat{\tau}} \tag{31 a}$$

$$d\hat{P}_{Xbt} = \left[f\hat{P}_{Xbt} - \frac{h^2}{\sigma_v^2} \hat{P}_{Xt}\hat{P}_{Xbt} \right] dt + \frac{h}{\sigma_v^2} s_{X^2 bt} dv_t \quad t \geqslant \hat{\tau},\ \hat{P}_{Xb\hat{\tau}} \tag{31 b}$$

The conditional covariance matrix $\hat{P}_{bt} = E^{F_t} \tilde{b}_t^2$ is again given by

$$d\hat{P}_{bt} = \left[\frac{-h^2}{\sigma_v^2} \hat{P}_{Xbt}^2 \right] dt + \frac{h}{\sigma_v^2} s_{Xb^2 t} dv_t \quad t \geqslant \hat{\tau},\ \hat{P}_{b\hat{\tau}} \tag{31 c}$$

where we have defined

$$s_{X^3t} = E^{F^t}\tilde{X}_t^3,\quad s_{X^2bt} = E^{F^t}\tilde{X}_t^2\tilde{b}$$

and

$$s_{Xb^2t} = E^{F_t}\tilde{X}\tilde{b}^2$$

We can now apply the adaptive algorithm to be described in the next section to yield a closed form sub-optimal filter.

Note : There is no stochastic differential equation corresponding to $\hat{P}_{Zt}$ in the set of eqns. 26 because $Z_t^2 = Z_t$, whereas we do have a stochastic differential equation for $\hat{P}_{bt}$ since $b_t^2 \neq b_t$.

5. Summary of sub-optimal filter scheme

The system equations before fault are

$$\begin{aligned}
dX_t &= fX_t\,dt + (g + bZ_t)\,dW_t & t < \tau\ X_0 \\
dZ_t &= \lambda(1 - Z_t)\,dt + dM_t & t < \tau\ Z_0 = 0 \\
dY_t &= hX_t\,dt + dV_t & t \in T\ Y_0 = 0
\end{aligned} \tag{32}$$

The corresponding filter equations are

$$\begin{aligned}
d\hat{X}_t &= f\hat{X}_t dt + \frac{h}{\sigma_v^2}\hat{P}_{Xt}dv_t\quad t < \hat{\tau},\ \hat{X}_0 \\
d\hat{Z}_t &= \lambda(1 - \hat{Z}_t)\,dt + \frac{h}{\sigma_v^2}\hat{P}_{XZt}dv_t\quad t < \hat{\tau},\ \hat{Z}_0 = 0 \\
d\hat{P}_{Xt} &= \left\{2f\hat{P}_{Xt} - \frac{h^2}{\sigma_v^2}\hat{P}_{Xt}^2 + \sigma_w^2[g^2 + (2gb + b^2)\hat{Z}_t]\right\}dt \\
&\quad + \frac{h}{\sigma_v^2}s_{X^3t}dv_t,\ t < \hat{\tau},\ \hat{P}_{X0} \\
d\hat{P}_{XZt} &= \left[(f - \lambda)\hat{P}_{XZt} - \frac{h^2}{\sigma_v^2}\hat{P}_{Xt}\hat{P}_{XZt}\right]dt + \frac{h}{\sigma_v^2}s_{X^2Zt}dv_t\quad t < \hat{\tau},\ \hat{P}_{XZ0}
\end{aligned} \tag{33}$$

The system equations after the occurrence of fault are

$$\begin{aligned}
dX_t &= fX_t\,dt + (g + b)\,dW_t & t \geq \tau\quad X_\tau \\
db_t &= 0 & t \geq \tau\quad b
\end{aligned} \tag{34}$$

The corresponding filter equations are

$$d\hat{X}_t = f\hat{X}_t\, dt + \frac{h}{\sigma_v^2}\hat{P}_{Xt}\, dv_t, \quad t \geqslant \hat{\tau}, \quad \hat{X}_{\hat{\tau}}$$

$$d\hat{b}_t = \frac{h}{\sigma_v^2}\hat{P}_{Xbt}\, dv_t \qquad , \quad t \geqslant \hat{\tau}, \ \hat{b}$$

$$d\hat{P}_{Xt} = \left\{ f\hat{P}_{Xbt} - \frac{h^2}{\sigma_v^2}\hat{P}_{Xt}^2 + \sigma_w^2[(g + \hat{b})^2 + \hat{P}_{bt}] \right\} dt$$

$$+ \frac{h}{\sigma_v^2} s_{X^3 t}\, dv_t, \ t \geqslant \hat{\tau}, \ \hat{P}_{X\hat{\tau}}$$

$$d\hat{P}_{Xbt} = \left[f\hat{P}_{Xbt} - \frac{h^2}{\sigma_v^2}\hat{P}_{Xt}\hat{P}_{Xbt} \right] dt + \frac{h}{\sigma_v^2} s_{X^2 bt}\, dv_t, \ t \geqslant \hat{\tau}, \ \hat{P}_{Xb\hat{\tau}},$$

$$d\hat{P}_{bt} = \left[-\frac{h^2}{\sigma_v^2}\hat{P}_{Xbt}^2 \right] dt + \frac{h}{\sigma_v^2} s_{Xb^2 t}\, dv_t, \ t \geqslant \hat{\tau}, \ \hat{P}_{b\hat{\tau}} \tag{35}$$

In the next section, we shall describe an adaptive scheme to obtain better estimates of the Kalman filter eqns. (33) and (35).

6. Adaptive algorithm

Different types of adaptive algorithms for Kalman filters have been described by Mehra[6]. We shall illustrate an adaptive algorithm for the problem under discussion. Let the signal process $\{X_t, t \in T\}$ be given by

$$dX_t = fX_t\, dt + g(X_t)\, dW_t \tag{36}$$

and the observation process $\{Y_t, t \in T\}$ by

$$dY_t = hX_t dt + dV_t \tag{37}$$

where f and h are constants and the σ-algebras $\{\mathcal{B}_t, t \in T\}$ and $\{F_t, t \in T\}$ are as defined previously.

The optimal filter equations are

$$d\hat{X}_t = f\hat{X}_t\, dt + \frac{K_t}{\sigma_v^2} dv_t \ t \in T, \ \hat{X}_0 \tag{38}$$

$$d\hat{P}_t = \left[2f\hat{P}_t - \frac{h^2}{\sigma_v^2}\hat{P}_t^2 + \sigma_w^2 E^{F_t} g^2(X_t) \right] dt + \frac{h}{\sigma_v^2} s_t dv_t \ t \in T, \ \hat{P}_0 \tag{39}$$

where

$$K_t = P_t h, \; s_t = E^{F_t} \tilde{X}_t^3,$$

σ_v^2 and σ_w^2 are the variance parameters associated with the Brownian motion processes $\{V_t\}$ and $\{W_t\}$ respectively and ν_t is the innovations process given by

$$d\nu_t = dY_t - h\hat{X}_t dt \tag{40}$$

We can again write an optimal filter equation for s_t by the now familiar method of writing the stochastic differential equation for $d\tilde{X}_t^3$ using the Ito–Doleans-Dade-Meyer rule and then using the nonlinear filtering equation. Thus,

$$s_t = \left\{3fs_t - \frac{3h^2}{\sigma_v^2}\hat{P}_t s_t + 3\sigma_w^2 E^{F_t}[\tilde{X}_t g^2(X_t)]\right\} dt + \frac{h}{\sigma_v^2}\left[E^{F_t}\tilde{X}_t^4 - 3\hat{P}_t^2\right] d\nu_t \tag{41}$$

Several approximation schemes are in existence for eqn. 39 or 41. We can set $E^{F_t} g^2(X_t) = g^2(\hat{X}_t)$ (Extended Kalman) in which case $s_t = 0$. A second approximation (Jazwinski[6]) is to set

$$E^{F_t} g^2(X_t) = g^2(\hat{X}_t) + [g_x^2(\hat{X}_t) + g(\hat{X}_t)g_{xx}(\hat{X}_t)]\hat{P}_t$$

and $s_t = 0$.

A third approximation (Gran and Kozin[7]) is to set $E^{F_t}\tilde{X}_t^4 = 3\hat{P}_t^2$ and expand $g^2(X_t)$ to any suitable order. The essential feature of the above schemes is that once the approximations are made the gain K_t in eqn. 38 is fixed. The resulting filters perform adequately in certain ranges of state space and system parameters. In certain other case they may diverge.

In the adaptive filtering algorithm to be described below the gain K_t is varied by feeding back the present information about the filter. This information is obtained from the innovations process which is a Brownian motion process under optimal conditions, having the same statistics[8] as the observation noise process $\{V_t\}$. With a sub-optimal filter, $\hat{X}_t$ is no longer an optimal estimate and hence the innovations process is no longer a Brownian motion. The non-Brownian motion nature of the innovation process is utilized to vary the gain K_t. It is expected that the sub-optiaml filter may not be too far away from the optimal filter and as a consequence the sub-optimal innovations process ν_t can be modelled by a simple first order autoregressive process

$$d\nu_t' = a_t \nu_t' \, dt + dV_t \; \nu_{t_0}' \tag{42}$$

where a_t is a parameter which is varying slowly with respect to time. If τ is the time interval of estimation then a_t may be considered to be a constant in the time interval τ and can be estimated by matching ν_t' as closely as possible to the observed innovations process v_t in the mean square sense in the time interval [to, to $+\tau$]. If v_t' is the solution to the differential equation (42) given by

$$v_t' = e^{a_t(t-t_o)} v_{t_o}' + \int_{t_o}^{t} e^{a_t(t-\xi)} dV_\xi \tag{43}$$

then $\hat{a}_\tau$, the estimate of a_τ is obtained from

$$\hat{a}_\tau = \arg \left\{ \min_a \frac{1}{\tau} E \int_t^{t+\tau} (v_\xi' - v_\xi)^2 d\xi \right\} \tag{44}$$

If we treat s_t in eqn. (39) as the control then setting it to any given value control K_t in eqn. (38) and thus $\hat{X}_t$ which in turn produces an innovations process v_t (eqn. 40) which again can be modelled by an a_t (eqn. (42)). However, the exact relationship between a_t and s_t is unknown and hence we can model a_t by

$$a_t' = Aa_t' + Ak(s - s_o) \quad a_{to}' \tag{45}$$

where A, k, s_o are parameters to be determined using the mean square error criterion. We have already stated that a_t is slowly varying with respect to time and hence the estimation interval T_o of eqn. (45) is an integral multiple of τ the estimation interval of eqn. (42). Thus the estimates $\hat{A}$, $\hat{k}$, $\hat{s}_o$ are obtained from

$$\hat{A}, \hat{k}, \hat{\hat{s}}_o = \arg \left\{ \min_{A, k, \hat{s}_o} \int_t^{t+T_o} (a_\xi' - \hat{a}_\xi)^2 d\xi \right\} \tag{46}$$

Having obtained $\hat{A}$, $\hat{k}$, $\hat{\hat{s}}_o$, we can solve eqn. (45) for the new control $\hat{s}$ from the condition, that a_t' must be driven to zero in the time interval T_o, *i.e.*, $a_{t+T_o}' = 0$. This condition yields the result

$$\hat{s} = \frac{e^{\hat{A}T_o} a_{t_o}'}{\hat{k}(l - e^{\hat{A}T_o})} + \hat{\hat{s}}_o \tag{47}$$

The estimation intervals τ and T_o must be properly chosen.

The sequence of operations for the algorithm can be given as follows and shown in fig. 1.

1. Choose the observation interval τ and the update interval T_o (T_o is an integral multiple of τ).

2. Initialize the procedure by choosing a 'suitable' value for $\hat{s}$ in eqn. 39 for the initial time t_o.

3. Observe and record the resulting innovations process v_t from the initial time t_o to the time $t_o + T_o = t_1$. Estimate the coefficient $\hat{a}_t$ from eqn. (44) for each observation interval $\tau \in T_o$.

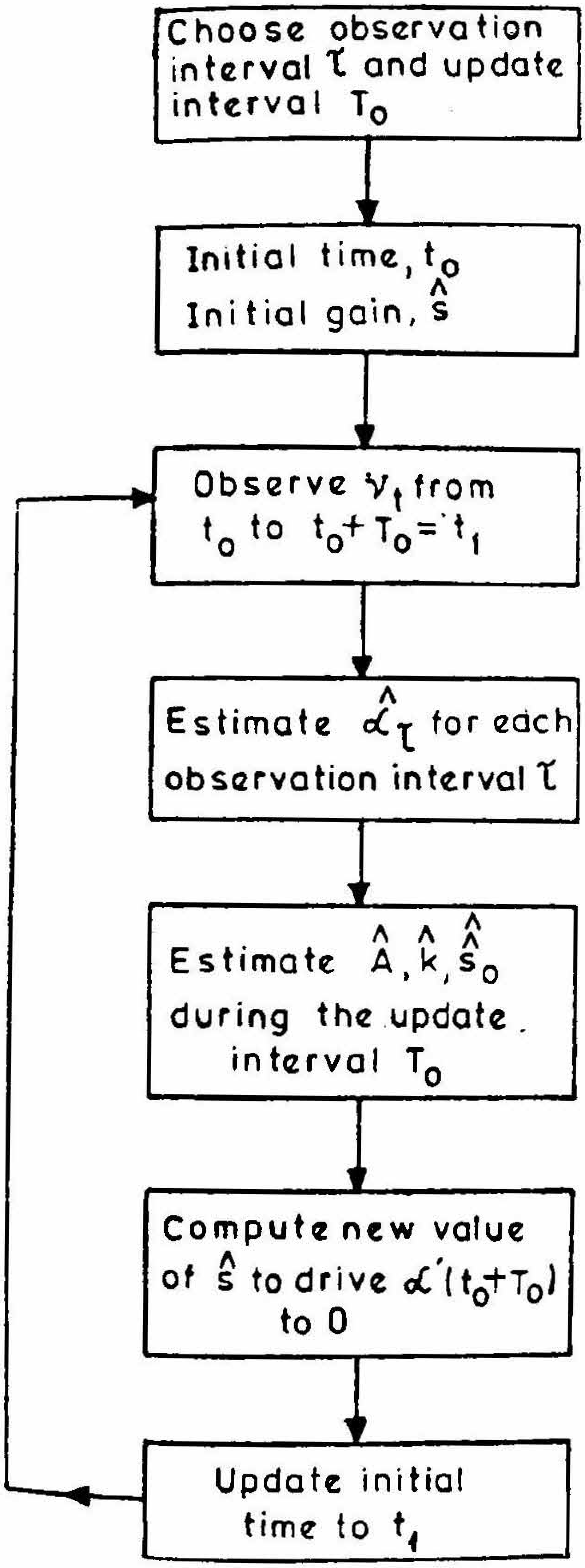

FIG. 1. The sequence of operations for the algorithm suggested as a possible method to improve the Kalman filter estimates for the fault detection problem.

4. Estimate the coefficients A, k, $\hat{s}_e$ in eqn. (45) during the update interval $[t_0, t_e + T_e]$, using the least squares minimization procedure of eqn. 46.
5. Compute the new value of $\hat{s}$ which will drive a_4' to zero in the interval T_e.
6. Consider t_1 to be the initial time and repeat the procedure from 3 onwards.

The above algorithm is suggested as a possible method of improving the Kalman filter estimates for the fault detection problem. It has not been implemented nor its stability analysed.

7. Conclusions

We have here presented for a class of fault detection problems how one finds the estimate for the time of fault and the magnitude of the fault. Even though the Kalman filter for eqn. (12), given the observation process, will eventually track after the occurrence of the fault, the problem will be the presence of unusually large errors immediately after the occurrence of the fault. Therefore, the estimate of the amount of fault is important so that the Kalman filter can be reparametized after the occurrence of the fault as given by eqn. 34. The question of setting the threshold value γ by some other criterion of performance is necessary to find the estimate of fault time. The adaptive algorithm described here gives a closed form solution to the nonlinear filter problem. In any case simulation studies have to be performed to verify the stability of the adaptive filter scheme at least in a practical situation.

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