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On functions of bounded kth variation

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Abstract

Russell introduced the concept of functions of bounded kth variation (BV_k functions) and obtained some properties. Das and Lahiri gave the definition of functions of related absolute continuity (AC_k functions) and proved interrelations between AC_k and BV_k functions. In this paper the authors consider the concept of BV_k and AC_k functions on a bounded set E dense in itself and prove that these functions admit extensions to BV_k and AC_k functions on an interval $[a, b]$ containing the closure of the set E.

Key words: BV_k functions, AC_k functions, k-convex functions, kth Riemann (Riemann*) derivative.

1. Introduction

Russell[1] introduced the concept of functions of bounded kth variation (BV_k functions) on $[a, b]$ and studied in detail some of its fundamental properties. Whenever a function has been defined in the sense of bounded variation (of any type) there is always an attempt to introduce the concept of absolutely continuous functions (in some respects). This was done by Das and Lahiri[2] where they defined absolutely kth continuous functions (AC_k functions) on $[a, b]$ and obtained basic properties of these functions including the interrelations with BV_k functions as in Russell[1]. In the last two decades many papers were published[3-8] where the authors defined various types of functions of bounded variation on a set or relative to a set instead of some continuous interval $[a, b]$. It appears, therefore, reasonable to study BV_k functions as well as AC_k functions defined on a set instead of an interval, which we have attempted in this paper. We further show that a function which is BV_k on a set E can be extended to a function (not necessarily unique) which is BV_k on an interval containing the closure of the set.

Let a, b be fixed real numbers such that $a < b$ and let k be a positive integer greater than 1. By E we shall always mean a subset of $[a, b]$ dense in itself. The greatest lower bound and the least upper bound of E will, respectively, be denoted by α and β. The Lebesgue measure of a set A will be denoted by mA. The ordinary kth order derivative of f at x will be denoted by $f^k(x)$.

Definition 1.1[1] : Let $x_0, x_1, \ldots, x_k$ be $k+1$ distinct points, not necessarily in the linear order, belonging to $[a, b]$. Define the kth divided difference of f as

$$Q_k(f; x_0, x_1, \ldots, x_k) = \sum_{i=0}^{k} [f(x_i) / \prod_{\substack{j=0 \\ j\neq i}}^{k} (x_i - x_j)].$$

Definition 1.2 : A function f is said to be k-convex on E if and only if $Q_k(f; x_0, x_1, \ldots, x_k) \geqslant 0$ for all choices of the points $x_0, x_1 \ldots, x_k$ in E.

Definition 1.3[1]: Let $x, x_1, \ldots, x_k$ be $k+1$ distinct points in $[a, b]$. Suppose that $h_i = x_i - x$, $i = 1, 2, \ldots, k$ and that

$$0 < |h_1| < |h_2| < \ldots < |h_k|.$$

Then define the kth Riemann*-derivative of f at x by $D^k f(x) = k! \lim_{h_k \to 0} \lim_{h_{k-1} \to 0} \ldots \lim_{h_1 \to 0} Q_k(f; x, x_1, \ldots, x_k)$ if the iterated limit exists. The right and the left kth Riemann*-derivatives $D^k_+ f(x)$ and $D^k_- f(x)$ are defined in the obvious way.

When the kth Riemann derivative, in the sense of Bullen[9], exists for $h_0 = 0$ it coincides with the kth Riemann*-derivative. The kth Riemann derivative in Bullen[9] will be denoted by $\mathscr{D}^k f(x)$. The right and the left kth Riemann derivatives will be denoted by $\mathscr{D}^k_+ f(x)$ and $\mathscr{D}^k_- f(x)$ respectively.

If in Definition 1.3 the points $x, x_1, \ldots, x_k$ are in E, we say that $D^k f(x)$ exists at $x \in E$ over the points of E. The existence of $\mathscr{D}^k f(x)$ at x over the points of E is analogously understood. Whenever we say $D^k f(x)$ or $\mathscr{D}^k f(x)$ or $f^k(x)$ exists on E we mean their existence over the points of E.

By a π subdivision of E, we mean a finite set of points $x_0, x_1, \ldots, x_n$ in E, with $x_0 < x_1 < \ldots < x_n$ and we denote it by $\pi(x_0, x_1, \ldots, x_n)$.

Definition 1.4 : The total kth variation of f in E is defined by

$$V_k[f; E] = \sup_{\pi} \sum_{i=0}^{n-k} (x_{i+k} - x_i) \mid Q_k(f; x_i, x_{i+1}, \ldots, x_{i+k}) \mid.$$

If $V_k[f; E] < +\infty$ we say that f is of bounded kth variation (BV_k) on E and write $f \in BV_k[E]$.

Let $x_{1,0} < x_{1,1} < \ldots < x_{1,k} \leqslant x_{2,0} < x_{2,1} < \ldots < x_{2,k} \leqslant \ldots \leqslant x_{n,0} < x_{n,1} < \ldots < x_{n,k}$ be any subdivision of E. We say that the intervals $(x_{i,0}, x_{i,k})$, $i = 1$,

$2, \ldots, n$ form an elementary system I, say, in E. The system is denoted by $I(x_{i,1}, \ldots, x_{i,k-1}) : (x_{i,0}, x_{i,k})$, $i = 1, 2, \ldots, n$. The elementary system consisting of the intervals $(\alpha, x_{1,0})$, $(x_{1,k}, x_{2,0})$, $\ldots$, $(x_{n,k}, \beta)$ is said to be the elementary system complementary to I and will be denoted by I_c. It is to be noted that I and I_c together form an elementary system of $E \cup \{\alpha, \beta\}$.

Definition 1.5 : The function f is said to be absolutely kth continuous on E if for an arbitrary $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for any elementary system $I(x_{i,1}, \ldots, x_{i,k-1}) : (x_{i,0}, x_{i,k})$, $i = 1, 2, \ldots, n$ in E with $mI = \sum_{i=1}^{n} (x_{i,k} - x_{i,0}) < \delta$ the relation

$$\sigma | I | = \sum_{i=1}^{n} (x_{i,k} - x_{i,0}) \; | Q_k(f ; x_{i,0}, x_{i,1}, \ldots, x_{i,k}) | < \epsilon$$

is satisfied. In this case, we say that f is AC_k on E and we write $f \in AC_k[E]$.

Following Das and Lahiri[2], Theorem 1, it is easy to prove the following result :

Theorem 1.1: If $f \in AC_k[E]$, then $f \in BV_k[E]$.

That the converse of the above theorem is not necessarily true is shown by an example in Section 2.

2. BV_k-and AC_k-functions on E

Theorem 2.1: If f is k-convex on E, and $\mathcal{D}_+^{k-1} f(\alpha)$, $\mathcal{D}_-^{k-1} f(\beta)$ both exist, then $f \in BV_k[E]$. (*cf.* Russell[1], Corollary following Theorem 17).

Theorem 2.2 : If the kth Riemann* derivative of a function $f(x) \in AC_k[E]$ is zero almost everywhere in E, then the function $f(x)$ is a polynomial of degree $(k-1)$ atmost (*cf.* Das and Lahiri[2], Theorem 2).

The proofs of the above two theorems are omitted.

Remark 2.1: For $k = 1$, Theorem 2.2 demands a simpler statement.

If the derivative of a function $f(x) \in AC[E]$ is zero almost everywhere in E, then $f(x)$ is constant on E.

Theorem 2.3 : If f is AC_k on E, then f has continuous $(k-1)$th Riemann*-derivatives, $D^{k-1} f(x)$, at each point x of E $(x \neq \alpha, \beta)$.

Proof : Let c be a point of $E (c \neq \alpha, \beta)$ and let $\epsilon > 0$ be arbitrary. Since f is AC_k on E, there exists $\delta_1(c) > 0$ such that the condition of the definition of AC_k on E is satisfied with ϵ replaced by $\epsilon/(k-1)!\ 3k$. We choose points $z_{p-k+1} < z_{p-k+2} < \ldots < z_{p-1} < z_p = c < d = z_{p+1} < \ldots < z_{p+k-1} < z_{p+k}$ of E such that $(z_{p+k} - z_{p-k+1}) < \delta_1$. Choose a positive integer i such that $p - k + 1 \leqslant i \leqslant p$ and consider the elementary system consisting of a single interval

$I(z_{i\,1}, \ldots, z_{i\,k-1}) : (z_i, z_{i\,k})$.

Using Lemma 4 of Russell[1], we get

$$| Q_{k-1}(f ; z_{i\,1}, \ldots, z_{i\,k}) - Q_{k-1}(f ; z_i, \ldots, z_{i\,k-1}) |$$
$$= (z_{i\,k} - z_i) \, | Q_k(f ; z_i, \ldots, z_{i\,k}) | < \epsilon/(k-1)!\,3k.$$

This inequality is true for each $i = p - k + 1, \ldots, p$. Proceeding as in the proof of Lemma 1 of Das and Lahiri[2], we obtain

$$| (k-1)!\, Q_{k-1}(f ; c, z_{p-1}), \ldots, z_{p-k+1}) - (k-1)!\, Q_{k-1}(f ; d, z_{p\,2}, \ldots, z_{p\,k}) | < \epsilon/3. \tag{1}$$

Since $f \in AC_k[E]$, $D^{k-1}f(x)$ exists at each $x \in E\,(x \neq \alpha, \beta)$ (*cf.* Das and Lahiri[2], Lemma 1). There exist $\delta_2(\epsilon) > 0$ and $\delta_3(\epsilon) > 0$ such that

$$| D^{k-1}f(c) - (k-1)!\, Q_{k-1}(f ; c, z_{p-1}, \ldots, z_{p-k+1}) | < \epsilon/3 \,; \tag{2}$$
$$| D^{k-1}f(d) - (k-1)!\, Q_{k-1}(f ; d, z_{p+2}, \ldots, z_{p+k}) | < \epsilon/3$$

where $c - z_{p-k+1} < \delta_2$ and $z_{p+k} - d < \delta_3$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then from (1) and (2) we obtain $| D^{k-1}f(c) - D^{k-1}f(d) | < \epsilon$ whenever $d - c < \delta$. This proves that $D^{k-1}f(x)$ is continuous at c from the right. Similarly, we obtain the left continuity of $D^{k-1}f(x)$ at c. Since c is an arbitrary point of E, the theorem follows.

Theorem 2.4 : If $f \in BV_k[E]$, then $f^k(x)$ exists almost everywhere in E.

Proof : Since $f \in BV_k[E]$, we have $f(x) = p(x) - q(x)$ where $p(x)$ and $q(x)$ are k-convex functions on E (*cf.* Russell[1], Theorem 15). It follows that $p^k(x)$ and $q^k(x)$ exist almost everywhere in E [*cf.* Bullen[9], Corollary 15 (*b*)]. Let $p^k(x)$ and $q^k(x)$ exist on the set E_1 and E_2 respectively. Then we see that $f^k(x)$ exists on the set $A = E_1 \cap E_2$ and that $m(E \setminus A) = 0$. This proves the theorem.

In view of Theorem 1.1, we obtain Corollary 2.1. If $f \in AC_k[E]$, then $f^k(x)$ exists almost everywhere in E.

Since the existence of ordinary derivative $f^k(x)$ implies the existence of $I^k f(x)$ we have

Corollary 2.2 : If $f \in AC_k[E]$, then $D^k f(x)$ exists almost everywhere in E.

Theorem 2.5 : If $f \in BV_{k+1}[E]$, then $f \in AC_k[E]$.

Proof : Let $f \in BV_{k+1}[E]$. Then, it follows that $Q_k(f ; x_0, x_1, \ldots x_k)$ is bounded where $x_i \in E$, $i = 0, 1, 2, \ldots, k$ (*cf.* Russell[1], Theorem 4). Hence there exists a constant M such that

$$| Q_k(f ; x_0, x_1, \ldots, x_k) | \leqslant M \tag{3}$$

where $x_i \in E$, $i = 0, 1, \ldots, k$. Let $\epsilon > 0$ be arbitrary. Then for any elementary system

$$I(x_{i,1}, \ldots, x_{i,k-1}) : (x_{i,0}, x_{i,k}), \quad i = 1, 2, \ldots, n$$

in E, we see, using (3), that

$$\sum_{i=1}^{n} (x_{i,k} - x_{i,0}) \quad | Q_k(f ; x_{i,0}, x_{i,1}, \ldots, x_{i,k}) | < \epsilon$$

whenever

$$\sum_{i=1}^{n} (x_{i,k} - x_{i,0}) < \epsilon/M.$$

This proves the theorem. Utilising Theorem 1.1 and 2.5 we obtain.

Corollary 2.3 (*cf.* Russell[1], Theorem 10) : If $f \in AC_{k+1}[E]$, then $f \in AC_k[E]$.

Remark 2.2 : Corollary 2.3 shows the decreasing nature of the sequence of sets $\{AC_k[E]\}$.

Theorem 2.6 : If f is AC_{k+1} on E, then f' is AC_k on E.

Proof: Let $\epsilon > 0$ be arbitrary. There exists a $\delta(\epsilon) > 0$ such that the condition of the definition of AC_{k+1} functions on E is satisfied with ϵ replaced by $\epsilon/2k(5 + 2k)$ We choose points

$$x_{1,0} < x_{1,1} < \ldots < x_{1,2k} \leqslant x_{2,0} < x_{2,1} < \ldots < x_{2,2k} \leqslant \ldots \leqslant x_{n,0}$$
$$< x_{n,1} < \ldots < x_{n,2k} \text{ of } E, \text{ such that}$$

$$\sum_{i=1}^{n} (x_{i,2k} - x_{1,0}) < \delta.$$

Consider an elementary system

$$I(x_{i,1}, \ldots, x_{i,k-1}) : (x_{i,0}, x_{i,k}) \; ; \; i = 1, 2, \ldots, n.$$

In view of Theorem 8 of Russell[1]

$$\sum_{i=1}^{n} | Q_{k-1}(f' ; x_{i,0}, \ldots, x_{i,k-1}) - Q_{k-1}(f' ; x_{i,1}, \ldots x_{i,k}) |$$

$$= \sum_{i=1}^{n} \Big| \sum_{t=0}^{k-1} [Q_k(f ; x_{i,0}, \ldots x_{i,t}, x_{i,t}, \ldots, x_{i,k-1})$$
$$- Q_k(f ; x_{i,1}, \ldots, x_{i,t+1}, x_{i,t+1}, \ldots, x_{i,k})] \Big|$$

$$\leqslant \sum_{i=1}^{n} \sum_{t=0}^{k-1} | Q_k(f ; x_{i,0}, \ldots, x_{i,t}, x_{i,t}, \ldots, x_{i,k-1})$$
$$- Q_k(f ; x_{i,1}, \ldots, x_{i,t+1}, x_{i,t+1}, \ldots, x_{i,k}) |$$

$$\leqslant \sum_{i=1}^{n} \sum_{t=0}^{k-1} \{ | Q_k(f ; x_{i,0}, \ldots, x_{i,t}, x_{i,t}, \ldots, x_{i,k-1})$$
$$- Q_k(f ; x_{i,0}, \ldots, x_{i,t}, \xi_{i,t}, \ldots, x_{i,k-1}) |$$
$$+ | Q_k(f ; x_{i,1}, \ldots, x_{i,t+1}, x_{i,t+1}, \ldots, x_{i,k})$$

$$- Q_k(f; x_{i,1}, \ldots, x_{i,t+1}, \xi_{i,t+1}, \ldots, x_{i,k})|$$
$$+ |Q_k(f; x_{i,0}, \ldots, x_{i,t}, \xi_{i,t}, \ldots, x_{i,k-1})$$
$$- Q_k(f; x_{i,1}, \ldots, x_{i,t+1}, \xi_{i,t+1}, \ldots, x_{i,k})|\}$$

where $x_{i,s} < \xi_{i,s} < x_{i,s+1}$ and $\xi_{i,s} \in E$ for each $s = 0, 1, \ldots, k$. The existence of $f'(x)$ on E is ensured by Theorem 2.3 and the fact that $D^k f = f^k$ for $k=1$ if either exists.

Furthermore $\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,k}$; $i = 1, 2, \ldots, n$ can be chosen such that

$$|Q_k(f; x_{i,0}, \ldots, x_{i,t}, x_{i,t}, \ldots, x_{i,k-1})$$
$$- Q_k(f; x_{i,0}, \ldots, x_{i,t}, \xi_{i,t}, \ldots, x_{i,k-1})| < \epsilon/k.2^{i+2}$$

and

$$Q_k(f; x_{i,1}, \ldots, x_{i,t+1}, x_{i,t+1}, \ldots, x_{i,k})$$
$$- Q_k(f; x_{i,1}, \ldots, x_{i,t+1}, \xi_{i,t+1}, \ldots, x_{i,k})| < \epsilon/k.2^{i+2}$$

when $i = 1, 2, \ldots, n$ and $t = 0, 1, \ldots, k-1$.

Therefore,

$$\sum_{i=1}^{n} |Q_{k-1}(f'; x_{i,0}, \ldots, x_{i,k-1}) - Q_{k-1}(f'; x_{i,1}, \ldots, x_{i,k})|$$
$$< \epsilon/2 + \sum_{t=0}^{k-1}\sum_{i=1}^{n} |Q_k(f; x_{i,0}, \ldots, x_{i,t}, \xi_{i,t}, \ldots, x_{i,k-1})$$
$$- Q_k(f; x_{i,1}, \ldots, x_{i,t+1}, \xi_{i,t+1}, \ldots, x_{i,k})|. \quad (4)$$

We now consider

$$\sum_{i=1}^{n} |Q_k(f; x_{i,0}, \ldots, x_{i,t}, \xi_{i,t}, \ldots, x_{i,k-1})$$
$$- Q_k(f; x_{i,1}, \ldots, x_{i,t+1}, \xi_{i,t+1}, \ldots, x_{i,k})|$$

for a fixed t. For the sake of simplicity we present the case for $t = 0$. Thus taking $t = 0$ **we** have

$$\sum_{i=1}^{n} |Q_k(f; x_{i,0}, \xi_{i,0}, \ldots, x_{i,k-1}) - Q_k(f; x_{i,1}, \xi_{i,1}, \ldots, x_{i,k})|$$
$$\leq \sum_{i=1}^{n} \{|Q_k(f; x_{i,0}, \xi_{i,0}, \ldots, x_{i,k-1}) - Q_k(f; \xi_{i,0}, x_{i,1}, \ldots, x_{i,k})|$$
$$+ |Q_k(f; \xi_{i,0}, x_{i,1}, \ldots, x_{i,k}) - Q_k(f; x_{i,1}, x_{i,2}, \ldots, x_{i,k+1})|$$
$$+ |Q_k(f; x_{i,1}, \xi_{i,1}, \ldots, x_{i,k}) - Q_k(f; \xi_{i,1}, x_{i,2}, \ldots, x_{i,k+1})|$$
$$+ |Q_k(f; \xi_{i,1}, x_{i,2}, \ldots, x_{i,k+1}) - Q_k(f; x_{i,2}, x_{i,3}, \ldots, x_{i,k+2})|$$
$$+ |Q_k(f; x_{i,2}, x_{i,2}, \ldots, x_{i,k+2}) - Q_k(f; x_{i,1}, x_{i,2}, \ldots, x_{i,k+1})|\}.$$

Consider now elementary systems

$$I_1(\xi_{i,0}, \ldots, x_{i,k-1}) : (x_{i,0}, x_{i,k}) ;$$

$$I_2(x_{i,1}, \ldots, x_{i,k}) : (\xi_{i,0}, x_{i,k-1}) ;$$

$$I_3(\xi_{i,1}, \ldots, x_{i,k}) : (x_{i,1}, x_{i,k+1}) ;$$

$$I_4(x_{i,2}, \ldots, x_{i,k+1}) : (\xi_{i,1}, x_{i,k-2}) ;$$

$$I_5(x_{i,2}, \ldots, x_{i,k+1}) : (x_{i,1}, x_{i,k-2}) ;$$

$i = 1, 2, \ldots, n$. We then have

$$\sigma \,|\, I_j \,| < \epsilon/2k\,(5 + 2k)$$

for each $j = 1, 2, \ldots, 5$.

Hence

$$\sum_{i=1}^{n} | Q_k(f ; x_{i,0}, \xi_{i,0}, \ldots, x_{i,k-1}) - Q_k(f ; x_{i,1}, \xi_{i,1}, \ldots, x_{i,k}) | < \epsilon/2k.$$

Now let t vary between $0 \leqslant t \leqslant k - 1$ and consider $5 + 2t$ elementary systems so that the sum

$$\sum_{i=1}^{n} | Q_k(f ; x_{i,0}, \ldots, x_{i,t}, \xi_{i,t}, \ldots, x_{i,k-1})$$

$$- Q_k(f ; x_{i,1}, \ldots, x_{i,t+1}, \xi_{i,t+1}, \ldots, x_{i,k}) | < \epsilon/2k$$

for each t, $0 \leqslant t \leqslant k - 1$. Hence the double sum on the right of (4) is less than $\epsilon/2$. Thus from (4)

$$\sum_{i=1}^{n} | Q_{k-1}(f' ; x_{i,0}, \ldots, x_{i,k-1}) - Q_{k-1}(f' ; x_{i,1}, \ldots, x_{i,k}) | < \epsilon$$

whenever $\sum_{i=1}^{n} (x_{i,k} - x_{i,0}) < \delta$ and the theorem is proved.

Corollary 2.4 : If $f \in AC_k\,[E]$, then $D^{k-1} f \in AC\,[E]$.

We now present examples of BV_k and AC_k functions (on a relevant set).

Example 2.1 : We consider the function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ on a dense set E. If $n < k$, then by Lemma 1 of Russell[1] it fiollows immediately that $f(x) \in AC_k\,[E]$. We, therefore, assume $n \geqslant k$. To show that $f(x)$ is AC_k on E it is sufficient to show that x^n is AC_k on E. Let $x_0, x_1, \ldots, x_k$ be a set of $(k + 1)$ points on E. Then by § 1.31 (p.7) of Milne-Thomson[10] it follows that

$$Q_k(x^n ; x_0, x_1, \ldots, x_k) = \sum x_0^{a_0} x_1^{a_1} \ldots x_k^{a_k}$$

where the summation is extended to all positive integers including zero which satisfy the relation $a_0 + a_1 + \ldots + a_k = n - k$. Since the above sum contains finite number of terms it follows that there exists a positive number M such that

$$Q_k(x^n; x_0, x_1, \ldots, x_k) < M\beta^{a_0+a_1+\cdots+a_k} = M\beta^{n-k}, \ \beta \text{ being the l.u.b. of } E.$$

Let $\epsilon > 0$ be arbitrary and consider an elementary system $I(x_{i,1}, \ldots, x_{i,k-1}) : (x_{i,0}, x_{i,k})$, $i = 1, 2, \ldots, p$ in E. Then we must have

$$\sum_{i=1}^{p} (x_{i,k} - x_{i,0}) \mid Q_k(x^n; x_{i,0}, x_{i,1}, \ldots, x_{i,k}) \mid < M \mid \beta^{n-k} \mid \sum_{i=1}^{p} (x_{i,k} - x_{i,0}) < \epsilon$$

whenever $\sum_{i=1}^{p} (x_{i,k} - x_{i,0}) < \epsilon/M \mid \beta^{n-k} \mid$. This proves that x^n is AC_k on E and hence $f(x)$ is AC_k on E.

We next present an example which is a $BV_k[E]$ function but not an $AC_k[E]$ function To construct the example we need the following two results from Russell[1] which we state for ready reference.

Theorem A (*cf.* Russell[1], Theorem 13) : If $k \geqslant 0$ and f is k-convex on $[a, b]$, then

$f(x) = \int_c^x f(t)\, dt$, where $a < c < b$, is $(k+1)$-convex on $[a, b]$.

Theorem B (*cf.* Russell[1], Corollary to Theorem 17) : If $k \geqslant 1$, f is k-convex on $[a, b]$, and $D_+^{k-1} f(a)$ and $D_-^{k-1} f(b)$ both exist, then $f \in BV_k[a,b]$.

Example 2.2 : To simplify the situation we assume that $k = 2$. Let P_0 be the Cantor perfect set and G_0 be the Cantor open set. Let $f(x)$ be the function $\theta(x)$ as described in Natanson[11], (Example, p. 213). Then $f(x)$ is defined everywhere on $[0, 1]$ and is non-decreasing and hence BV on $[0, 1]$.

By Theorems A and B, it follows that the function $F(x)$ defined by

$$F(x) = \int_0^x f(t)\, dt, \quad 0 \leqslant x \leqslant 1,$$

is BV_2 on $[0, 1]$. We now observe that $F'(x) = f(x)$ on a certain set, say $E \subset [0, 1]$ with $mE = 1$. Clearly then $F(x)$ is BV_2 on E. If possible, let $F(x)$ be AC_2 on E. Then by Corollary 2.4, $F'(x) = f(x)$ is AC on E. Since $f'(x) = 0$ on $E \cap G_0$, by Remark 2.1, $f(x)$ is constant on E which however is not the case. This contradiction proves that $F(x)$ is not AC_2 on E.

3. Problems of extensions

In this section we prove three theorems of which the first two generalise two results in Lemma 4.1 of Saks[12] (p. 221).

Theorem 3.1 : If f is k-convex on E and $\mathscr{D}_+^{k-1} f(\alpha)$, $\mathscr{D}_-^{k-1} f(\beta)$ exists, then f admits an extension to F on $[a, b]$ where $F(x)$ is a function k-convex on $[a, b]$.

Proof: It is easy to show that $D^{k-1} f(x)$ exists on a set $E' \subset E$ such that $mE' = mE$ and $D^{k-1} f(x)$ is non-decreasing on E'. Therefore by Lemma 4.1 of Saks[12] (p. 221), there exists a function $G(x)$ which is non-decreasing on the whole real line R_1 and coincides with $D^{k-1} f(x)$ on E'. By repeated application of Theorem 13 of Russell[1] the function

$$g(x) = \int_c^x \int_c^{x_{k-2}} \cdots \int_c^{x_2} \int_c^{x_1} G(t)\, dt \ldots dx_{k-3}\, dx_{k-2},$$

$a < c < b$, is k-convex on $[a, b]$. We see that $D^{k-1} g(x) = G(x) = D^{k-1} f(x)$ on E'' where $E'' \subset E'$ and $mE'' = mE' = mE$. This implies

$$D^{k-1} [f(x) - g(x)] = 0 \qquad (5)$$

on E''.

In fact $g(x)$ is k-convex on any closed interval containing $[a, b]$ and so by Theorem 7 of Bullen[9] $D_+^{k-1} g(a)$ and $D_-^{k-1} g(b)$ exist. This by Corollary to Theorem 17 of Russell[1] $g(x)$ is BV_k on $[a, b]$ and hence, by Theorem 2.5, $g(x)$ is AC_{k-1} on $[a, b]$. Also by Theorem 2.1 and 2.5, $f(x)$ is AC_{k-1} on E. Therefore from (5) and Theorem 2.2, it follows that for $x \in E$, $f(x) - g(x) =$ a polynomial of degree $(k-2)$ atmost $= p(x)$ say.

Clearly $p(x)$ is k-convex on $[a, b]$. Now we see that the function $F(x)$ defined by $F(x) = g(x) + p(x)$ is k-convex on $[a, b]$ and $f(x) = F(x)$ on E. This proves the theorem.

Theorem 3.2: If f is BV_k on E, then f admits an extension to F on $[a, b]$ where $F(x)$ is BV_k on $[a, b]$.

Proof: Let f be BV_k on E. It is easy to show that $D^{k-1} f(x)$ exists on a set $E' \subset E$ such that $mE' = mE$ and $D^{k-1} f(x)$ is BV on E'. For each x, let $E'_{(x)} = (-\infty, x] \cap E'$. For each x we define

$$V(x) = V(D^{k-1} f; E'_{(x)}) \quad \text{for } E'_{(x)} \neq \phi$$
$$= 0 \qquad\qquad \text{for } E'_{(x)} = \phi.$$

Clearly $V(x)$ is non-decreasing on the whole real line R_1. For $x_1 > x_2$ and $x_1, x_2 \in E'$ we have

$$\begin{aligned}
&[V(x_1) - D^{k-1} f(x_1)] - [V(x_2) - D^{k-1} f(x_2)] \\
&= [V(x_1) - V(x_2)] - [D^{k-1} f(x_1) - D^{k-1} f(x_2)] \\
&= [V(D^{k-1} f; E'_{(x_1)}) - V(D^{k-1} f; E'_{(x_2)})] - [D^{k-1} f(x_1) - D^{k-1} f(x_2)] \\
&= V(D^{k-1} f; E' \cap [x_1, x_2]) - [D^{k-1} f(x_1) - D^{k-1} f(x_2)]
\end{aligned}$$

$\geqslant 0$.

This shows that $V(x) - D^{k-1} f(x)$ is non-decreasing on E'. Hence, by Lemma 4.1 of Saks[12] (p. 221) there exists a function $G(x)$ which is non-decreasing on R_1 and coincides with $V(x) - D^{k-1} f(x)$ on E'. By repeated application of Theorem 13 of Russell[1] we see that the functions

$$g(x) = \int_c^x \int_c^{x_{k-2}} \dots \int_c^{x_2} \int_c^{x_1} G(t)\, dt \dots dx_{k-3}\, dx_{k-2}, \quad a < c < b, \text{ and}$$

$$v(x) = \int_c^x \int_c^{x_{k-2}} \dots \int_c^{x_2} \int_c^{x_1} V(t)\, dt \dots dx_{k-3}\, dx_{k-2}, \quad a < c < b,$$

are k-convex on $[a, b]$. Also we see that

$$D^{k-1}\, g(x) = G(x) = V(x) - D^{k-1} f(x) = D^{k-1}\, v(x) - D^{k-1} f(x)$$

on E'', where $E'' \subset E'$ and $mE'' = mE' = mE$.

Hence

$$D^{k-1}\, [g(x) + f(x) - v(x)] = 0 \tag{6}$$

on E''. Now $g(x)$ and $v(x)$ are k-convex on $[a, b]$ and by the same argument as in the proof of the previous theorem, $D_+^{k-1}\, g(a)$, $D_-^{k-1}\, g(b)$, $D_+^{k-1}\, v(a)$, $D_-^{k-1}\, v(b)$ exist. So by Theorem 19 of Russell[1] the function $s(x) = v(x) - g(x)$ is BV_k on $[a, b]$. Also $f(x)$ is BV_k on E. Thus by Theorem 2.5, $s(x)$ and $f(x)$ are AC_{k-1} on E. Hence by Theorem 2.2 and the relation in (6) it follows that for $x \in E$ $f(x) - s(x) = a$ polynomial of degree $(k-2)$ atmost $= q(x)$, say.

A polynomial of degree $(k-2)$ atmost is always a BV_k function and hence the theorem is proved with $F(x) = s(x) + q(x)$ on $[a, b]$.

Theorem 3.3 : If f is AC_k on E, then f admits an extension to F on $[a, b]$ where $F(x)$ is AC_k on $[a, b]$.

Proof : By Theorem 1.1, it follows that $f \in BV_k\,[E]$. Also one may get analogues of Theorems 13 and 19 of Russell[1] by replacing the interval $[a, b]$ by the set E. The proof now follows taking into consideration Theorems 2.4 and 2.5.

We now demonstrate below an example of a function defined on a dense subset showing the behaviour of the function after extension. Before going to the example we further remark that it comes out easily that an extension of a polynomial function is the polynomial function.

Example 3.1 : For simplicity of presentation we assume that $k = 2$. Let E be the rational subset of $[-\frac{1}{2}, \frac{1}{2}]$. Consider the function $f(x) = 1 - \sqrt{(1 - x^2)}$ on E. Clearly $f(x)$ is BV_2 on E. We now obtain an extension of $f(x)$ on $[a, b]$,

$$-\infty < a \leqslant -\tfrac{1}{2} < \tfrac{1}{2} \leqslant b < \infty.$$

We see that $f'(x) = x/\sqrt{(1 - x^2)}$ on E. Clearly $f'(x)$ is increasing on E and is bounded on E. We take $E_{(x)} = [-\infty, x] \cap E$ and for each x let

$$G(x) = \text{l.u.b. of } f'(x) \text{ on } E_{(x)}, \text{ if } E_{(x)} \neq \phi,$$
$$= \text{g.l.b. of } f'(x) \text{ on } E, \text{ if } E_{(x)} = \phi.$$

Then we have

$$G(x) = -\frac{1}{\sqrt{3}}, \quad -\infty < x < -\tfrac{1}{2},$$

$$= x/\sqrt{(1 - x^2)}, \quad -\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2},$$
$$= \frac{1}{\sqrt{3}}, \quad \tfrac{1}{2} < x < \infty.$$

Clearly $G(x)$ is increasing on $(-\infty, \infty)$ and hence, by Theorem 13 of Russell[1], it follows that the function $F(x)$ defined by

$$F(x) = \int_{a}^{x} G(t)\, dt, \quad a \leqslant x \leqslant b$$

is 2-convex on $[a, b]$. This implies

$$F(x) = -\frac{1}{\sqrt{3}} x, \quad a \leqslant x < -\tfrac{1}{2},$$
$$= 1 - \sqrt{(1 - x^2)}, \quad -\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2},$$
$$= \frac{1}{\sqrt{3}} x, \quad \tfrac{1}{2} < x \leqslant b$$

is 2-convex on $[a, b]$. Further $F'(x)$ exists at a and b and so, by Theorem 19 of Russell[1] $F(x)$ is BV_2 on $[a, b]$. Therefore $F(x)$ is an extension of $f(x)$ on $[a, b]$.

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