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Short Communication

Two-dimensional wave motion of a viscous fluid of infinite depth due to an applied shearing stress on the surface

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Abstract

The surface elevation of the wave produced by the applied shearing stress of the general type admitting of the transient and spatially periodic has been obtained in a closed form by a method involving integral representation. The integral has been numerically evaluated in particular case by Filon's method.

Key words : Wave motion, surface elevation, Filon's method, applied shearing stress, propagation, viscous incompressible flow.

1. Introduction

The classical problem of the effect of viscosity on infinitesimal waves in deep sea was solved by Basset[1] and Lamb[2]. Basset assumed that both the normal and shear stresses on the surface were zero and the wave motion was propagated by a train of disturbances attributed to the velocities. Lamb[2] considered the effect of surface stresses in two-dimensional wave motion of a viscous liquid.

Wave motion of liquid in a rectangular duct due to variable pressure has been investigated by Das[3]. It is well known that in a non-rotating system deep water waves are

dispersive whereas the shallow water waves are non-dispersive. Debnath and Rosenblat[4] have made initial value investigation into the development of surface waves on a homogeneous ocean of finite and infinite depth which is either at rest or in uniform motion. They have investigated the dispersive wave phenomena and discussed the principal features of the steady and the transient wave motions. Long waves on a rotating sea due to atmospheric disturbances has been investigated by Crease[5]. Bagchi and Debnath[6] considered travelling wind stress distributions on the surface of the ocean. Debnath[7] considered the wind driven currents in a non-uniformly rotating shallow ocean with dissipating effect due to the bottom friction.

The present paper is concerned with the study of surface wave in a viscous incompressible fluid of infinite depth. The tangential stress over an area of the surface is assumed to be of general type admitting of the transient and spatially periodic. This problem is considered in connection with the flow system in ocean when air blows over its free surface.

2. Formulation of the problem

We take the origin of coordinates on the free surface of the viscous fluid and y-axis vertically upwards. The x and z axes are taken on the surface. The governing equations of motion are (*cf.* Lamb[2])

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla_1^2 u, \tag{1}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g + \nu\nabla_1^2 v, \tag{2}$$

where u and v are velocities in x and y directions respectively, ν is the kinematic coefficient of viscosity, p is the pressure and ∇_1^2 denotes the operator

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3}$$

introducing the non-dimensional quantities :

$$u' = \frac{uL}{\nu},\ v' = \frac{vL}{\nu},\ x' = \frac{x}{L},\ y' = \frac{y}{L},\ t' = \frac{\nu t}{L^2},\ p' = \frac{pL^2}{\rho\nu^2},\ g' = \frac{gL^3}{\nu^2}$$

in equations (1), (2) and (3), we get (dropping dash)

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \nabla^2 u, \tag{4}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} - g + \nabla^2 v, \tag{5}$$

and $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{6}$$

where ∇^2 denote the non-dimensional form of ∇_1^2.

We assume

$$u = -\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}, \tag{7}$$

$$v = -\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x}, \tag{8}$$

Pressure p is determined from the Bernouli's equation .

$$p = \frac{\partial \phi}{\partial t} - gy. \tag{9}$$

Substituting equations (7), (8) and (9) in equations (4), (5) and (6) we get the following expressions for ϕ and ψ as

$$\nabla^2 \phi = 0 \tag{10}$$

$$\frac{\partial \psi}{\partial t} = \nabla^2 \psi \tag{11}$$

The boundary conditions are

(i) $\phi, \psi \to 0$ as $y \to -\infty$ (12)

(ii) neglecting the surface tension, the stress conditions on the surface $y = 0$ are

(*a*) normal stress $p_{yy} = 0$ (13)

(*b*) tangential stress over an area is given by

$$\left.\begin{aligned} p_{xy} &= f(x, t), \quad |x| \leqslant 1 \\ &= 0\ , \quad |x| > 1 \end{aligned}\right\} \tag{14}$$

where $f(x, t)$ is any function of time t and even function of the space co-ordinate x.

Equations (10) and (11) are to be solved in accordance with (12), (13) and (14).

3. Method of solution

As a general solution of Laplace's equations (10) and (11), we consider

$$\phi = e^{-nt} \int_0^\infty [A\, e^{My} + B\, e^{-My}] \sin Mx\, dM, \tag{15}$$

$$= e^{-nt} \int_0^\infty [C\, e^{Ny} + D\, e^{-Ny}] \cos Mx\, dM, \tag{16}$$

where A, B, C and D are functions independent of y and N is to be determined. Substituting (16) in (11), we get

$$N^2 = M^2 - n, \tag{17}$$

where N is the positive root of equation (17).

The boundary condition (12) will be satisfied if

$$B = D = 0.$$

Hence equations (15) and (16) become

$$\phi = e^{-nt} \int_0^\infty A\, e^{My} \sin Mx\, dM. \tag{18}$$

$$\psi = e^{-nt} \int_0^\infty C\, e^{Ny} \cos Mx\, dM. \tag{19}$$

Substituting ϕ and ψ in equations (7) and (8) we get

$$u = -e^{-nt} \int_0^\infty [MA\, e^{My} - CN\, e^{Ny}] \cos Mx\, dM. \tag{20}$$

$$v = -e^{-nt} \int_0^\infty [MA\, e^{My} - CM\, e^{Ny}] \sin Mx\, dM. \tag{21}$$

Denoting the free surface elevation by η, we have the kinematical relation as

$$\frac{\partial \eta}{\partial t} = v \quad \text{on} \quad y = 0. \tag{22}$$

Integrating (21), we get

$$\eta = \frac{e^{-nt}}{n} \int_0^\infty M(A - C) \sin Mx\, dM. \tag{23}$$

The values of A and C will be determined from the boundary conditions (13) and (14) and equation (9).

Expressions for surface stresses are

$$\begin{aligned} p_{yy} &= -p + 2\frac{\partial v}{\partial y} \\ &= -\frac{\partial \phi}{\partial t} + g\eta - 2\frac{\partial v}{\partial y} \end{aligned} \tag{24}$$

$$p_{yy}\Big|_{y=0} = \frac{e^{-nt}}{n} \int_0^\infty [(n^2 - gM - 2M^2 n) A - (gM - 2MNn) C] \sin Mx\, dM.$$

Using boundary condition (13), we have

$$C = \frac{n^2 + gM - 2M^2 n}{2nMN - gM} A. \tag{25}$$

Again, $p_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$

and $$p_{xy}\Big|_{y=o} = -e^{-nt} \int_0^\infty [2M^2 A - (2M^2 - n)C] \cos Mx\, dM \tag{26}$$

Substitution of (25) into (26), results as

$$p_{xy}\Big|_{y=0} = -ne^{-nt}\int_0^\infty A\left[\frac{4M^3N-(2M^2-n)^2-gM}{(2nN-g)\,M}\right]\cos Mx\,dM. \quad (27)$$

Determination of A :

The applied shearing stress can be written in the form

$$f(x,t) = \frac{2}{\pi}\int_0^\infty \cos Mx \int_0^1 f(\beta,t)\cos M\beta\,d\beta\,dM \quad (28)$$

Using the condition in (14), we can find out the value of A from equations (27) and (28) as

$$A = \frac{2e^{nt}}{n\pi}\,\frac{(2n\,N-g)\,M}{-4M^3N+(2\,M^2-n)^2+gM}\int_0^1 f(\beta,t)\cos M\beta\,d\beta. \quad (29)$$

Knowing A, the expression for surface elevation can be found out from equation (23).

Special case

We consider

$$f(x,t) = S\,e^{-nt}\cos kx. \quad (30)$$

So
$$A = \frac{2S}{n\pi}\,\frac{1}{k^2-M^2}\,\frac{(2n\,N-g)\,M}{gM+(2M^2-n)^2-4M^3N}$$
$$\times\,[k\sin k\cos M - M\cos k\sin M] \quad (31)$$

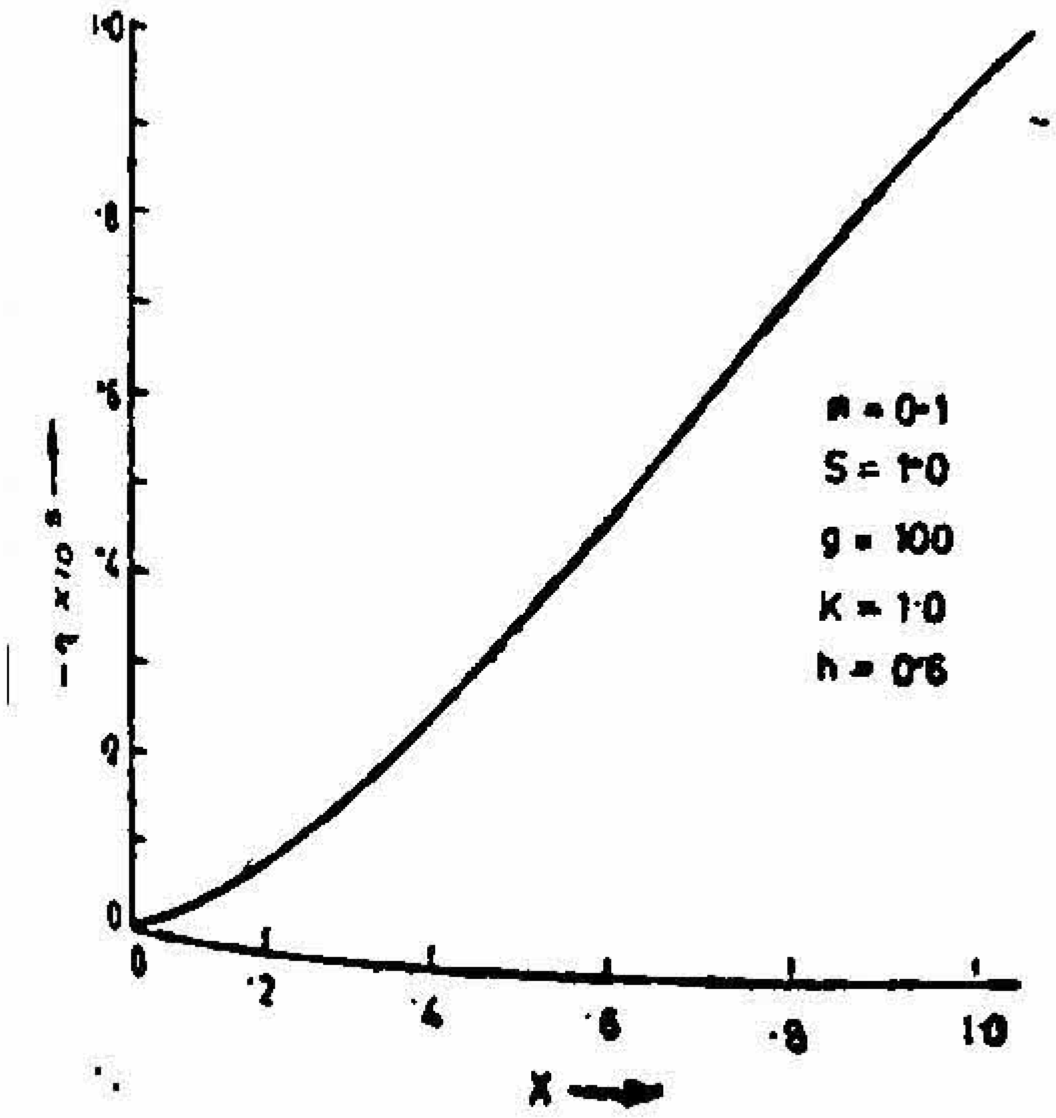

FIG 1. Variation of the surface elevation with x at $t = 3$.

Substituting the value of A in equation (23), we get the surface elevation η as

$$\eta = \frac{2S\,e^{-nt}}{n\pi} \int_0^\infty \frac{M}{K^2 - M^2} \frac{n - 2M^2 + 2MN}{gM + (2M^2 - n)^2 - 4M^3N}$$

$$\times [k \sin k \cos M - M \cos k \sin M] \sin Mx \, dM. \tag{32}$$

It is clear from equation (32) that the surface elevation decays with time. Considering the case when $t = 3$, the above integral is computed by Filon's method and the different values of the surface elevation η are plotted against x in fig. 1.

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