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# PROPAGATION OF MICROWAVES THROUGH A CYLINDRICAL METALLIC GUIDE FILLED COAXIALLY WITH TWO DIFFERENT DIELECTRICS

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## ABSTRACT

The field components for the $TE_{01}$ mode in a cylindrical metallic guide completely filled with two different dielectrics have been derived from Maxwell's equations. The attenuation constant for this mode has been calculated with the help of the Poynting vector and the field components.

## INTRODUCTION

The theory of microwave propagation through a metallic guide involves the solution of Maxwell's equations for certain boundary conditions at the boundary wall of the guide. The simplest case is that of a guide of uniform cross-section with infinitely conducting walls and filled inside with a homogeneous, isotropic dielectric.

The theory of wave propagation in a dielectric guide has been studied by Hondros and Debye (1910), Zahn (1916), Carson, Mead, and Schelkunoff (1936). The experimental study in dielectric guides has been made by Schriever (1920) and Kašpar (1938). The propagation of electromagnetic waves in a metallic guide with a dielectric inside has been studied by Frank (1942), Pincherle (1944) and Frankel (1948).

The paper reports on the study of the theory of the propagation of microwaves ($TE_{01}$ mode) in a cylindrical metallic guide of uniform cross-section, and completely filled coaxially with two homogeneous, isotropic layers of dielectrics of dielectric constant $\epsilon_1$ and $\epsilon_2$.

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## MAXWELL'S EQUATIONS

The equations of Maxwell in m.k.s. units are expressed as follows:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{D} &= \rho \end{aligned} \tag{1}$$

The general solutions of these field equations has been obtained by Bromwich (1919) and Ledinegg (1942). The scalar components of E and H expressed in general orthogonal co-ordinates $u^1$, $u^2$, $u^3$ are given (Stratton, 1941) as

$$\begin{aligned} &\frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u^2}(h_3 E_3) - \frac{\partial}{\partial u^3}(h_2 E_2)\right] + \frac{\partial B_1}{\partial t} = 0 \\ &\frac{1}{h_3 h_1}\left[\frac{\partial}{\partial u^3}(h_1 E_1) - \frac{\partial}{\partial u^1}(h_3 E_3)\right] + \frac{\partial B_2}{\partial t} = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{2}$$

Putting the values of the metrical coefficients $h_1$, $h_2$, $h_3$ equal to unity and $u^1 = r$, $u^2 = \theta$, $u^3 = z$ and assuming that the conductivities of the metallic wall and the dielectric inside the guide are infinity and zero respectively, the equations (2) can be expressed in cylindrical co-ordinates $(r, \theta, z)$ as follows, with the further assumption that $E = \hat{E}e^{j\omega t}$

$$\begin{aligned} \frac{1}{r}\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z} &= j\omega\epsilon E_r \\ \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} &= j\omega\epsilon E_\theta \\ \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) - \frac{1}{r}\frac{\partial H_r}{\partial \theta} &= j\omega\epsilon E_z \\ \frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} &= -j\omega\mu H_r \\ \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} &= -j\omega\mu H_\theta \\ \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta} &= -j\omega\mu H_z \end{aligned} \tag{3}$$

where $\mathbf{B} = \mu \mathbf{H}$ and $\mathbf{D} = \epsilon \mathbf{E}$.

## Field Components of $TE_{01}$ Mode

For this mode the lines of the electric field form a system of concentric circles about the $z$-axis in the dielectric. This is associated with uniform current flow in the $\theta$ direction on the surface of the perfect conductor. The components $E_r$, $E_z$ and $H_\theta$ are each equal to zero. The normal mode field components for the $TE_{01}$ mode are then obtained from (3) as follows:

$$\begin{aligned} \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} &= j\omega\epsilon E_\theta \\ \frac{\partial E_\theta}{\partial z} &= j\omega\mu H_r \\ \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) &= -j\omega\mu H_z \end{aligned} \tag{4}$$

From (4) the following partial differential equation in $E_\theta$ is obtained

$$\frac{\partial^2 E_\theta}{\partial z^2} + \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta)\right] + \omega^2\mu\epsilon E_\theta = 0. \tag{5}$$

This equation can be solved by the method of separation of variables. We attempt to find a solution which will consist of the product of a function of $r$ alone and a function of $z$ alone. Let the solution be expressed as

$$E_\theta = Re^{-\gamma z}, \tag{6}$$

where $R = f(r)$ only and $\gamma$ indicates the propagation constant in the $z$ direction which is the direction of transmission. Substituting (6) in (5) and putting $K^2 = \gamma^2 + \omega^2\mu\epsilon$, the following differential equation is obtained:

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rR)\right] + K^2R = 0 \tag{7}$$

as $e^{-\gamma z} \neq 0$.

The equation (7) when solved gives R in terms of the Bessel functions of the first and the second kind.

$$R = AJ_1(Kr) + BY_1(Kr), \tag{8}$$

where A and B are constants to be evaluated from the boundary conditions. The following field components are obtained from (4), (6) and (8).

$$\begin{aligned} E_\theta &= [AJ_1(Kr) + BY_1(Kr)]\, e^{-\gamma z} \\ H_r &= -\frac{\gamma}{j\omega\mu}\left[AJ_1(Kr) + BY_1(Kr)\right] e^{-\gamma z} \\ H_z &= -\frac{K}{j\omega\mu}\left[AJ_0(Kr) + BY_0(Kr)\right] e^{-\gamma z} \end{aligned} \tag{9}$$

The functions $J_p$'s are finite and continuous everywhere in the two mediums but the function $Y_p$'s have infinite discontinuities in the axial region of the guide. In other words, near the axis of the guide as $r \to 0$ the functions $Y_1(r) \to -\infty$ and $Y_0(r) \to -\infty$. So, the function $Y_p$'s cannot be used to express physically any finite field in the neighbourhood of the origin. Therefore, the terms involving $Y_p$'s can be omitted from the expressions for the field components in the inner dielectric ($\epsilon_2$). The field components for both the mediums can then be written from (9) as

$$E_{\theta 1} = [A_1 J_1(K_1 r) + B_1 Y_1(K_1 r)]\, e^{-\gamma_1 z}$$

$$H_{r1} = -\frac{\gamma_1}{j\omega\mu_1}\Big[A_1 J_1(K_1 r) + B_1 Y_1(K_1 r)\Big]e^{-\gamma_1 z}$$

$$H_{z1} = -\frac{K_1}{j\omega\mu_1}\Big[A_1 J_0(K_1 r) + B_1 Y_0(K_1 r)\Big]e^{-\gamma_1 z} \qquad (10\ a)$$

where

$$r_1 \geqslant r \geqslant r_2$$

$$E_{\theta 2} = [A_2 J_1(K_2 r)]\, e^{-\gamma_2 z}$$

$$H_{r2} = -\frac{\gamma_2}{j\omega\mu_2}[A_2 J_1(K_2 r)]\, e^{-\gamma_2 z} \qquad (10\ b)$$

$$H_{z2} = -\frac{K_2}{j\omega\mu_2}[A_2 J_0(K_2 r)]\, e^{-\gamma_2 z}$$

where $r_2 \geqslant r \geqslant 0.$

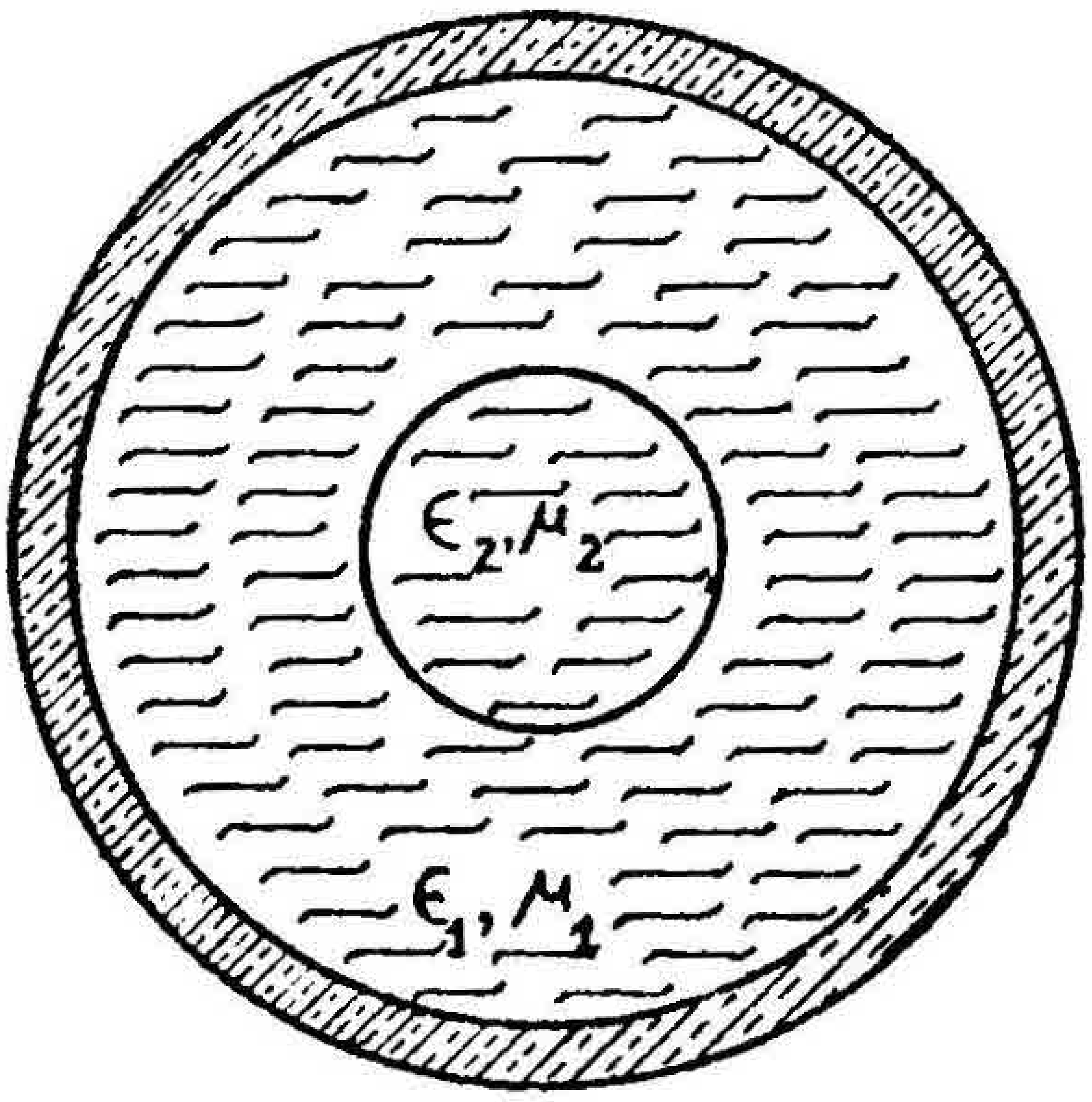


FIG. 1

The following boundary conditions are satisfied

$$H_{z1} = H_{z2} \text{ at } r = r_2$$

$$\mu_1 H_{r1} = \mu_2 H_{r2} \text{ at } r = r_2 \text{ as } \nabla \cdot \mathbf{B} = 0$$

$$E_{\theta 1} = 0 \text{ at } r = r_1.$$

Applying the boundary conditions, the following equations are obtained from (10 *a*) and (10 *b*).

$$\frac{K_1}{j\omega\mu_1}\left[A_1J_0(K_1r)+B_1Y_0(K_1r)\right]e^{-\gamma_1 z}$$
$$=\frac{K_2}{j\omega\mu_2}\left[A_2J_0(K_2r)\right]e^{-\gamma_2 z} \tag{11 a}$$

$$\frac{\gamma_1}{j\omega}\left[A_1J_1(K_1r)+B_1Y_1(K_1r)\right]e^{-\gamma_1 z}$$
$$=\frac{\gamma_2}{j\omega}\left[A_2J_1(K_2r)\right]e^{-\gamma_2 z} \tag{11 b}$$

$$A_1J_1(K_1r)+B_1Y_1(K_1r)=0. \tag{11 c}$$

In order that propagation may take place through the guide, the propagation constant $\gamma$ should be imaginary. This requires that the dielectrics should be perfect in that the attenuation constants of the two mediums are zero. So, $\gamma_1 = j\beta_1$ and $\gamma_2 = j\beta_2$, where $\beta_1$ and $\beta_2$ are the phase constants for the mediums 1 and 2 respectively.

Let $\mu_2/\mu_1 = \mu$, $\beta_1/\beta_2 = \beta'$, $\beta_2 - \beta_1 = \beta$.

The equations (11 *a*) to (11 *c*) reduce to the following:

$$\begin{aligned}K_1A_1\mu\, J_0(K_1r_2)\,e^{j\beta z} + K_1\mu B_1Y_0(K_1r_2)\,e^{j\beta z} - K_2A_2\,J_0(K_2r_2) &= 0\\ A_1J_1(K_1r_1)+B_1Y_1(K_1r_1) &= 0\\ \beta' A_1J_1(K_1r_2)\,e^{j\beta z} + B_1\beta' Y_1(K_1r_2)\,e^{j\beta z} - A_2J_1(K_2r_2) &= 0\end{aligned} \tag{12}$$

In order that $A_1$, $B_1$ and $A_2$ are not zero, the determinant of their coefficients must vanish or in other words, the following conditions must hold good:

$$\begin{vmatrix} K_1\mu J_0(K_1r_2)\,e^{j\beta z} & K_1\mu Y_0(K_1r_2)\,e^{j\beta z} & -K_2J_0(K_2r_2)\\ J_1(K_1r_1) & Y_1(K_1r_1) & \\ \beta' J_1(K_1r_2)\,e^{j\beta z} & \beta' Y_1(K_1r_2)\,e^{j\beta z} & -J_1(K_2r_2)\end{vmatrix}=0$$

From (12) $A_1$ and $B_1$ can be expressed in terms of $A_2$ as follows:

$$A_1\left[(J_0(K_1r_2)Y_1(K_1r_2)-Y_0(K_1r_2)J_1(K_1r_2)\right]K_1\mu\beta' e^{j\beta z}$$
$$=A_2\left[K_2\beta' J_0(K_2r_2)Y_1(K_1r_2)-K_1\mu Y_0(K_1r_2)J_1(K_2r_2)\right] \tag{13 a}$$

$$B_1\left[J_1(K_1r_2)Y_0(K_1r_2)-Y_1(K_1r_2)J_0(K_1r_2)\right]K_1\mu\beta' e^{j\beta z}$$
$$=A_2\left[K_2\beta' J_0(K_2r_2)J_1(K_1r_2)-K_1\mu J_1(K_2r_2)J_0(K_1r_2\right] \tag{13 b}$$

The equations (13 *a*) and (13 *b*) can be written (McLachlan, 1948) as

$$A_1=-A_2\frac{\pi r_2}{2\mu\beta'}\left[K_2\beta' J_0(K_2r_2)Y_1(K_1r_2)-K_1\mu Y_0(K_1r_2)J_1(K_2r_2)\right]e^{-j\beta z} \tag{13 c}$$

$$B_1=A_2\frac{\pi r_2}{2\mu\beta'}\left[K_2\beta' J_0(K_2r_2)J_1(K_1r_2)-K_1\mu J_0(K_1r_2)J_1(K_2r_2)\right]e^{-j\beta z} \tag{13 d}$$

Or, (13 *c*) and (13 *d*) can be written as

$$A_1 = A_2.A' \text{ and } B_1 = A_2.B' \tag{14}$$

where

$$A' = -\frac{\pi r_2}{2\mu\beta'}\begin{vmatrix} K_2\beta' J_0(K_2r_2) & K_1\mu Y_0(K_1r_2) \\ J_1(K_2r_2) & Y_1(K_1r_2) \end{vmatrix} e^{-j\beta z} \tag{14 a}$$

and

$$B' = \frac{\pi r_2}{2\mu\beta'}\begin{vmatrix} K_2\beta' J_0(K_2r_2) & K_1\mu J_0(K_1r_2) \\ J_1(K_2r_2) & J_1(K_1r_2) \end{vmatrix} e^{-j\beta z} \tag{14 b}$$

## EVALUATION OF $A_2$

The constant $A_2$ can be evaluated by integrating the Poynting vector over any cross-section normal to the axis of the guide. The average power flowing through the guide is given by the sum of

$$P_1 = \tfrac{1}{2}\int_{r_2}^{r_1}\int_0^{2\pi} E_{\theta 1}H_{r1}^{*}\, r d\theta\, dr$$

and

$$P_2 = \tfrac{1}{2}\int_{0}^{r_2}\int_0^{2\pi} E_{\theta 2}H_{r2}^{*}\, r d\theta\, dr$$

where the subscripts 1 and 2 refer to the two mediums. The peak power flowing through the guide is therefore given by the following expression:

$$\hat{P} = \int_0^{r_2}\int_0^{2\pi} E_{\theta 2}H_{r2}^{*}\, r d\theta\, dr + \int_{r_2}^{r_1}\int_0^{2\pi} E_{\theta 1}H_{r1}^{*}\, r d\theta\, dr$$

$$= 2\pi A_2^2\ \frac{\beta}{\mu\omega}\left[\int_0^{r_2} rJ_1^2(K_2r)\, dr - \int_{r_2}^{r_1} r\left[A'J_1(K_1r)+B'Y_1(K_1r)\right]^2 dr \tag{15}$$

In the above expression for the peak power flow $K^2 = \gamma^2 + \omega^2\mu\epsilon$. For both the mediums $K_1$ and $K_2$ can be expressed in terms of the phase velocities $v_p = \omega/\beta$ of the two mediums as

$$K_1^2 = \beta_1^2\left(\frac{v_{p1}^2}{c_1^2} - 1\right) \text{ and } K_2^2 = \beta_2^2\left(\frac{v_{p2}^2}{c_2^2} - 1\right),$$

where $c_1 = (1/\mu_1\epsilon_1)^{\frac{1}{2}}$; $c_2 = (1/\mu_2\epsilon_2)^{\frac{1}{2}}$; $v_{p1} = \omega/\beta_1$; $v_{p2} = \omega/\beta_2$. The argument of the Bessel functions in the integrals for $\hat{P}$ may be real or imaginary. Different cases will arise depending on the relative values of $v_p$'s with respect to $c_p$'s. However, the method of analysis remains unaltered irrespective of the value of $(v_p/c_p)$. Let us consider that the constants of the two mediums are such that $K_2$ is imaginary and $K_1$ is real. Let $K_2 = jK'$. The field components in the second medium, therefore, involve Bessel functions of complex argument and hence it is convenient to replace $J_0$ and $J_1$ for the second medium by the modified Bessel functions $I_0$ and $I_1$ as follows:

$$J_1(K_2 r) = jI_1(K'r)$$
$$J_0(K_2 r) = I_0(K'r)$$

The integrals in equation (15) can be evaluated as follows:

$$\int_0^{r_2} rJ_1^2(K_2 r)\,dr$$
$$= \tfrac{1}{2} r_2^2 \left[ I_0^2(K'r_2) - I_1^2(K'r_2) - \frac{2I_0(K'r_2)\, I_1(K'r_2)}{K'r_2} \right] \qquad (16\,a)$$

$$\int_{r_2}^{r_1} r\,[A'J_1(K_1 r)]^2\,dr$$
$$= \frac{\pi^2 r_2^2}{8\mu\beta'} \begin{vmatrix} jK_2'\beta' I_0(K'r_2) & K_1\mu Y_0(K_1 r_2) \\ jI_1(K'r_2) & Y_1(K_1 r_2) \end{vmatrix}^2$$
$$\times \left[ r^2 \{J_1^2(K_1 r) - J_0(K_1 r)\, J_2(K_1 r)\} \right]_{r_2}^{r_1} e^{-2j\beta z} \qquad (16\,b)$$

$$\int_{r_2}^{r_1} r\,[B'Y_1(K_1 r)]^2\,dr$$
$$= \frac{\pi^2 r_2^2}{8\mu\beta'} \begin{vmatrix} jK'\beta' I_0(K'r_2) & K_1\mu J_0(K_1 r_2) \\ jI_1(K'r_2) & J_1(K_1 r_2) \end{vmatrix}^2$$
$$\times \left[ r^2 \{Y_1^2(K_1 r) - Y_0(K_1 r)\, Y_2(K_1 r)\} \right]_{r_2}^{r_1} e^{-2j\beta z} \qquad (16\,c)$$

$$2\int_{r_2}^{r_1} A'B'rJ_1(K_1 r)\, Y_1(K_1 r)\,dr$$
$$= \frac{\pi^2 r_2^2}{2\mu^2\beta'^2} \begin{vmatrix} jK'\beta' I_0(K'r_2) & K_1\mu Y_0(K_1 r_2) \\ jI_1(K'r_2) & Y_1(K_1 r_2) \end{vmatrix} \begin{vmatrix} j\beta'K' I_0(K'r_2) & K_1\mu J_0(K_1 r_2) \\ jI_1(K'r_2) & J_1(K_1 r_2) \end{vmatrix}$$
$$\times \left[ \tfrac{1}{4} r^2 \{2J_1(K_1 r)\, Y_1(K_1 r) - J_0(K_1 r)\, Y_2(K_1 r) - J_2(K_1 r)\, Y_0(K_1 r)\} \right]_{r_2}^{r_1} e^{-2j\beta z}$$
$$(16\,d)$$

The above expression show that the peak power is proportional to some function of $K'$, $K_1$, $r_1$ and $r_2$. So, the expression for the peak power flow through the guide can be written from (15) and (16) as

$$\hat{P} = 2\pi A_2^2 \frac{\beta}{\mu\omega} F(K', K_1, r_1, r_2) \qquad (17)$$

The constant $A_2$ is therefore given by

$$A_2 = \left( \frac{\hat{P}\mu f}{\beta F} \right)^{\frac{1}{2}} \qquad (18)$$

So, the field components for the two mediums can be written from (10), (14 and (18) as follows:

$$E_{\theta 1} = \left(\frac{\hat{P}\mu f}{\beta F}\right)^{\frac{1}{2}} [A'J_1(K_1 r) + B'Y_1(K_1 r)]\, e^{-j\beta z}$$

$$H_{r1} = -\frac{\beta}{\omega\mu}\left(\frac{\hat{P}\mu f}{\beta F}\right)^{\frac{1}{2}} [A'J_1(K_1 r) + B'Y_1(K_1 r)]\, e^{-j\beta z}$$

$$H_{z1} = -\frac{K_1}{j\omega\mu}\left(\frac{\hat{P}\mu f}{\beta F}\right)^{\frac{1}{2}} [A'J_0(K_1 r) + B'Y_0(K_1 r)]\, e^{-j\beta z}$$

$$E_{\theta 2} = j\left(\frac{\hat{P}\mu f}{\beta F}\right)^{\frac{1}{2}} I_1(K' r)\, e^{-j\beta z}$$

$$H_{r2} = -\frac{j\beta}{\omega\mu}\left(\frac{\hat{P}\mu f}{\beta F}\right)^{\frac{1}{2}} I_1(K' r)\, e^{-j\beta z}$$

$$H_{z2} = -\frac{K'}{\omega\mu}\left(\frac{\hat{P}\mu f}{\beta F}\right)^{\frac{1}{2}} I_0(K' r)\, e^{-j\beta z}$$

Or, expressed in real parts the field components can be written as

$$E_{\theta 1} = \left(\frac{\hat{P}\mu f}{\beta F}\right)^{\frac{1}{2}} [A'J_1(K_1 r) + B'Y_1(K_1 r)] \cos \beta z$$

$$H_{r1} = -\frac{1}{2\pi}\left(\frac{\hat{P}\beta}{\mu f F}\right)^{\frac{1}{2}} [A'J_1(K_1 r) + B'Y_1(K_1 r)] \cos \beta z$$

$$H_{z1} = \frac{K_1}{2\pi}\left(\frac{\hat{P}}{\mu f \beta F}\right)^{\frac{1}{2}} [A'J_0(K_1 r) + B'Y_0(K_1 r)] \sin \beta z$$

$$E_{\theta 2} = \left(\frac{\hat{P}\mu f}{\beta F}\right)^{\frac{1}{2}} I_1(K' r) \sin \beta z$$

$$H_{r2} = -\frac{1}{2\pi}\left(\frac{\hat{P}\beta}{\mu f F}\right)^{\frac{1}{2}} I_1(K' r) \sin \beta z$$

$$H_{z2} = -\frac{K'}{2\pi}\left(\frac{\hat{P}}{\mu f \beta F}\right)^{\frac{1}{2}} I_0(K' r) \cos \beta z \qquad (19)$$

## Attenuation Constant

In the ideal case when the conductivity of the boundary wall of the guide is infinity and that of the dielectric is zero, the boundary condition is simply $\mathbf{n} \times \mathbf{E} = 0$ at the surface where $\mathbf{n}$ is the outward normal to the surface. But if the dissipation due to the finite conductivity of the wall and the dielectric is considered, the boundary conditions are simply the continuity of both $\mathbf{E}_{tan}$ and $\mathbf{H}_{tan.}$ The transmission takes place in the direction of the guide axis ($z$) and the amplitude varies as $e^{-\gamma z}$ where the propagation constant $\gamma$ involves the attenuation constant $\alpha$ per unit length. The attenuation constant $\alpha$ is given in terms of the mean energy $\bar{W}$ transmitted through the guide by the following equation

$$\alpha = -\frac{1}{2}\frac{1}{\bar{W}}\frac{\partial \bar{W}}{\partial z} \tag{20 a}$$

or in terms of **E** and **H**

$$\alpha = Re\left[\frac{-\int ds\, (\mathbf{E}.\mathbf{H}^*)_r}{2\int ds\, (\mathbf{E}.\mathbf{H}^*)_z}\right] \tag{20 b}$$

The subscripts $r$ and $z$ in (20 $b$) indicate the direction of the power flow, *i.e.*, power dissipated and transmitted respectively. The attenuation is due to imperfect conductivity of the boundary wall of the guide and also due to finite conductivities of the two dielectric mediums. For the metal boundary $\sigma >> \omega\epsilon$ except at extremely high frequencies and the following relation holds good:

$$\mathbf{E}_{tan} = \eta\, \mathbf{H}_{tan},$$

where the intrinsic impedance $\eta$ of the metal is given by $Re(\eta) = (\pi f\mu/\sigma)^{\frac{1}{2}}$ and the direction of $\mathbf{E}_{tan}$ is such that the Poynting vector is directed into the metal. The equation (20 $b$) can be written as

$$\alpha = Re\left[\frac{-\iint (\mathbf{E}\cdot\mathbf{H}^*)\, ds}{2\iint (E_r'H_\theta' - E_\theta'H_r')_z\, ds}\right] \tag{20 c}$$

which in the present case can be reduced to

$$\alpha = \tfrac{1}{2} Re \left[\frac{\int_{r_2}^{r_1}\int_0^{2\pi}(H_{z1}')^2\, d\theta dr + \int_{r_2}^{r_1}\int_0^{2\pi} E_{\theta 1}'H_{z1}'\, rd\theta\, dr + \int_0^{r_2}\int_0^{2\pi} E_{\theta 2}'\, H'_{z2}\, rd\theta\, dr}{\int_{r_2}^{r_1}\int_0^{2\pi} E_{\theta 1}'\, H_{r1}'\, rd\theta\, dr + \int_{r_2}^{r_1}\int_0^{2\pi} E_{\theta 1}'\, H_{r1}'\, rd\theta\, dr + \int_0^{r_2}\int_0^{2\pi} E_{\theta 2}'H'_{r2} rd\theta dr}\right] \tag{20 d}$$

The prime on E's and H's indicate amplitude values of the field components. The integrals in (20 $d$) can be evaluated with the help of (19) as follows:

$$\int_{r_2}^{r_1}\int_0^{2\pi} E'_{\theta 1}H'_{r1}\, rd\theta\, dr$$

$$= -\frac{\hat{P}}{F}\int_{r_2}^{r_1} r\,[A'^2 J_1^2(K_1 r) + B'^2 Y_1^2(K_1 r) + 2A'B'J_1(K_1 r)\, Y_1(K_1 r)]\, dr \tag{21}$$

$$\int_0^{r_2}\int_0^{2\pi} E_{\theta 2}\, H_{r2}'\, rd\theta\, dr$$

$$= -\frac{\hat{P}}{F}\int_0^{r_2} rI_1^2(K'r)\, dr \tag{22}$$

which have been evaluated in (16).

$$\int_{r_2}^{r_1}\int_0^{2\pi} E'_{\theta 1} H'_{z1}\, r\, d\theta\, dr$$

$$= \frac{K_1 \hat{P}}{\beta F}\int_{r_2}^{r_1} r\,[A'^2 J_1(K_1 r)\, J_0(K_1 r) + B'^2 Y_1(K_1 r)\, Y_0(K_1 r)$$

$$+ A'B' J_1(K_1 r)\, Y_0(K_1 r) + A'B' Y_1(K_1 r)\, J_0(K_1 r)]\, dr \quad (23)$$

The products of two Bessel functions have been expressed in terms of infinite series by Orr (1900), Nielsen (1904), Watson (1922), Whittaker and Watson (1927). The integrals in (23) have been evaluated term by term and the results are as follows:

$$\int_{r_2}^{r_1} r J_1(K_1 r)\, J_0(K_1 r)\, dr$$

$$= \left[\sum_{s=0}^{\infty} \pi_{s,1} \frac{r^{3+2s}}{3+2s}\right]_{r_2}^{r_1}, \quad (23\,a)$$

where $\pi_{s,1} = \dfrac{(-1)^s (1+2s)!\, K_1^{1+2s}}{2^{1+2s}(s!)^2[(1+s)!]^2}$

$$\int_{r_2}^{r_1} r J_1(K_1 r)\, Y_0(K_1 r)\, dr$$

$$= -\frac{1}{\pi}\int_{r_2}^{r_1} \sum_{0}^{-1} \pi_{s,2} r^{2+2s} dr + \frac{1}{\pi}\int_{r_2}^{r_1} \sum_{0}^{\infty} \pi_{s,3}\, r^{2+2s}\{2\log\tfrac{1}{2}(K_1 r) + \pi_{s,4}\}\, dr$$

$$= -\frac{1}{\pi}\left[\sum_{0}^{-1} \pi_{s,2} \frac{r^{3+2s}}{3+2s}\right]_{r_2}^{r_1} + \frac{1}{\pi}\left[\sum_{0}^{\infty}\left\{2\pi_{s,3}\left(\frac{r^{3+2s}}{3+2s}\log\tfrac{1}{2}(K_1 r) - \frac{r^{3+2s}}{(3+2s)^2}\right)\right.\right.$$

$$\left.\left. + \pi_{s,3}\pi_{s,4}\frac{r^{3+2s}}{3+2s}\right\}\right]_{r_2}^{r_1} \quad (23\,b)$$

where

$$\pi_{s,2} = \frac{(-s-1)!\, K_1^{1+2s}(2+s)_s}{2^{1+2s}(s!)\,\Gamma(s+2)}$$

$$\pi_{s,3} = \frac{(-1)^s (s+2)_s\, K_1^{1+2s}}{2^{1+2s}(s!)^2\,\Gamma(s+2)}$$

$$\pi_{s,4} = 2\,[\psi(2s+2) - \psi(s+2) - \psi(s+1)],$$

where $\psi$ represents the logarithmic derivate of the Gamma function and $\Gamma$ represents the Gamma function (Nielsen, 1906).

$$\int_{r_2}^{r_1} r Y_1(K_1 r)\, J_0(K_1 r)\, dr$$

$$= \frac{1}{\pi}\int_{r_2}^{r_1} \sum_{s=0}^{\infty} \pi_{s,5}\{2r^{2s+2}\log\tfrac{1}{2}(K_1 r) + \pi_{s,4} r^{2s+2}\}\, dr$$

$$= \frac{1}{\pi}\sum_{s=0}^{\infty} \pi_{s,5}\left[2\left(\frac{r^{2s+3}}{2s+3}\log\tfrac{1}{2}(K_1 r) - \frac{r^{2s+3}}{(2s+3)^2}\right) + \pi_{s,4}\frac{r^{2s+3}}{2s+3}\right]_{r_2}^{r_1} \quad (23\,c)$$

where

$$\pi_{s,5} = \frac{(-1)^s K_1^{2s+1} (s+2)_s}{2^{2s+1} s!(1+s)!\,\Gamma(1+s)}$$

$$\int_{r_2}^{r_1} r Y_1(K_1 r)\, y_0(K_1 r)\, dr$$

$$= \int_{r_2}^{r_1} \Big[\frac{2}{\pi}\gamma r J_0(K_1 r) + \frac{2}{\pi} r \log \tfrac{1}{2}(K_1 r))J_0(K_1 r)$$

$$- \sum_{s=1}^{\infty} s_s r^{1+2s}\Big] \times \Big[\frac{2}{\pi}\gamma J_1(K_1 r) + \frac{2}{\pi}\log \tfrac{1}{2}(K_1 r)\, J_1(K_1 r)$$

$$- \sum_{s=0}^{\infty} S_{1s} r^{1+2s}\Big] dr \qquad (23\,d)$$

where $\gamma$ = Euler's constant = 0·5772

$$S_s = \frac{2}{\pi}\frac{(-1)^s K_1^{2s}}{2^{2s}(s\,!)^2}\left\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots\ldots + \frac{1}{s}\right\}$$

$$S_{1s} = \frac{1}{\pi}\frac{(-1)^s K_1^{1+2s}}{2^{1+2s} s!\,(1+s)\,!}\left\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots\ldots + \frac{1}{s} + 1 + \tfrac{1}{2} + \ldots\ldots + \frac{1}{1+s}\right\}$$

The expression in (23*d*) have been integrated term by term as follows:

$$\int_{r_2}^{r_1} r J_0(K_1 r)\, J_1(K_1 r) \log \tfrac{1}{2}(K_1 r)\, dr$$

$$= \left[\log \tfrac{1}{2}(K_1 r) \sum_{s=0}^{\infty} \pi_{s,1} \frac{r^{3+2s}}{3+2s} - \sum_{s=0}^{\infty} \pi_{s,1} \frac{r^{3+2s}}{(3+2s)^2}\right]_{r_2}^{r_1} \qquad (23\,d_1)$$

$$\int_{r_2}^{r_1} r J_0(K_1 r) \sum_{s=0}^{\infty} S_{1s}\, r^{1+2s}\, dr$$

$$= \int_{r_2}^{r_1} \sum_{s=0}^{\infty} \pi_{s,6}\, r^{1+2s} \sum_{s=0}^{\infty} S_{1,s}\, r^{1+2s}\, dr$$

The above integral can be written as (Bromwich, 1926; Phillips, 1939)

$$= \left[\sum_{s=0}^{\infty} c_s' \frac{r^{2+2s}}{2+2s}\right]_{r_2}^{r_1}, \qquad (23\,d_2)$$

where

$$\pi_{s,6} = \frac{(-1)^s K_1^{2s}}{2^{2s}(s\,!)^2}$$

$$c_0' = \pi_{0,6}\, S_{1,0}$$

$$c_s' = \pi_{0,6}\, S_{1,s} + \pi_{1,6}\, S_{1,(s-1)} + \ldots\ldots + \pi_{s,6}\, S_{1,0}$$

$$\int_{r_2}^{r_1} r J_0(K_1 r) J_1(K_1 r) [\log \tfrac{1}{2}(K_1 r)]^2 \, dr$$

$$= \Bigg[\Big\{\log \tfrac{1}{2}(K_1 r)\Big\}^2 \sum_{s=0}^{\infty} \pi_{s,1} \frac{r^{3+2s}}{3+2s} - 2\Big\{\log \tfrac{1}{2}(K_1 r) \sum_{s=0}^{\infty} \pi_{s,1} \frac{r^{3+2s}}{(3+2s)^2}\Big\}$$

$$+ 2\sum_{s=0}^{\infty} \pi_{s,1} \frac{r^{3+2s}}{(3+2s)^3}\Bigg]_{r_2}^{r_1} \qquad (23\ d_3)$$

$$\int_{r_2}^{r_1} r \log \tfrac{1}{2}(K_1 r) J_0(K_1 r) \sum_{s=0}^{\infty} S_{1,s} r^{1+2s} \, dr$$

$$= \Bigg[\log \tfrac{1}{2}(K_1 r) \sum_{s=0}^{\infty} c_s' \frac{r^{2+2s}}{2+2s} - \sum_{s=0}^{\infty} c_s' \frac{r^{2+2s}}{(2+2s)^2}\Bigg]_{r_2}^{r_1} \qquad (23\ d_4)$$

$$\int_{r_2}^{r_1} J_1(K_1 r) \sum_{s=1}^{\infty} S_s r^{1+2s} \, dr$$

$$= \Bigg[\sum_{s=0}^{\infty} c_s'' \frac{r^{2+2s}}{2+2s} - \sum_{s=0}^{\infty} \pi_{s,7} S_1 \frac{r^{5+2s}}{5+2s} - \sum_{s=0}^{\infty} \pi_{s,7} S_0 \frac{r^{3+2s}}{3+2s}\Bigg]_{r_2}^{r_1} \qquad (23\ d_5)$$

where

$$\pi_{s,7} = \frac{(-1)^s K_1^{1+2s}}{2^{1+2s} s!(1+s)!}$$

$$c_0'' = \pi_{0,7} S_0$$

$$c_s'' = \pi_{0,7} S_s + \ldots + \pi_{s,7} S_0$$

$$\int_{r_2}^{r_1} \log \tfrac{1}{2}(K_1 r) J_1(K_1 r) \sum_{s=1}^{\infty} S_s r^{1+2s} \, dr$$

$$= \Bigg[\log \tfrac{1}{2}(K_1 r) \sum_{s=0}^{\infty} c_s'' \frac{r^{2+2s}}{2+2s} - \sum_{s=0}^{\infty} c_s'' \frac{r^{2+2s}}{(2+2s)^2}$$

$$- \log \tfrac{1}{2}(K_1 r) \sum_{s=0}^{\infty} S_1 \pi_{s,7} \frac{r^{5+2s}}{5+2s} + \sum_{s=0}^{\infty} S_1 \pi_{s,7} \frac{r^{5+2s}}{(5+2s)^2}$$

$$- \log \tfrac{1}{2}(K_1 r) \sum_{s=0}^{\infty} S_0 \pi_{s,7} \frac{r^{3+2s}}{3+2s} + \sum_{s=0}^{\infty} S_0 \pi_{s,7} \frac{r^{3+2s}}{(3+2s)^2}\Bigg]_{r_2}^{r_1} \qquad (23\ d_6)$$

$$\int_{r_2}^{r_1} \sum_{1}^{\infty} S_s r^{1+2s} \sum_{0}^{\infty} S_{1,s} r^{1+2s} \, dr$$

$$= \Bigg[\sum_{s=0}^{\infty} c_s''' \frac{r^{2+2s}}{2+2s} - \sum_{s=0}^{\infty} S_{1,s} S_0 \frac{r^{3+2s}}{3+2s} - \sum_{s=0}^{\infty} S_{1,s} S_1 \frac{r^{5+2s}}{5+2s}\Bigg]_{r_2}^{r_1} \qquad (23\ d_7)$$

where

$$c_s''' = S_0 S_{1,s} + \ldots + S_s S_{1,0}$$
$$c_0''' = S_0 S_{1,0}$$

$$\int_0^{2\pi}\int_{r_2}^{r_1} \eta\,(H_{z1}')^2\,dr\,d\theta$$

$$= \frac{K_1^2 \hat{P}}{2\beta F\sqrt{\pi\mu f\sigma}} \int_{r_2}^{r_1} \Big[A'J_0(K_1 r) + B'Y_0(K_1 r)\Big]^2 dr \tag{24}$$

$$\int_{r_2}^{r_1} J_0^2(K_1 r)\,dr = \Big[\sum_{s=0}^{\infty} \pi_{s,8}\frac{r^{2s+1}}{2s+1}\Big]_{r_2}^{r_1}, \tag{24 a}$$

where

$$\pi_{s,8} = \frac{(-1)^s (2s)!\,K_1^{2s}}{2^{2s}(s!)^4}$$

$$\int J_0(K_1 r)\,Y_0(K_1 r)\,dr$$

$$= \frac{2}{\pi}\int_{r_2}^{r_1}\Big[J_0^2(K_1 r)\log\frac{K_1 r}{2} + \sum_{s=0}^{\infty}\pi_{s,9}r^{2s}\Big]dr$$

$$= \frac{2}{\pi}\Big[\log\tfrac{1}{2}(K_1 r)\sum_{s=0}^{\infty}\pi_{s,8}\frac{r^{2s+1}}{2s+1} - \sum_{s=0}^{\infty}\pi_{s,8}\frac{r^{2s+1}}{(2s+1)^2} - \sum_{s=0}^{\infty}\pi_{s,9}\frac{r^{2s+1}}{2s+1}\Big]_{r_2}^{r_1}, \tag{24 b}$$

where

$$\pi_{s,9} = \frac{(-1)^s\binom{2s}{s}}{(s!)^2}\,[\psi(s+\tfrac{1}{2}) - \psi(s+1)]\,(\tfrac{1}{2})^{2s}$$

$$\int_{r_2}^{r_1} Y_0^2(K_1 r)\,dr$$

$$= \frac{4}{\pi^2}\int_{r_2}^{r_1}\Big[\gamma^2 J_0^2(K_1 r) + J_0^2(K_1 r)\{\log\tfrac{1}{2}(K_1 r)\}^2$$
$$+ \Big\{\sum_{s=1}^{\infty} S_s r^{2s}\Big\}^2 + 2\gamma J_0^2(K_1 r)\log\tfrac{1}{2}(K_1 r)$$
$$- 2\gamma J_0(K_1 r)\sum_{s=1}^{\infty}S_s r^{2s} - 2\log\tfrac{1}{2}(K_1 r)\,J_0(K_1 r)\sum_{s=1}^{\infty}S_s r^{2s}\Big]\,dr \tag{24 c}$$

$$\int_{r_2}^{r_1} [\log \tfrac{1}{2} (K_1 r)]^2 J_0^2 (K_1 r)\, dr$$

$$= \Big[\{\log \tfrac{1}{2} (K_1 r)\}^2 \sum_{s=0}^{\infty} \pi_{s,8} \frac{r^{2s+1}}{2s+1} - 2 \log \tfrac{1}{2} (K_1 r) \sum_{s=0}^{\infty} \pi_{s,8} \frac{r^{2s+1}}{(2s+1)^2}$$

$$+ 2 \sum_{s=0}^{\infty} \frac{r^{2s+1}}{(2s+1)^3}\Big]_{r_2}^{r_1} \qquad (24\ c_1)$$

$$\int_{r_2}^{r_1} \Big[\sum_{s=1}^{\infty} S_s r^{2s}\Big]^2 dr = \Big[\sum_{1}^{\infty} c_s^{\mathrm{IV}} \frac{r^{2s+1}}{2s+1}\Big]_{r_2}^{r_1}, \qquad (24\ c_2)$$

where

$$c_s^{\mathrm{IV}} = c_0^{\mathrm{IV}} c_s^{\mathrm{IV}} + \ldots . + c_s^{\mathrm{IV}} c_0^{\mathrm{IV}}$$

$$\int_{r_2}^{r_1} J_0^2 (K_1 r) \log \tfrac{1}{2} (K_1 r)\, dr$$

$$= \Big[\log \tfrac{1}{2} (K_1 r) \sum_{s=0}^{\infty} \pi_{s,8} \frac{r^{2s+1}}{2s+1} - \sum_{s=0}^{\infty} \pi_{s,8} \frac{r^{2s+1}}{(2s+1)^2}\Big]_{r_2}^{r_1} \qquad (24\ c_3)$$

$$\int_{r_2}^{r_1} J_0 (K_1 r) \sum_{s=1}^{\infty} S_s r^{2s}\, dr$$

$$= \Big[\sum_{s=0}^{\infty} c_s^{\mathrm{V}} \frac{r^{2s+1}}{2s+1} - \sum_{s=0}^{\infty} \pi_{s,6} S_0 \frac{r^{2s+2}}{2s+2} - \sum_{s=0}^{\infty} \pi_{s,6} S_1 \frac{r^{2s+3}}{2s+3}\Big]_{r_2}^{r_1}, \qquad (24\ c_4)$$

where

$$c_s^{\mathrm{V}} = \pi_{0,6} S_s + \ldots . + \pi_{s,6} S_0$$

$$c_0^{\mathrm{V}} = \pi_{0,6} S_0$$

$$\int_{r_2}^{r_1} \log \tfrac{1}{2} (K_1 r) J_0 (K_1 r) \sum_{s=1}^{\infty} S_s r^{2s}\, dr$$

$$= \Big[\log \tfrac{1}{2} (K_1 r) \sum_{s=0}^{\infty} c_s^{\mathrm{V}} \frac{r^{2s+1}}{2s+1} - \sum_{s=0}^{\infty} c_s \frac{r^{2s+1}}{(2s+1)^2}$$

$$- \log \tfrac{1}{2} (K_1 r) \sum_{s=0}^{\infty} \pi_{s,6} \frac{r^{2s+2}}{2s+2} S_0 + \sum_{s=0}^{\infty} \pi_{s,6} \frac{r^{2s+2}}{(2s+2)^2} S_0$$

$$- \log \tfrac{1}{2} (K_1 r) \sum_{s=0}^{\infty} \pi_{s,6} \frac{r^{2s+3}}{2s+3} S_1 + \sum_{s=0}^{\infty} \pi_{s,6} \frac{r^{2s+3}}{(2s+3)^2} S_1\Big]_{r_2}^{r_1} \qquad (24\ c_5)$$

$$\int_0^{2\pi}\int_0^{r_2} E_{\theta 2}' H_{z2}'\, r\, d\theta\, dr$$

$$= -\frac{K'\hat{P}}{\beta F} \int_0^{r_2} r I_0 (K' r) I_1 (K' r)\, dr$$

$$= -\frac{K'\hat{P}}{\beta F} \Big[c_s^{\mathrm{VI}} \frac{r^{2+2s}}{2+2s}\Big]_0^{r_2}, \qquad (25)$$

where

$$c_0^{VI} = \frac{(K')^2}{2}$$

$$c_s^{VI} = K_1 \sum_{s=0}^{\infty} \frac{(K')^{1+2s}}{2^{1+2s}\, s\,!\,(1+s)\,!} + \left(\frac{K'}{2}\right)^2 \sum_{s=0}^{\infty} \frac{(K')^{2s-1}}{2^{2s-1}\,(s-1)\,!\,s\,!} +$$

$$+ \ldots + \frac{K'}{2} \sum_{s=0}^{\infty} \frac{(K')^{2s}}{2^{2s}\,(s\,!)^2}$$

So, the attenuation constant $\alpha$ can be calculated with the help of the equations (20 *d*) to (25). In the calculation of $\alpha$ by equation (20 *d*), the loss due only to the process of conduction has been taken into account. But when a radio frequency field is applied to a dielectric, the molecular process involve additional losses due to the turning of molecules with permanent dipole moments against viscous dragging forces. All these dissipative forces give rise to a current density in the medium which is in time phase with **E**. The losses due to the molecular processes can be taken into account by expressing $\epsilon$ in the expression for $\alpha$ as consisting of the effective dielectric constant $\epsilon'$ and the loss factor $\epsilon''$, *i.e.*, expressing $\epsilon$ as $\epsilon = \epsilon' - j\epsilon''$, where $\epsilon$ involves three terms (Manning and Bell, 1940) as given by the following relation

$$\epsilon = \epsilon_{\infty} + \epsilon_a + \epsilon_c,$$

where $\epsilon_{\infty}$ represents the geometric dielectric constant due to electronic polarisation which does not contribute to $\epsilon''$ but only to $\epsilon'$. $\epsilon_c$ is that part which arises due to ohmic conduction and hence contributes to $\epsilon''$ as the conduction current corresponding to $\epsilon_c$ is in phase with the applied voltage, $\epsilon_a$ is that part which arises due to dielectric absorption. As the current corresponding to $\epsilon_a$ is out of phase with the voltage by an angle less than 90° so $\epsilon_a$ contributes to both $\epsilon'$ and $\epsilon''$. The factor $\epsilon_a$ is frequency dependent and it depends on the characteristic of the dielectric material. So, the true attenuation will be greater than that calculated by the eq. (20 *d*).

(*To be continued.*)

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