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INTERACTION OF MODES IN A MICROWAVE CAVITY RESONATOR

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Abstract

The quantum mechanical concept of zero-point energy has been utilised to calculate the minimum energy differences between the desired and all the undesired modes appearing in a cylindrical echo-box when the latter is excited at 9 Kmc to operate in the TE_{0-1-19} mode. Expressions for the mutual energy and the coupling coefficient between the two companion modes TE_{0-1} — TM_{1-1} have been derived with the help of the field equations. The paper ends with a discussion of the significance of the total energy of a microwave cavity resonator.

Introduction

In a microwave cavity resonator when excited to operate in a particular mode, some interfering and cross-over modes invariably appear. Any irregularity in the cavity either in the boundary or the dielectric within it provides a coupling between some or all of the modes. If the coupling between the modes is loose, the performance of the cavity will not be appreciably altered. But the effect of strong coupling is to alter the electro-magnetic field configurations from that of the desired mode and also to lower the Q of the cavity considerably. An idea of the interaction between any two modes can be obtained from the energy difference between the two modes. The paper presents a report of the calculation of the minimum energy differences between the desired and the undesired modes in a 9 Kmc echo-box by considering each mode as analogous to a linear harmonic oscillator. It is also the object of the paper to calculate the coupling coefficient between the $TE_{0\text{-}1}$ and the $TM_{1\text{-}1}$ mode with the help of the field equations.

Resonant Frequencies of Different Modes

Let us consider the case of a cylindrical cavity resonator having infinitely conducting walls and end plates and enclosing a lossless dielectric. If this cavity is excited it will sustain electro-magnetic oscillations whose **E** and **H** components can be written as follows:

$$\mathbf{E} = \frac{1}{\sqrt{\epsilon}} \; \mathbf{e} \sin\left(\frac{k}{\sqrt{\epsilon\mu}}\, t + \phi\right)$$

$$\mathbf{H} = \frac{1}{\sqrt{\mu}} \; \mathbf{h} \cos\left(\frac{k}{\sqrt{\epsilon\mu}}\, t + \phi\right)$$

where k and ϕ are constants. The electric and magnetic field configurations are given by the mode vectors e and **h** which are vector functions of positions only. The constants of the medium are ϵ and μ. The electro-magnetic field satisfies Maxwell's equations if

$$\nabla \times \mathbf{h} = k\,\mathbf{e} \quad \text{and} \quad \nabla \times \mathbf{e} = k\,\mathbf{h}$$

within the cavity and

$$\mathbf{h}.\,d\mathbf{s} = \mathbf{e} \times d\mathbf{s} = 0$$

at the boundary of the cavity.

These equations when solved show that any given cavity can oscillate in an infinite number of modes having eigenfrequencies $k_1/2\pi\sqrt{\epsilon\mu}$, $k_2/2\pi\sqrt{\epsilon\mu}$ .... $k_n/2\pi\sqrt{\epsilon\mu}$ with eigenvalues k_1, k_2 .... k_n.

The resonant frequencies of a cavity depend on whether the cavity is excited in TE or TM mode as given by the following equation (Kinzer, 1943).

$$\nu_{l,m,n} = \sqrt{\left(\frac{cn}{2L}\right)^2 + (f_o)^2_{l,m}} \qquad (1)$$

The cut-off frequencies $(f_o)_{l,m}$ are given by

$$(f_o)_{l,m} = \frac{c\,k_{l,m}}{2\pi a} \text{ for } TE_{l,m} \text{ mode}$$

and $(f_o)_{l,m} = \dfrac{c\, k_{l,m}}{2\pi a}$ for $TM_{l,m}$ mode

where a and L represent the radius and the length of the cavity respectively. The quantities $k'_{l,m}$ and $k_{l,m}$ are the roots of the following equations :

$$J'_l\ (k'_{l,m}) = 0 \text{ for } TE_{l,m} \text{ mode}$$

and $$J_l\ (k_{l,m}) = 0 \text{ for } TM_{l,m} \text{ mode}$$

Minimum Energy of Modes

As a cavity resonator can support infinite number of modes, the electro-magnetic fields inside the cavity resonator can be considered to consist of an infinite number of harmonic oscillators. The energy of each oscillator expressed in generalised co-ordinates (p, q) is given by the following expression :

$$W = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$

where m and ω represent the mass and the angular frequency of the oscillator respectively. The uncertainty principle of Heisenberg states that $\Delta p\, \Delta q \approx h/2\pi$ where h is the Planck's constant and Δp, Δq are the r.m.s. deviations of the momentum and the co-ordinate. The minimum value of

$$\frac{h^2}{8\pi^2 m\, (\Delta q)^2} + \frac{m\omega^2}{2} (\Delta q)^2 \tag{2}$$

gives the lowest possible energy for an oscillator. The minimum value of (2)

$$W_{min.} \approx \frac{1}{2} h\nu \tag{3}$$

is obtained when $(\Delta q)^2 = h/2\pi m\omega$. The frequency of vibration of the oscillator is given by ν. The energy difference

$$\left(\frac{1}{2} h\nu\right)_{\text{Mode A}} \sim \left(\frac{1}{2} h\nu\right)_{\text{Mode B}}$$

can be interpreted as the minimum possible energy responsible for bringing interaction between the two modes. The frequency of vibration for the different modes can be calculated from eqn. (I).

Different Modes in a 9 Kmc Echo-Box (TE $_{0\text{-}1\text{-}12}$).

The figure shows the expanded mode chart (Kinzer, 1946) for a 9 Kmc echo-box designed to operate in the TE$_{0\text{-}1\text{-}12}$ mode.

It will be observed that the following interfering and crossing modes appear in addition to the companion mode TM$_{1\text{-}1\text{-}12}$.

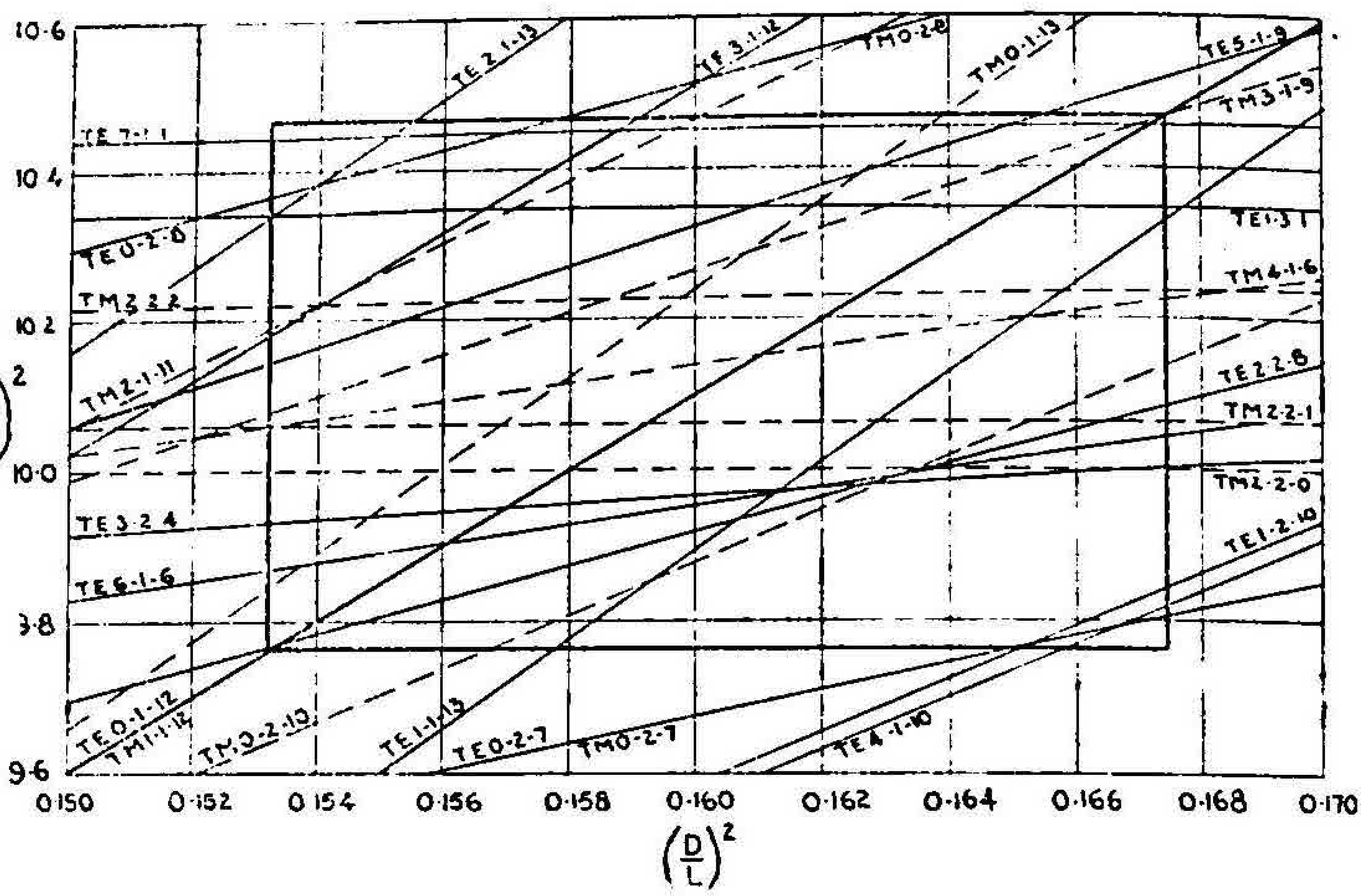

Fig. I. Expanded Mode Chart covering the Operating Area of a 9 Kmc Echo Box

Crossing Modes :

TE $_{6\text{-}1\text{-}6}$	TM $_{2\text{-}2\text{-}1}$	TE $_{7\text{-}1\text{-}1}$
TE $_{3\text{-}2\text{-}4}$	TM $_{4\text{-}1\text{-}6}$	TE $_{1\text{-}3\text{-}1}$
TM $_{2\text{-}2\text{-}0}$	TM $_{2\text{-}2\text{-}2}$	TE $_{2\text{-}2\text{-}8}$

Interfering Modes :

TE $_{4\text{-}1\text{-}10}$	TM $_{0\text{-}2\text{-}10}$	TE $_{0\text{-}2\text{-}8}$
TE $_{1\text{-}2\text{-}10}$	TM $_{0\text{-}1\text{-}13}$	TM $_{0\text{-}2\text{-}8}$
TE $_{0\text{-}2\text{-}7}$	TE $_{5\text{-}1\text{-}9}$	TE $_{2\text{-}1\text{-}13}$
TM $_{0\text{-}2\text{-}7}$	TM $_{2\text{-}1\text{-}11}$	TE $_{3\text{-}1\text{-}12}$
TE $_{1\text{-}1\text{-}13}$		TM $_{3\text{-}1\text{-}13}$

The minimum energy differences which may be responsible for the interaction of the different undesired modes with the desired one (TE $_{0\text{-}1\text{-}12}$) have been calculated from (1) and (3) and are given in the following table:

Mode	$k'_{l,m}$ or $k_{l,m}$	Minimum energy difference $\times 10^{18}$ ergs.	Mode	$k'_{l,m}$ or $k_{l,m}$	Minimum energy difference $\times 10^{18}$ ergs.
$TE_{0\text{-}1\text{-}12}$	3.8317	0	$TE_{0\text{-}2\text{-}7}$	7.0156	3.012
$TM_{1\text{-}1\text{-}12}$	3.8317	0	$TM_{0\text{-}2\text{-}7}$	5.5201	4.845
$TE_{6\text{-}16}$	7.503	0.196	$TE_{1\text{-}1\text{-}13}$	1.8412	0.229
$TE_{3\text{-}24}$	8.0152	0.164	$TM_{0\text{-}2\text{-}10}$	5.5201	0.262
$TM_{2\text{-}2\text{-}0}$	8.4172	0.098	$TM_{0\text{-}1\text{-}13}$	2.4048	0.262
$TM_{2\text{-}2\text{-}1}$	8.4172	0.033	$TE_{5\text{-}1\text{-}9}$	6.4156	0.360
$TM_{4\text{-}1\text{-}6}$	7.5883	0.098	$TM_{2\text{-}1\text{-}11}$	5.1356	0.589
$TM_{2\text{-}2\text{-}2}$	8.4172	0.229	$TE_{0\text{-}2\text{-}8}$	7.0156	0.655
$TE_{7\text{-}1\text{-}1}$	8.5778	0.524	$TM_{0\text{-}2\text{-}8}$	5.5201	3.404
$TE_{1\text{-}3\text{-}1}$	8.5363	0.393	$TE_{2\text{-}1\text{-}13}$	3.0542	0.982
$TE_{2\text{-}2\text{-}8}$	6.7061	0.229	$TE_{3\text{-}1\text{-}12}$	4.2012	0.556
$TE_{4\text{-}1\text{-}10}$	5.3176	0.753	$TM_{3\text{-}1\text{-}9}$	6.3802	0.262
$TE_{1\text{-}2\text{-}10}$	5.3314	0.687			

The second column of the table gives the different roots $k'_{l,m}$ and $k_{l,m}$ corresponding to the TE and the TM modes respectively in the echo-box. It will be observed from the above table that the difference in the minimum energy level between the two companion modes is zero. This means that very little energy is necessary to bring about the coupling of the two modes, or in other words, if the cavity is excited in the $TE_{0\text{-}1}$ mode, the $TM_{1\text{-}1}$ mode will be invariably associated with this mode or vice versa. It will however be interesting to study the interaction between these two modes with the help of the field equations.

Interaction of Modes

It has been shown by Wien (1897) that the interaction between the free vibrations of two resonators depends on the coupling coefficient and the ratio of the resonance frequencies of the two resonators. We shall apply the circuit concept of magnetic induction to calculate the mutual energy of interaction between the two companion modes $TE_{0\text{-}1}$ and $TM_{1\text{-}1}$. The total energy of the two modes is given (Smythe, 1950) in m.k.s. rationalised units as follows:

$$W = \frac{1}{2\mu}\int_V \left(B_1 + B_2\right) \cdot \left(B_1 + B_2\right) dv$$

$$= \frac{\mu}{2}\left[\int_V H_1^2\, dv + 2\int_V H_1 . H_2\, dv + \int_V H_2^2\, dv\right]$$

The first and the last term gives the energy stored in the desired and the companion modes respectively. The second term then gives the energy used in bringing the two modes into interaction. Or, in other words, the mutual energy between the two coupled modes is given by

$$W_{1,2} = \mu\int_V H_1 . H_2\, dv$$

or, for the cylindrical cavity having radius a and length L, the expression for the mutual energy is

$$W_{1,2} = \mu \int_0^a \int_0^{2\pi} \int_0^L H_1 . H_2\, r\, dr\, d\theta\, dz \tag{4}$$

The expressions for H_1 and H_2 for the two modes $TE_{0\text{-}1}$ and $TM_{1\text{-}1}$ respectively can be obtained from the field equations as follows (Chatterjee, 1952)

$$H_1 = \sqrt{\frac{k_3^2}{k^2}\left[J_0'(k_1 r)\right]^2 \cos^2 k_3 z + \frac{k_1^2}{k^2} J_0^2(k_1 r)\sin^2 k_3 z} \tag{5}$$

$$H_2 = \sqrt{\frac{J_1^2(k_1 r)}{k_1^2 r^2}\sin^2\theta \cos^2 k_3 z + \left[J_1'(k_1 r)\right]^2 \cos^2\theta \cos^2 k_3 z} \tag{6}$$

Substituting (5) and (6) in (4), using binomial expansions, making first order approximations and rearranging terms the following energy expression responsible for the interaction is obtained:

$$W_{1,2} = \frac{\mu k_1^3}{4\,kk_3} \int_0^a \int_0^L \int_0^{2\pi} \frac{r\,[J_1'(k_1r)]^2\,J_0^2(k_1r)}{J_1^2\,(k_1r)} \sin^2 k_3z \; \frac{\cos^2\theta}{\sin^2\theta} \; r\,d\theta\,dr\,dz$$

$$= -\frac{2\pi\mu\,k_1^3}{4\,kk_3}\left[\frac{L}{2} - \frac{\sin 2\,k_3L}{4\,k_3}\right]\int_0^a r^2\,\frac{[J_1'(k_1r)]^2\,J_0^2\,(k_1r)}{J_1^2\,(k_1r)}\,dr \qquad (7)$$

As the values of the arguments for the different Bessel functions in (7) are greater than unity, the following approximations (Dwight, 1949) can be made

$$J_0(k_1r) = \left(\frac{2}{\pi k_1 r}\right)^{\frac{1}{2}}\left[\cos\left(k_1r - \frac{\pi}{4}\right) + \frac{1}{8k_1r}\sin\left(k_1r - \frac{\pi}{4}\right)\right]$$

as $P_0\,(k_1r) \approx 1$ and $Q_0(k_1r) \approx -\dfrac{1}{8k_1r}$

$$J_1\,(k_1r) = \left(\frac{2}{\pi k_1 r}\right)^{\frac{1}{2}}\left[\cos\left(k_1r - \frac{\pi}{2} - \frac{\pi}{4}\right) - \frac{3}{8k_1r}\sin\left(k_1r - \frac{\pi}{2} - \frac{\pi}{4}\right)\right]$$

as $P_1\,(k_1r) \approx 1$ and $Q_1\,(k_1r) = \dfrac{3}{8k_1r}$

$$J_1'\,(k_1r) = -\left(\frac{2}{\pi k_1 r}\right)^{\frac{1}{2}}\left[\sin\left(k_1r - \frac{\pi}{2} - \frac{\pi}{4}\right) + \frac{7}{8k_1r}\cos\left(k_1r - \frac{\pi}{2} - \frac{\pi}{4}\right)\right]$$

as $P_1^{(1)}\,(k_1r) \approx 1$

and $Q_1^{(1)}\,(k_1r) = \dfrac{7}{8k_1r}$

Introducing the above approximations and after some simplifications, eqn. (7) reduces to the following:

$$W_{1,2} \simeq -\frac{2\pi\,\mu\,k_1^3}{4\,k\,k_3}\left[\frac{L}{2} - \frac{\sin 2\,k_3\,L}{4\,k_3}\right] \times 128\,\frac{k_1}{9\pi}\left[\frac{a^4}{8} + \int_0^a \frac{r^3}{2}\sin 2\,k_1\,r\,dr\right]$$

or,

$$W_{1,2} \simeq -\frac{2\pi\,\mu\,k_1^3}{4\,k\,k_3}\left[\frac{L}{2} - \frac{\sin 2\,k_3L}{4\,k_3}\right] \times \frac{128\,k_1}{9\,\pi}\left[\frac{a^4}{8} - \frac{1}{4\,k_1}a^3\cos 2\,k_1a\right.$$

$$+ \frac{3}{8\,k_1^2} a^2 \sin 2\,k_1 a + \frac{3}{8\,k_1^3} a \cos 2\,k_1 a - \frac{3}{16\,k_1^4} \sin 2\,k_1 a \Big] \qquad (8)$$

gives the mutual energy responsible for interaction between the two companion modes. The interaction can also be studied in terms of the coupling coefficient between the two modes.

Coupling Coefficient

The coupling coefficient between the two modes can be defined as

$$k' = \frac{W_{1,2}}{\sqrt{W_1 W_2}} \qquad (9)$$

where W_1 and W_2 represent the energies stored in the $TE_{0\text{-}1}$ and $TM_{1\text{-}1}$ modes respectively and $W_{1,2} = W_{2,1}$ represents the mutual energy or energy interchanged between the two modes.

The maximum energies stored in the electric field of the resonator operating in the TE and TM modes are given by the following expressions (Chatterjee, loc. cit.).

$$W_1 = \frac{\epsilon}{2} \int_0^a \int_0^L \int_0^{2\pi} r \left[J_0' \,(k_1 r) \right]^2 \sin^2 k_3 z \; dr\, d\theta\, dz$$

$$= \frac{\epsilon}{2} \frac{\pi a^2 L}{4} \epsilon\, J_0^2 \,(k_{01}) \qquad (10)$$

$$W_2 = \frac{\epsilon}{2} \Bigg[\frac{k_3^2}{k^2} \int_0^a \int_0^L \int_0^{2\pi} r \,[J_1' \,(k_1 r)]^2 \cos^2 \theta \sin^2 k_3 z \; dr\, d\theta\, dz$$

$$+ \frac{k_3^2}{k^2} \int_0^a \int_0^L \int_0^{2\pi} r \frac{J_1^2 \,(k_1 r)}{k_1^2 r^2} \sin^2 \theta \sin^2 k_3 z \; dr\, d\theta\, dz$$

$$= \frac{k_1^2}{k^2} \int_0^a \int_0^L \int_0^{2\pi} r\, J_1^2 \,(k_1 r) \cos^2 \theta \cos^2 k_3 z \; dr\, d\theta\, dz \Bigg]$$

$$= \epsilon \frac{L}{4} \left[\frac{\pi a^2}{2} - \frac{\pi k_3^2}{k^2 k_1^2} + \frac{\pi k_3^2}{k^2 k_1} \right] J_0^2 \,(k_{11}) \qquad (11)$$

The coupling coefficient can then be written from (8), (9), (10), (11) as follows:

$$k' = -256\mu\, k_1^4 \left[\frac{L}{2} - \frac{\sin 2k_3 L}{4k_3}\right]\left[\frac{a^4}{8} - \frac{1}{4k_1} a^3 \cos 2k_1 a + \frac{3}{8k_1^2} a^2 \sin 2k_1 a + \frac{3}{8k_1^3} a \cos 2k_1 a - \frac{3}{16k_1^4} \sin 2k_1 a\right]$$

$$\div\; 9a\,\epsilon L\, k k_3\, J_0(k_{01})\, J_0(k_{11}) \sqrt{\pi\left(\frac{\pi a^3}{2} - \frac{\pi k_3^2}{k^2 k_1^2} + \frac{\pi k^2_3}{k_1 k^2}\right)} \qquad (12)$$

Illustrations

The dimensions of the 9 Kmc echo-box as found from the centre of the expanded mode chart are $L \approx 21.6$ cms. and $a \approx 4.45$ cms. For this echo-box the coupling coefficient between the two companion modes, calculated from (12) is 0.11. This shows that any two modes which have identical frequencies need very little coupling for interaction. The two companion modes as they are loosely coupled will have their coupled frequencies equal to the uncoupled frequencies and the energy flow will be equally divided between the two modes. It can however be shown that the other undesired modes whose resonant frequencies are further apart from that of the desired mode require a much stronger coupling for interaction. When any two modes are strongly coupled, their coupled frequencies will be different from uncoupled frequencies and in this case the energy flow will not be equally divided between the two modes.

Total Energy of a Resonator

The total minimum energy of the above echo-box is $\frac{1}{2} \sum_{n\,(\text{definite})} h\nu$ where the summation is taken over all the twenty-five modes indicated within the operating area of the mode chart. In this case the highest frequency is limited to ν_m and the sum has a definite value and can be replaced by the following integral:

$$\frac{1}{2}\, h \int_{\nu=0}^{\nu=\nu_m} \nu\, \rho(\nu)\, d\nu$$

where $\rho(\nu)\,d\nu$ represents the distribution of frequencies between ν and $\nu + d\nu$. It is only under such circumstances that we can attach physical significance to the total energy of the resonator. But, in general when a lossless cavity is excited it can support an infinite number of modes. In this case the total energy of the resonator is $\frac{1}{2}\sum_{n} h\nu$ where the summation is extended over an infinite number of modes. The series becomes divergent and the total energy of the resonator loses its physical significance.

Though the sum is divergent and devoid of physical significance the difference in the minimum energy in the two states of the resonator can have some physical significance as discussed below. Under practical operating condition, the distance between the two end plates of a cavity resonator has to be adjusted for securing resonance to a particular frequency. If we consider any two positions of resonance as the two states of the resonator, then the change, if there is any, in the zero-point energy of the resonator in the two states

$$\delta W_0 = \frac{1}{2}\sum_{n,1} h\nu \sim \frac{1}{2}\sum_{n,2} h\nu$$

of the resonator can be written as

$$\delta W_0 = \lim_{l \to 0} \frac{1}{2}\left[\sum_{n,1} h\nu e^{-l\nu} \sim \sum_{n,2} h\nu e^{-l\nu}\right]$$

where l is the distance between the two infinitely conducting plates. Then the change δW_0 in the zero-point energy can have some physical significance as the divergent series has been converted into a convergent one by the introduction of the convergence factor $e^{-l\nu}$. The above expression for the change in the zero-point energy can be interpreted as giving rise to a zero-point electro-magnetic pressure (Casimir, 1948)

$$P = 0.013 \cdot \frac{1}{l_\mu^4}\ \text{dynes/cm}^2$$

exerted by the electro-magnetic wave under the restriction that the skin depth $< l$. This means that the change in the zero-point energy, if there is any, in the two states of the resonator gives rise to the concept of a force of attraction between the two end plates due to the presence of the electro-magnetic waves present inside the cavity.

The above discussion is based on the assumption that the cavity is lossless and as such there is no damping of the modes. For finite conductivity of the boundary walls of the cavity the different modes will suffer damping to a different degree. For instance, it can be shown with the help of the field equations that the TM mode will have higher attenuation than the TE mode. Consequently, under such circumstances the dicussions made above as regards the difference in the zero-point energy of any two modes will not hold good.

References

Kinzer, J. P. ... *B.T.L. case* 23458-5 (1943), Jan. 8, *B.S.T.J.* (1946), 25, 424

Smythe, W. R. ... *Static and Dynamic Electricity* (1950), p. 311, McGraw Hill Book Company, Inc.

Wien, M. ... *Ann. d. Physik. u. Chemie.* (1897), 61, 151.

Chatterjee, S. K. ... *Jour. Ind. Inst. Sci.*, (1952), 34, 43; *ibid*, 77.

Dwight, H. B. ... *Tables of Integrals and other Mathematical Data*, (1949), p. 178, MacMillan Co.

Casimir, H. B. G. ... *Proc. Kon. Ned. Akad. Wentsch. Amsterdam* (1948), 51, 793