

MATRIX ANALYSIS OF UNSYMMETRICAL MACHINES

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Abstract

The matrix method of analysis is applied to the complete analysis of an unsymmetrical machine by considering it as a system of coupled circuits whose relative positions can be varied. The problem thus reduces itself in evaluating at constant speed, the impedance dyadic or matrix Z , which, operating on the current vector, gives rise to the voltage vector. The method is general and is applicable to any type of machine. The final equations obtained by the method used in this paper are shown to agree with those obtained by Puchstein and Lloyd, by Burian, as also by Lyon and Kingsley.

The complete analysis of a single phase machine with two asymmetric windings on the stator and a balanced rotor was first obtained by Puchstein and Lloyd¹, and later by Lyon and Kingsley², and Burian³. Puchstein and Lloyd employed the double revolving field theory, while Lyon and Kingsley, and Burian made use of symmetrical components and cross field theory respectively. In the first two papers, the method of approach had been mainly to resolve the air gap flux into forward and backward rotating fields. An entirely different approach to this problem is to consider the machine as a system of coupled circuits, whose relative positions can be varied. Based on this method, called the matrix or dyadic analysis, the problem reduces itself in evaluating for any type of machine, at constant speed, the impedance dyadic or matrix Z which, operating on the current vector gives rise to voltage vector. Except for the slight differences in notation, matrix and dyadic analysis run on identical lines. The rotating machine is ultimately reduced, at constant speed, to an equivalent stationary network with an asymmetric impedance matrix. This was employed by Sah⁴ for analysing many types of machines including the single phase machine with unsymmetric stator windings.

Symmetrical components can be considered merely as a transformation from phase quantities to sequence quantities and the relationship between phase impedances and sequence impedances obtained.

In this paper, the symmetrical components employed by Lyon and Kingsley⁷ are considered as a suitable type of transformation, to be employed in order to resolve the rotor currents into two systems of balanced currents of frequencies ω_s and $\omega(2-s)$. By properly separating the total inductances into air gap and leakage values, the expressions for the air gap e.m.f. are obtained and the final equations obtained are shown to check with those of Lyon and Kingsley; and also of Puchstein and Lloyd¹. Expression is obtained for torque by the matrix method and shown to agree with that of Lyon and Kingsley.

For the special case, where the windings are in quadrature, the relationship between A, B, C, D the phase impedances and Z_M^+ , Z_M^- , Z_s^+ , Z_s^- , (sequence impedances employed by Suhr⁶) are obtained.

A simple but direct method of application of the matrix method to analyse this machine by the use of cross field theory is also developed at the end. The final performance equations thus obtained are identified with those of Burian⁸.

Assumptions.—The following assumptions are made throughout the treatments in the paper.

1. Hysteresis and eddy current losses are neglected.
2. The air gap is uniform.
3. The windings are so distributed that the mutual inductance between any two windings varies as the cosine of the angle between them. (This is equivalent to the assumption that the air gap flux is sinusoidally distributed).
4. The resistances and inductances have got constant values.
5. The saturation effects are neglected.
6. Friction and windage losses are neglected.

Voltages and currents are represented as row matrices and column matrices respectively. Thus they can be represented as vectors in space. The instantaneous power is represented by the matrix product of voltage and current and corresponds to the scalar product. The impedance matrix Z in the present case is a 2-matrix which operates on the current giving rise to the applied voltage. At steady state, when each of the components of a vector is a sinusoidal quantity, it is represented by a complex number.

Thus \bar{E}_a denotes the vector (E_a, E_b) whereas \dot{E}_a denotes the complex number representing the component $E_a \cos(\omega t + \phi)$. \bar{E}_a

ERRATA

Page 73	Line 19	V_b	should read	\dot{V}_b
	21	I_y	„	\dot{I}_y
	23	x_{la}, x_{lb}	„	$X_{la} X_{lb}$
	27	x_{la}	„	X_{la}
	29	x_{lb}	„	X_{lb}
	31	x_{ab}	„	X_{ab}
	32	x_{ab}	„	X_{ab}
x_{la}, x_{lb}	=	leakage reactances of a and b .		
$X_m = \omega M$	=	mutual inductive reactance of windings a and either of rotor windings when they are in line.		
$X_a = \omega L_a$	=	Total inductive reactance of winding a		
	=	$c X_m + x_{la}$.		
$X_b = \omega L_b$	=	Total inductive reactance of winding b		
	=	$n^2 c X_m + x_{lb}$.		
$X_{ab} = \omega M_{ab}$	=	Mutual inductive reactance a and b		
x_{ab}	=	mutual leakage reactance of a and b		
X_{ab}	=	$n c X_m \cos \alpha + x_{ab}$ (because of assumption 3)		
r_x	=	rotor resistance per winding		

- $s = \text{slip} = \frac{\omega - \nu}{\omega}$
 $S = \frac{\nu}{\omega} = \text{Speed of the rotor expressed as a fraction of synchronous speed.}$
 $X_x = \omega L_x = \text{Total reactance of a rotor phase.}$
 $m_{ax} = \text{instantaneous mutual impedance dyadic between the stator and rotor.}$
 $p = d/dt.$

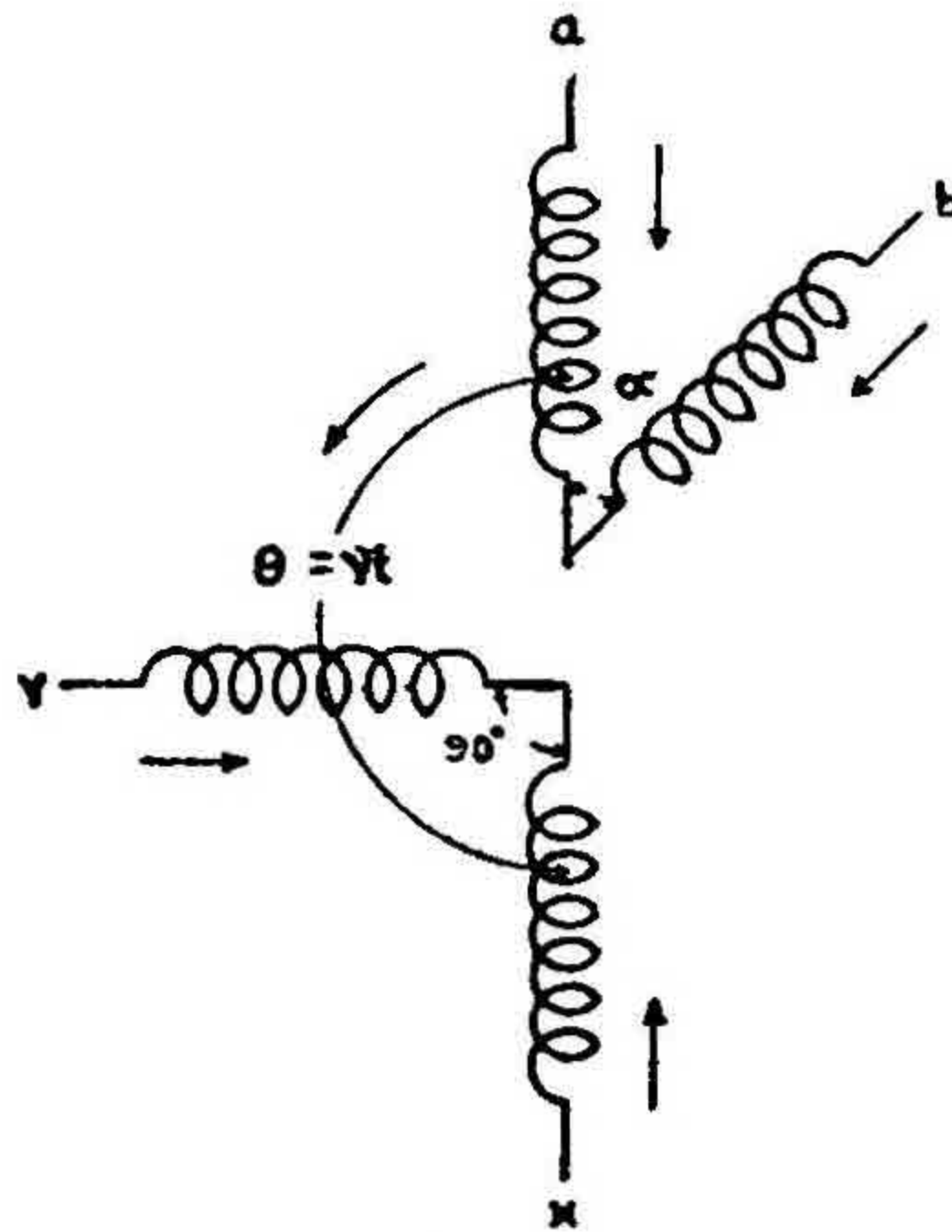


FIG. 1

$$\bar{V}_a = Z_{aa} \cdot \bar{I}_a + p (m_{ax} \cdot \bar{I}_x) \quad (1)$$

Z_{aa} is the matrix representing the self-impedance of the stator and is given by

$$Z_{aa} = \begin{bmatrix} r_a + L_a p & p M_{ab} \\ p M_{ab} & r_b + L_b p \end{bmatrix} \quad (2)$$

$$\bar{I}_x = -y_{xx} p (m_{xa} \cdot \bar{I}_a) \quad (3)$$

y_{xx} is the admittance of the rotor and is given by

$$y_{xx} = \frac{1}{r_x + L_x p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (4)$$

Substituting (3) in (1)

$$\bar{V}_a = Z \bar{I}_a \tag{5}$$

where $Z = Z_{aa} - p m_{ax} y_{xx} p m_{xa}$ (6)

and is called the instantaneous impedance matrix.

From Fig. 1, by inspection, is obtained

$$m_{ax} = M \begin{bmatrix} \cos \theta & \sin \theta \\ n \cos (\theta + \alpha) & n \sin (\theta + \alpha) \end{bmatrix} \tag{7}$$

and m_{xa} is its transpose.

$$\begin{bmatrix} I_x \\ I_y \end{bmatrix} = - \frac{p M}{r_x + L_x p} \begin{bmatrix} \cos \theta & n \cos (\theta + \alpha) \\ \sin \theta & n \sin (\theta + \alpha) \end{bmatrix} \begin{bmatrix} I_a \\ I_b \end{bmatrix} \tag{8}$$

„ 74 Line 1 s „ s
 „ 75 „ 8 m_{xa} „ m_{xa}
 „ 75 Last line inside matrix

$$I_a e^{j\omega t} \text{ and } I_b e^{j\omega t} \quad \dots \quad -j n e^{j\alpha} \quad \dots \quad -j -j n e^{j\alpha}$$

represents the instantaneous quantity of the current. It is customary, however, in alternating current theory, to omit the term $e^{j\omega t}$, since it is common for all the quantities involving the same frequency. But it will be retained here because of the different frequencies which exist in the rotor coils simultaneously. Thus replacing I_a and I_b in eqn. (8) by $\dot{I}_a e^{j\omega t}$ and $\dot{I}_b e^{j\omega t}$, and performing the usual operations in eqn. (8),

$$\begin{bmatrix} I_x \\ I_y \end{bmatrix} = - \frac{p M e^{-j\theta} e^{j\omega t}}{2 (r_x + L_x p)} \begin{bmatrix} 1 & n e^{-j\alpha} \\ j & j n e^{-j\alpha} \end{bmatrix} \begin{bmatrix} \dot{I}_a \\ \dot{I}_b \end{bmatrix} \\ - \frac{p M e^{j\theta} e^{j\omega t}}{2 (r_x + L_x p)} \begin{bmatrix} 1 & n e^{j\alpha} \\ -j & -j n e^{j\alpha} \end{bmatrix} \begin{bmatrix} \dot{I}_a \\ \dot{I}_b \end{bmatrix} \tag{9}$$

Employing the shifting theorem :

$$F(p) \left[e^{kt} f(t) \right] = e^{kt} F(p+k) f(t) \quad (10)$$

and since $\theta = \nu t$ and $\omega - \nu = s\omega$ one gets,

$$\begin{aligned} \begin{bmatrix} I_x \\ I_y \end{bmatrix} &= \frac{-jX_m e^{js\omega t}}{2\left(\frac{r_x}{s} + jX_x\right)} \begin{bmatrix} 1 & ne^{-ja} \\ j & jne^{-ja} \end{bmatrix} \begin{bmatrix} \dot{I}_a \\ \dot{I}_b \end{bmatrix} \\ &+ \frac{\left(-X_m e^{j(2-s)\omega t}\right)}{2\left(\frac{r_x}{2-s} + jX_x\right)} \begin{bmatrix} 1 & ne^{ja} \\ -j & -jne^{ja} \end{bmatrix} \begin{bmatrix} \dot{I}_a \\ \dot{I}_b \end{bmatrix} \end{aligned} \quad (11)$$

From the above equation and by applying the transformation, one finds,

$$\begin{bmatrix} \dot{I}_a \\ \dot{I}_b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{n} e^{-ja} & -\frac{1}{n} e^{ja} \end{bmatrix} \begin{bmatrix} \dot{I}_{a_1} \\ \dot{I}_{a_2} \end{bmatrix} \quad (12)$$

The rotor currents are easily resolved into two systems of currents as follows :

$$\begin{aligned} \begin{bmatrix} I_x \\ I_y \end{bmatrix} &= \frac{X_m \sin \alpha e^{-ja} e^{js\omega t}}{\frac{r_x}{s} + jX_x} \begin{bmatrix} \dot{I}_{a_1} \\ j\dot{I}_{a_1} \end{bmatrix} \\ &- \frac{X_m \sin \alpha e^{ja} e^{j(2-s)\omega t}}{\frac{r_x}{2-s} + jX_x} \begin{bmatrix} \dot{I}_{a_2} \\ -j\dot{I}_{a_2} \end{bmatrix} \end{aligned} \quad (13)$$

The first term represents the balanced system of positive sequence currents of frequency $s\omega$ and the second term that of negative sequence currents of frequency $(2-s)\omega$. \dot{I}_{a_1} and \dot{I}_{a_2} are symmetrical components of currents employed by Lyon and Kingsley.

$$\left. \begin{aligned} \text{Writing } \dot{I}_s &= \frac{X_m \sin \alpha e^{-ja} \dot{I}_{a_1}}{\frac{r_x}{s} + jX_x} \\ \text{and } \dot{I}_{2-s} &= \frac{X_m \sin \alpha e^{ja} \dot{I}_{a_2}}{\frac{r_x}{2-s} + jX_x} \end{aligned} \right\} \quad (14)$$

equation (13) takes the form:

$$\begin{bmatrix} I_x \\ I_y \end{bmatrix} = \begin{bmatrix} \dot{I}_s \\ j \dot{I}_s \end{bmatrix} e^{js\omega t} + \begin{bmatrix} \dot{I}_{2-s} \\ -j \dot{I}_{2-s} \end{bmatrix} e^{j(2-s)\omega t} \quad (15)$$

For calculating the voltages induced in the stator due to rotor reaction (corresponding to the second term of eqn. (6)), let these

„ 76 2nd part of equation (11)

$$\left(-X_m e^{j(2-s)\omega t} \right) \quad \text{„} \quad -jX_m e^{j(2-s)\omega t}$$

„ 77 Lower row of 2nd matrix in (16)

$$= p M \begin{bmatrix} n \cos(\theta + \alpha) & n \sin(\theta + \alpha) \end{bmatrix} \begin{bmatrix} \dot{I}_s e^{js\omega t} & \dot{I}_s e^{js\omega t} \\ j \dot{I}_s e^{js\omega t} - j \dot{I}_{2-s} e^{j(2-s)\omega t} \end{bmatrix} \quad (16)$$

On expressing sines and cosines in terms of exponentials and again using the shifting theorem, one gets:

$$\begin{bmatrix} E_a \\ E_b \end{bmatrix} = e^{j\omega t} j\omega M \begin{bmatrix} \dot{I}_s \\ ne^{ja} \dot{I}_s \end{bmatrix} + e^{j\omega t} j\omega M \begin{bmatrix} \dot{I}_{2-s} \\ ne^{-ja} \dot{I}_{2-s} \end{bmatrix} \quad (17)$$

Therefore, the steady state voltages induced in the stator due to rotor reaction are:

$$\begin{bmatrix} \dot{E}_a \\ \dot{E}_b \end{bmatrix} = \begin{bmatrix} j X_m (\dot{I}_s + \dot{I}_{2-s}) \\ jn X_m (e^{ja} \dot{I}_s + e^{-ja} \dot{I}_{2-s}) \end{bmatrix} \quad (18)$$

The final equations at steady state can now be written as

$$\begin{bmatrix} \dot{V}_a \\ \dot{V}_b \end{bmatrix} = \begin{bmatrix} r_a + jX_a & jX_{ab} \\ jX_{ab} & r_b + jX_b \end{bmatrix} \begin{bmatrix} \dot{I}_a \\ \dot{I}_b \end{bmatrix} + \begin{bmatrix} jX_m (\dot{I}_s + \dot{I}_{2-s}) \\ jnX_m (e^{j\alpha} \dot{I}_s + e^{-j\alpha} \dot{I}_{2-s}) \end{bmatrix} \quad (19)$$

On separating Z_{aa} into air gap and leakage values as

$$Z_{aa} = \begin{bmatrix} Z_m & jX_{ab} \\ jX_{ab} & Z_s \end{bmatrix} + \begin{bmatrix} jX_m c & jnc X_m \cos \alpha \\ jnc X_m \cos \alpha & jn^2 c X_m \end{bmatrix} \quad (20)$$

where $Z_m = r_a + jX_{la}$

$Z_s = r_b + jX_{lb}$

If \dot{E}_{ag} and \dot{E}_{bg} denote the total air gap voltages induced in a and b , these are given by the relations

$$\left. \begin{aligned} \dot{E}_{ag} &= jX_m c \dot{I}_a + jnc X_m \cos \alpha \dot{I}_b + jX_m (\dot{I}_s + \dot{I}_{2-s}) \\ \dot{E}_{bg} &= jnc X_m \cos \alpha \dot{I}_a + jn^2 c X_m \dot{I}_b + jnX_m (e^{j\alpha} \dot{I}_s + e^{-j\alpha} \dot{I}_{2-s}) \end{aligned} \right\} \quad (21)$$

Substituting for \dot{I}_s and \dot{I}_{2-s} in terms of \dot{I}_a and \dot{I}_b as in eqn. (12) and then by use of the relations in eqn. (14) the expression for \dot{E}_{ag} becomes

$$\dot{E}_{ag} = \frac{\dot{I}_a}{2} (Z_1 + Z_2) + \frac{n \dot{I}_b}{2} (Z_1 e^{-j\alpha} + Z_2 e^{j\alpha}) \quad (23)$$

where $Z_1 = jc X_m + \frac{X_m^2}{\frac{r_x}{s} + jX_x}$

$$Z_2 = jc X_m + \frac{X_m^2}{\frac{r_x}{2-s} + jX_x}$$

Eqn. (23) can be further simplified and made to correspond to eqn. (31) of Lyon and Kingsley, if it is written as

$$\begin{aligned} \dot{E}_{ag} &= \frac{Z_1}{2} (\dot{I}_a + n \dot{I}_b e^{-ja}) + \frac{Z_2}{2} (\dot{I}_a + n \dot{I}_b e^{ja}) \\ &= Z_1 \dot{I}'_{a_1} + Z_2 \dot{I}'_{a_2} \\ &= \dot{E}_{a_1} + \dot{E}_{a_2} \end{aligned} \quad (24)$$

\dot{I}'_{a_1} and \dot{I}'_{a_2} correspond to \dot{I}'_{m_1} and \dot{I}'_{m_2} used by Lyon and Kingsley. Since Z_1, Z_2 are easily seen to be positive- and negative-sequence impedances of a two-phase symmetrical motor with the same number of turns in a and b ; \dot{I}'_{a_1} and \dot{I}'_{a_2} can be considered as the vectorial currents which, when flowing through the rotor of the symmetrical two-phase motor, produce the same induced e.m.f. in

„ 79 Last line equation (25)

$$\begin{aligned} \dot{E}_{bg} &= \frac{n Z_1}{2} e^{ja} (\dot{I}_a + n \dot{I}_b e^{-ja}) + \frac{n Z_2 e^{-ja}}{2} (\dot{I}_a + n \dot{I}_b e^{ja}) \\ &= n Z_1 e^{ja} \dot{I}'_{a_1} + n Z_2 e^{-ja} \dot{I}'_{a_2} \\ &= n e^{ja} \dot{E}_{a_1} + n e^{-ja} \dot{E}_{a_2} \end{aligned} \quad (25)$$

Equations (24) and (25) check with eqns. (31) and (32) of Lyon and Kingsley if it is remembered that $x_\phi = c X_m$; $r_r = c^2 r_x$ and $x_r = c^2 X_x - c X_m$; E_{a_1} and E_{a_2} are the symmetrical components of the voltages of Lyon and Kingsley.

The final equations for the line currents \dot{I}_a and \dot{I}_b can be written down as :

$$\left. \begin{aligned} \dot{V}_a &= \dot{I}_a Z_m + j \dot{I}_b X_{ab} + \dot{E}_{ag} \\ \dot{V}_b &= \dot{I}_b Z_s + j X_{ab} \dot{I}_a + \dot{E}_{bg} \end{aligned} \right\} \quad (26)$$

Substituting for \dot{E}_{ag} and \dot{E}_{bg} from (23) and (24) one gets :

$$\left. \begin{aligned} \dot{V}_a &= \frac{\dot{I}_a}{2} (Z_1 + Z_2 + 2Z_m) + \frac{\dot{I}_b}{2} (nZ_1 e^{-ja} + nZ_2 e^{ja} + 2jX_{ab}) \\ &= A \dot{I}_a + C \dot{I}_b \\ \dot{V}_b &= \frac{\dot{I}_a}{2} (nZ_1 e^{ja} + nZ_2 e^{-ja} + 2jX_{ab}) + \frac{\dot{I}_b}{2} (n^2 Z_1 + n^2 Z_2 + 2Z_s) \\ &= B \dot{I}_a + D \dot{I}_b \end{aligned} \right\} (27)$$

It may be noted that A, B, C, D correspond to the phase impedances obtained by Sah.

On remembering that the rotor constants used by Puchstein and Lloyd in their paper are half the test values, one can easily see on simplification that the above equations for line currents check also with those of Puchstein and Lloyd.

Torque :—

$$\begin{aligned} \text{Instantaneous power input} &= \bar{I}_a \cdot \bar{V}_a \\ &= \bar{I}_a \cdot Z_{aa} \cdot \bar{I}_a + \bar{I}_a m_{ax} (p \bar{I}_x) + \bar{I}_a (p m_{ax}) \bar{I}_x \\ &= \bar{I}_a \cdot R_{aa} \cdot \bar{I}_a + \bar{I}_a L_{aa} p \bar{I}_a + \bar{I}_a m_{ax} p \bar{I}_x + \bar{I}_a (p m_{ax}) \bar{I}_x \quad (28) \end{aligned}$$

The first term in the above equation represents the stator copper loss, the second and the third terms represent the power stored as magnetic energy. It is only the fourth term which represents the instantaneous gross output. The term $(p m_{ax}) \cdot \bar{I}_x$ represents the induced e.m.f. in the stator due to rotor magnetic field and corresponds to the back e.m.f. This can be expressed in terms of \dot{I}_s and \dot{I}_{2-s} on substitution for m_{ax} and \dot{I}_x from equations (7) and (15). After simplification, this becomes the vector $\frac{vj X_m}{\omega} \left[\dot{I}_s - \dot{I}_{2-s}, ne^{ja} \dot{I}_s - ne^{-ja} \dot{I}_{2-s} \right]$ at steady state.

The steady state complex expression for power, therefore, is given by the Dot product of (\dot{I}_a, \dot{I}_b) and complex conjugate of the above vector.

∴ Complex power output

$$= \begin{bmatrix} \dot{I}_a \\ \dot{I}_b \end{bmatrix} \cdot \frac{vjX_m}{\omega} \begin{bmatrix} \dot{I}_s^* - \dot{I}_{2-s}^* \\ ne^{-j\alpha}\dot{I}_s^* + ne^{j\alpha}\dot{I}_{2-s}^* \end{bmatrix} \quad (29)$$

Expressing \dot{I}_a and \dot{I}_b in terms of \dot{I}_{a1} and \dot{I}_{a2} with the help of eqns. (12) and (14), and simplifying and taking the real part, the torque in synchronous watts is given by the expression

Torque

$$= 2X_m^2 \sin^2\alpha \left[\frac{|I_{a1}|^2 r_x}{s \left\{ \left(\frac{r_x}{s} \right)^2 + X_x^2 \right\}} - \frac{|I_{a2}|^2 r_x}{(2-s) \left\{ \left(\frac{r_x}{2-s} \right)^2 + X_x^2 \right\}} \right] \quad (30)$$

,, 81 First equation in (32)

$$\left(A - \frac{C}{n} \right) \dot{I}_{a2} \quad ,, \quad \left(A - \frac{jC}{n} \right) \dot{I}_{a2}$$

motor discussed by Suhr, eqn. (12) reduces to :

$$\left. \begin{aligned} \dot{I}_a &= \dot{I}_{a1} + \dot{I}_{a2} \\ \dot{I}_b &= \frac{j}{n} (\dot{I}_{a1} - \dot{I}_{a2}) \end{aligned} \right\} \quad (31)$$

Substituting (31) in eqn. (27) we get :

$$\dot{V}_a = \left(A + \frac{jC}{n} \right) \dot{I}_{a1} + \left(A - \frac{C}{n} \right) \dot{I}_{a2} \quad (32)$$

and since in this case $B = -C$,

$$\dot{V}_b = \left(\frac{jD}{n} - C \right) \dot{I}_{a1} - \left(C + \frac{jD}{n} \right) \dot{I}_{a2}$$

Comparing the above equations with (13a) and (14a) of Suhr, the following relations can be written :

$$Z_M^+ = A + \frac{jC}{n}; \quad Z_M^- = A - \frac{jC}{n}$$

$$Z_S^+ = D + jnC; \quad Z_S^- = D - jnC$$

When $n = 1$, this reduces to the case of 2-phase motor where :

$$Z_M^+ = Z_S^+ = Z_1 = A + jC;$$

$$Z_M^- = Z_S^- = Z_2 = A - jC;$$

The above relations can be verified on substituting the values for the various impedances.

Crossfield Theory.—The above method of approach to the problem can also be utilised to analyse the same machine from crossfield theory. The fundamental assumption in the crossfield theory is to consider the rotor to be equivalent to two short circuited coils in quadrature as shown in Fig. 2, stationary in space with its conductors having an instantaneous velocity.

The physical equivalent of this is a balanced rotor with 2 sets of brushes as shown in Fig. 3.

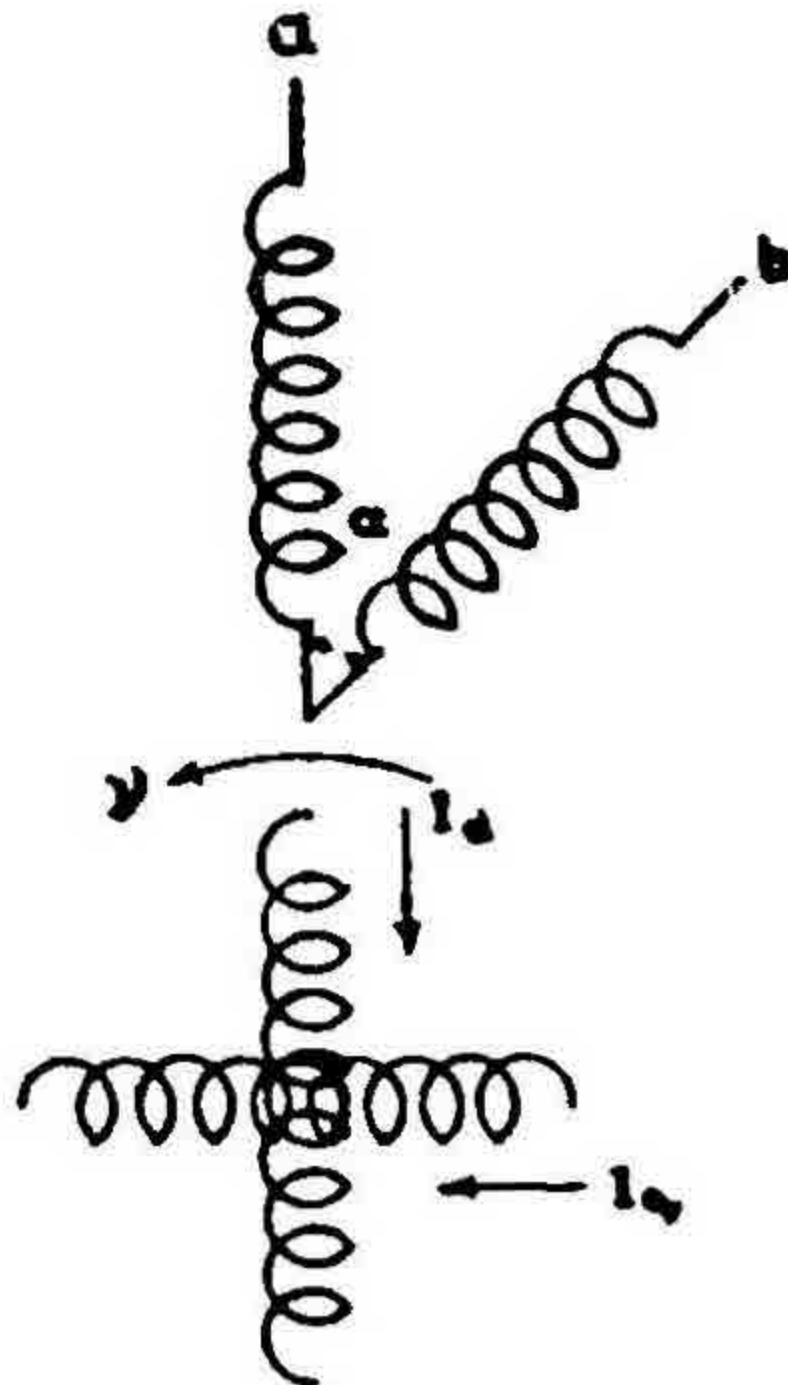


FIG. 2

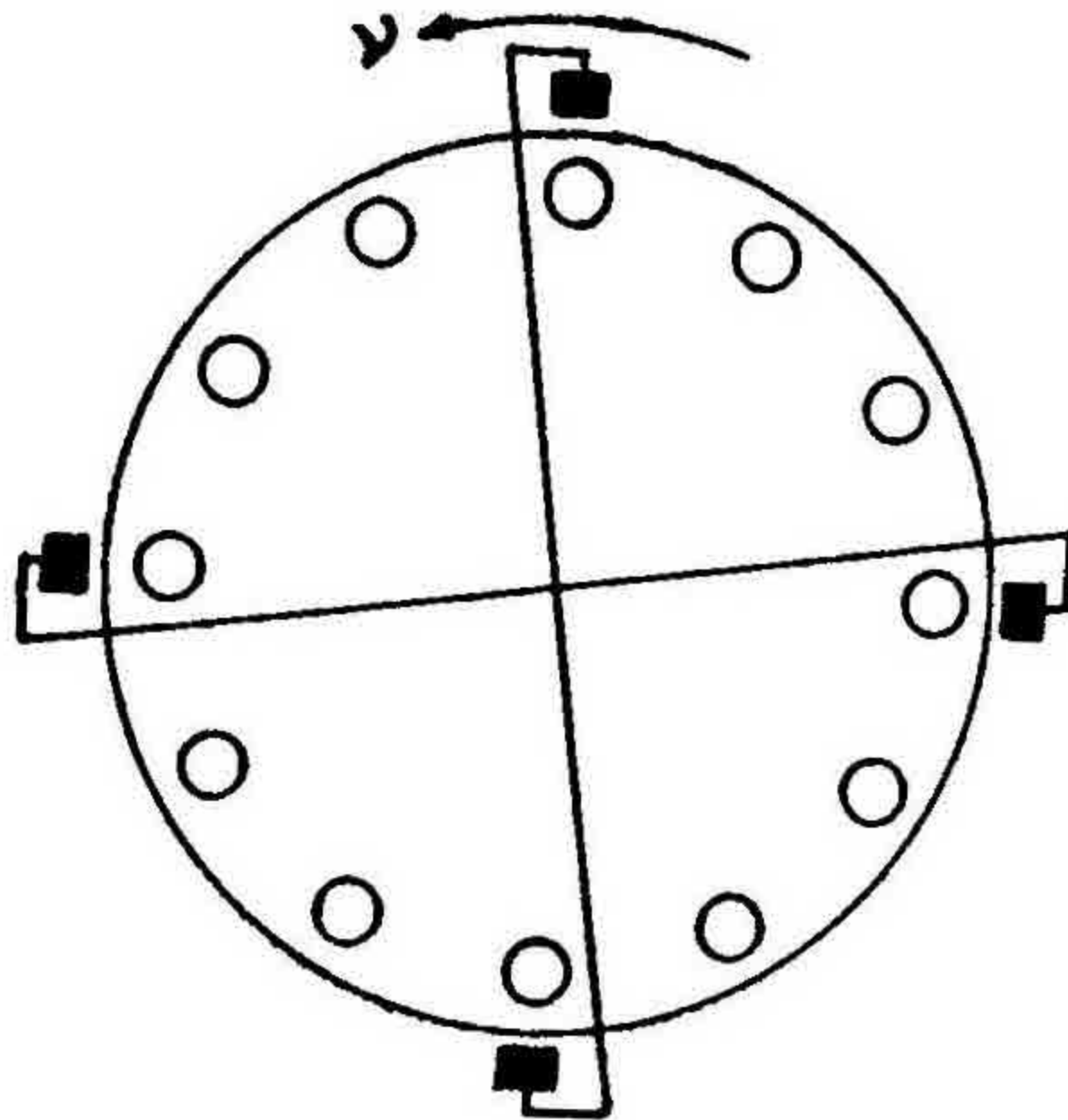
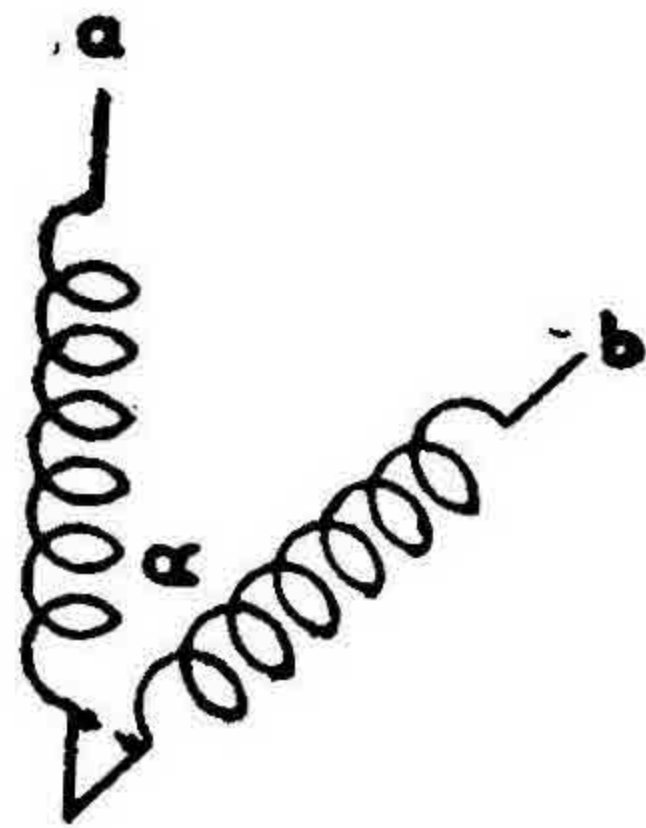


FIG. 3

$$\left. \begin{aligned} \bar{V}_a &= Z_{aa} \cdot \bar{I}_a + p (m_{ad} \bar{I}_d) \\ 0 &= Z_{dq} \cdot \bar{I}_d + p (m_{da} \bar{I}_a) \end{aligned} \right\} \quad (33)$$

where, as usual, \bar{I}_a and \bar{V}_a represents the vectors (I_a, I_b) and (V_a, V_b) . \bar{I}_d represents the vector (I_d, I_q) . Z_{dq} is the self-impedance of the coils (d, q) .

The above equations can be written as

$$\left. \begin{aligned} \bar{V}_a &= Z_{aa} \cdot \bar{I}_a + (p m_{ad}) \cdot \bar{I}_d + m_{ad} (p \bar{I}_d) \\ 0 &= Z_{dq} \cdot \bar{I}_d + (p m_{da}) \cdot \bar{I}_a + m_{da} (p \bar{I}_a) \end{aligned} \right\} \quad (34)$$

The second term in both equations represents the voltage induced due to the instantaneous variation of mutual inductance and is called the speed e.m.f. The third term represents the voltage induced due to instantaneous variation of currents and is the transformer e.m.f.

Since the field produced by the coils d and q is stationary as far as the stator is concerned, m_{ad} is constant and is equal to

$$m_{ad} = M \begin{bmatrix} 1 & 0 \\ n \cos \alpha & n \sin \alpha \end{bmatrix} \quad (35)$$

m_{da} is the transpose of the above matrix. pm_{da} , however, is given by differentiation of m_{xa} given by (7) and setting $\theta = 0$. This is because of the relative velocity of conductors of d and q with reference to the stator.

$$\therefore (p m_{da}) = M \begin{bmatrix} 0 & -n v \sin \alpha \\ v & n v \cos \alpha \end{bmatrix} \quad (36)$$

The term $Z_{dq} \cdot \bar{I}_d$ in (34) can be expanded as follows :

$$\begin{aligned} Z_{dq} \cdot \bar{I}_d &= \begin{bmatrix} r_x + p L_x & p M_{dq} \\ p M_{dq} & r_x + p L_x \end{bmatrix} \begin{bmatrix} I_d \\ I_q \end{bmatrix} \\ &= \begin{bmatrix} r_x + p L_x & 0 \\ 0 & r_x + p L_x \end{bmatrix} \begin{bmatrix} I_d \\ I_q \end{bmatrix} \\ &+ \begin{bmatrix} 0 & p M_{dq} \\ p M_{dq} & 0 \end{bmatrix} \begin{bmatrix} I_d \\ I_q \end{bmatrix} \end{aligned} \quad (37)$$

The second term in the above equation can be written as

$$m_{dq} (p \bar{I}_d) + (p m_{dq}) \cdot \bar{I}_d$$

where

$$m_{dq} = \begin{pmatrix} 0 & M_{dq} \\ M_{dq} & 0 \end{pmatrix} \quad (38)$$

But $M_{dq} = 0$ since d and q are in quadrature. $(p M_{dq})$, however, is different from zero because of the relative velocity of the conductors in either of the windings d and q . It is obtained by setting $\theta = 90$ deg. in (pM_{xy}) where

$$M_{xy} = L_x \begin{bmatrix} 0 & \cos \theta \\ \cos \theta & 0 \end{bmatrix} \quad (39)$$

represents the mutual inductance between two coils, similar to d and q , placed at an angle θ .

$$(p m_{dq}) = L_x \begin{bmatrix} 0 & -\nu \\ \nu & 0 \end{bmatrix} \quad (40)$$

$d\theta/dt$ in the above deduction is taken as $-\nu$ in the lower row of the above matrix since the velocity of q as shown in Fig. 2 corresponds to a decrease in the angle between d and q . Thus the instantaneous velocity of conductors tends to decrease the mutual inductance between d and q , whereas it tends to increase the same between q and d . L_x is the mutual inductance between d and q , placed in line with each other, and is taken as equivalent to the self-inductance of either of them.

Substituting for m_{ad} , pm_{ad} , and pm_{dq} in eqn. (34) and remembering that $\nu = \omega s$, and $p = \omega$ at steady state; the final equations are of the form:

$$\left. \begin{aligned} - (r_x + j X_x) \dot{I}_d &= j X_m (\dot{I}_a + n \dot{I}_b \cos \alpha) - S X_x \dot{I}_q \\ &\quad - S X_m n \dot{I}_b \sin \alpha \\ - (r_x + j X_x) \dot{I}_q &= j X_m n \dot{I}_b \sin \alpha + S X_x \dot{I}_d \\ &\quad + S X_m (\dot{I}_a + n \dot{I}_b \cos \alpha) \end{aligned} \right\} \quad (41a)$$

where $S = \nu/\omega =$ speed of the rotor expressed as a fraction of the synchronous speed.

$$\left. \begin{aligned} \dot{V}_a &= (r_a + j X_a) \dot{I}_a + j X_m \dot{I}_d \\ \dot{V}_b &= (r_b + j X_b) \dot{I}_b + j X_m (\dot{I}_d \cos \alpha + \dot{I}_q \sin \alpha) \end{aligned} \right\} \quad (41b)$$

These equations can easily be recognised to be of the same form as those obtained by Burian. It is to be noted, however, that signs of currents \dot{I}_d and \dot{I}_q in this paper are opposite to those taken by Burian. On solving the above equations for \dot{I}_d and \dot{I}_q and substituting for \dot{I}_a and \dot{I}_b in terms of \dot{I}_s and \dot{I}_{2-s} from eqns. (12) and (14) one finds on simplification

$$\left. \begin{aligned} \dot{I}_d &= \dot{I}_s + \dot{I}_{2-s} \\ \dot{I}_q &= j (\dot{I}_s + \dot{I}_{2-s}) \end{aligned} \right\} \quad (42)$$

If \dot{I}_d and \dot{I}_q are expressed in terms of \dot{I}_s and \dot{I}_{2-s} in the last two equations of (41), the final equation of performance obtained by crossfield theory agrees with eqn. (19) obtained from circuit considerations alone.

It will be seen from the treatment developed in the paper that the performance equations of an unsymmetrical machine are obtainable in a more direct and simple manner with the help of matrix algebra as compared with the crossfield theory or the method of symmetrical components. The method is indicated how the equations developed by the matrix method can be ultimately reduced to the form obtained by the other methods. The relative advantages of a particular method depends upon the actual quantities, such as phase quantities, sequence quantities or the direct and quadrature values, in which one is interested. The advantage of the matrix or dyadic method is that all the different aspects of a problem fall under one general treatment, namely, the transformation of co-ordinates. The solutions by the other methods are only particular aspects of the general method of solution.

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