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PROPAGATION OF MICROWAVES THROUGH A CYLINDRICAL METALLIC GUIDE FILLED COAXIALLY WITH TWO DIFFERENT DIELECTRICS—PART II

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Abstract

The field components for the TM mode in a cylindrical metallic guide filled completely with two coaxial dielectrics have been derived from Maxwell's equations. It has been shown that all the other modes except the TM_0 are hybrid. By applying boundary conditions the propagation characteristics for the TM mode have been derived from the field components. Hollow wave guide excited in the TM mode has been treated as a special case.

Introduction

In a recent paper (Chatterjee, 1953) the field components of microwaves (TE_{01}) passing through a cylindrical metallic guide filled completely with two different coaxial dielectrics have been derived from the field equations. The present paper is a continuation of the previous one and presents a report of the theoretical investigations on the propagation characteristics of microwaves (TM mode) passing through the same type of guide. The result indicates that all the other modes except the circularly symmetrical mode TM_0 are hybrid.

Field Components (TM Mode)

Maxwell's equations in cylindrical co-ordinates (r, θ, z) for the TM mode are expressed in m.k.s. rationalised units as follows:

$$\left.\begin{aligned}
&H_z = 0 \\
&-\frac{\partial H_\theta}{\partial z} = j\omega\epsilon\, E_r \\
&\frac{\partial H_r}{\partial z} = j\omega\epsilon\, E_\theta \\
&\frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) - \frac{1}{r}\frac{\partial H_r}{\partial \theta} = j\omega\epsilon\, E_z \\
&\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} = -\,j\omega\mu\, H_r \\
&\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\,j\omega\mu\, H_\theta \\
&\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta} = 0
\end{aligned}\right\} \tag{1}$$

From eq. (1) the following equation is obtained:

$$\frac{\partial^2 H_\theta}{\partial z^2} + \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rH_\theta)\right] + \omega^2\mu\epsilon\, H_\theta - \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial H_r}{\partial \theta}\right] = 0 \tag{2}$$

The electric **E** and the magnetic **H** field intensities are expressed in terms of the Hertz vector Π as follows:

$$\mathbf{H} = \left(\epsilon\frac{\partial}{\partial t} + \sigma\right)\nabla \times \Pi$$

$$\mathbf{E} = \nabla \times \nabla \times \Pi$$

Let us consider propagation in the z direction only, then

$$\Pi_1 = \Pi_2 = 0 \text{ and hence}$$

$$\left.\begin{aligned}
H_r &= \left(\epsilon\frac{\partial}{\partial t} + \sigma\right)\frac{1}{r}\frac{\partial \Pi_z}{\partial \theta} \\
H_\theta &= -\left(\epsilon\frac{\partial}{\partial t} + \sigma\right)\frac{\partial \Pi_z}{\partial r}
\end{aligned}\right\} \tag{3}$$

Substituting (3) in (2) the following differential equation is obtained:

$$\left[\epsilon\frac{\partial}{\partial t} + \sigma\right]\left[\frac{\partial^2}{\partial z^2}\left(\frac{\partial \Pi_z}{\partial r}\right) + \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Pi_z}{\partial r}\right)\right\}\right.$$
$$\left. + \omega^2\mu\epsilon\left(\frac{\partial \Pi_z}{\partial r}\right) + \frac{\partial}{\partial r}\left\{\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\partial \Pi_z}{\partial \theta}\right)\right\}\right] = 0 \tag{4}$$

The time variation is expressed by $e^{j\omega t}$. So,

$$\epsilon \frac{\partial}{\partial t} + \sigma = j\omega\epsilon + \sigma$$

As $j\omega\epsilon + \sigma \neq 0$, the following differential equation is obtained from (4).

$$\frac{\partial^2 \Pi_z}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Pi_z}{\partial r}\right) + \omega^2\mu\epsilon\Pi_z + \frac{1}{r^2}\frac{\partial^2 \Pi_z}{\partial \theta^2} = \text{constant} \tag{5}$$

Let $\pi_z = \mathrm{R}\Theta\mathrm{Z}$ (5 *a*)

where $\mathrm{R} = f(r)$, $\Theta = f(\theta)$ and $\mathrm{Z} = f(z)$ only.

From equations (5) and (5 *a*) and making the constants of integration in (5) zero, the following differential equation is obtained.

$$\frac{1}{\mathrm{R}}\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathrm{R}}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2}\frac{d^2\Theta}{d\theta^2} + \frac{1}{\mathrm{Z}}\frac{d^2\mathrm{Z}}{dz^2} + \omega^2\mu\epsilon = 0 \tag{6}$$

Let $\frac{1}{\mathrm{Z}}\frac{d^2\mathrm{Z}}{dz^2} = \gamma^2$ and $\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -m^2$ (6 *a*)

where γ and m are constants independent of r, θ, z.

From (6 *a*)

$$\mathrm{Z} = {}^{\cosh}_{\sinh}(\gamma z) \text{ and } \Theta = {}^{\cos}_{\sin}(m\theta) \tag{7}$$

From (6) and (6 *a*)

$$\frac{d^2\mathrm{R}}{dr^2} + \frac{1}{r}\frac{d\mathrm{R}}{dr} + \mathrm{R}\left[(\omega^2\mu\epsilon + \gamma^2) - \frac{m^2}{r^2}\right] = 0 \tag{7 a}$$

The equation (7 *a*) when solved gives

$$\mathrm{R} = \mathrm{A}\mathrm{J}_m\left(r\sqrt{\omega^2\mu\epsilon + \gamma^2}\right) + \mathrm{B}\mathrm{Y}_m\left(r\sqrt{\omega^2\mu\epsilon + \gamma^2}\right) \tag{8}$$

From (5 *a*), (7) and (8), Π_z is given as follows:

$$\Pi_z = \left[\mathrm{A}\mathrm{J}_m\left(r\sqrt{\omega^2\mu\epsilon + \gamma^2}\right) + \mathrm{B}\mathrm{Y}_m\left(r\sqrt{\omega^2\mu\epsilon + \gamma^2}\right)\right] {}^{\cos}_{\sin}(m\theta)\, \mathrm{Z}(z) \tag{8 a}$$

(8 *b*)

Let $\mathrm{Z}(z) = e^{-\gamma z}$, $k = \sqrt{\omega^2\mu\epsilon + \gamma^2}$

From (8 *a*) and (8 *b*)

$$\Pi_z = \left[\mathrm{A}\mathrm{J}_m(kr) + \mathrm{B}\mathrm{Y}_m(kr)\right] {}^{\cos}_{\sin}\, m\theta e^{-\gamma z} \tag{9}$$

The components of **H** and **E** are obtained from equations (3), (9) and (10), (1) respectively as follows:

$$\left.\begin{aligned} H_r &= j(j\omega\epsilon+\sigma)\frac{m}{r}\,[AJ_m(kr)+BY_m(kr)]\,{}^{\sin}_{\cos}\, m\theta e^{-\gamma z} \\ H_\theta &= -(j\omega\epsilon+\sigma)\,[AkJ'_m(kr)+BkY'_m(kr)]\,{}^{\cos}_{\sin}\, m\theta e^{-\gamma z} \end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned} E_r &= -\frac{j\gamma(j\omega\epsilon+\sigma)}{j\omega\epsilon}\,[AkJ'_m(kr)+BkY'_m(kr)]\,{}^{\cos}_{\sin}\, m\theta e^{-\gamma z} \\ E_\theta &= -\frac{\gamma(j\omega\epsilon+\sigma)}{j\omega\epsilon}\frac{m}{r}\,[AJ_m(kr)+BY_m(kr)\,{}^{\sin}_{\cos}\, m\theta e^{-\gamma z} \\ E_z &= -\frac{(j\omega\epsilon+\sigma)}{j\omega\epsilon}\,[\{Ak^2J''_m(kr)+Bk^2Y''_m(kr)\} \\ &\quad +\frac{1}{r}\{AkJ'_m(kr)+BkY'_m(kr)\} \\ &\quad -\frac{m^2}{r^2}\{AJ_m(kr)+BY_m(kr)\}]\,{}^{\cos}_{\sin}\, m\theta e^{-\gamma z} \end{aligned}\right\} \quad (10\,a)$$

In the second medium Y_m's have infinite discontinuities in the axial region which is physically inadmissible. The field components for the second medium are given as follows from (10) and (10 *a*)

$$0 \leqslant r \leqslant r_2$$

$$\left.\begin{aligned} H_{z2} &= 0 \\ H_{r2} &= -j(j\omega\epsilon_2+\sigma)\frac{m}{r}\,[A_2J_m(k_2r)]\,{}^{\sin}_{\cos}\, m\theta e^{-\gamma_2 z} \\ H_{\theta 2} &= -(j\omega\epsilon_2+\sigma)\,[A_2k_2J'_m(k_2r)]\,{}^{\cos}_{\sin}\, m\theta e^{-\gamma_2 z} \\ E_{r2} &= -\frac{\gamma_2(j\omega\epsilon_2+\sigma)}{j\omega\epsilon_2}\,[A_2k_2J'_m(k_2r)]\,{}^{\cos}_{\sin}\, m\theta e^{-\gamma_2 z} \\ E_{\theta 2} &= \frac{j\gamma_2(j\omega\epsilon_2+\sigma)}{j\omega\epsilon_2}\frac{m}{r}\,[A_2J_m(k_2r)]\,{}^{\sin}_{\cos}\, m\theta e^{-\gamma_2 z} \\ E_{z2} &= -\frac{(j\omega\epsilon_2+\sigma)}{j\omega\epsilon_2}\,[\{A_2k_2^2J''_m(k_2r)\}+\frac{1}{r}\{A_2k_2J'_m(k_2r)\} \\ &\quad -\frac{m^2}{r^2}\{A_2J_m(k_2r)\}]\,{}^{\cos}_{\sin}\, m\theta e^{-\gamma_2 z} \end{aligned}\right\} \quad (11)$$

In the first medium, the axial region is excluded, so the field components are

$$r_2 \leqslant r \leqslant r_1$$

$$\left.\begin{aligned}
H_{z1} &= 0 \\
H_{r1} &= -j(j\omega\epsilon_1 + \sigma)\frac{m}{r}[A_1J_m(k_1r) + B_1Y_m(k_1r)]\,{}^{\sin}_{\cos}\, m\theta e^{-\gamma_1 z} \\
H_{\theta 1} &= -(j\omega\epsilon_1+\sigma)[A_1k_1J'_m(k_1r) + B_1k_1Y'_m(k_1r)]\,{}^{\cos}_{\sin}\, m\theta e^{-\gamma_1 z} \\
E_{r1} &= -\frac{\gamma_1(j\omega\epsilon_1+\sigma)}{j\omega\epsilon_1}[A_1k_1J'_m(k_1r) + B_1k_1Y'_m(k_1r)]\,{}^{\cos}_{\sin}\, m\theta e^{-\gamma_1 z} \\
E_{\theta 1} &= \frac{j\gamma_1(j\omega\epsilon_1+\sigma)}{j\omega\epsilon_1}\frac{m}{r}[A_1J_m(k_1r) + B_1Y_m(k_1r)\,{}^{\sin}_{\cos}\, m\theta e^{-\gamma_1 z} \\
E_{z1} &= -\frac{(j\omega\epsilon_1+\sigma)}{j\omega\epsilon_1}[\{A_1k_1^2J''_m(k_1r) + B_1k_1^2Y''_m(k_1r)\} \\
&\qquad + \frac{1}{r}\{A_1k_1J'_m(k_1r) + B_1k_1Y'_m(k_1r)\} \\
&\qquad - \frac{m^2}{r^2}\{A_1J_m(k_1r) + B_1Y_m(k_1r)\}]\,{}^{\cos}_{\sin}\, m\theta e^{-\gamma_1 z}
\end{aligned}\right\} \quad (12)$$

The following boundary conditions hold good:

$$E_{\theta 1} = E_{z1} = 0 \qquad \text{at } r = r_1$$

$$H_{\theta 1} = H_{\theta 2} \qquad \text{at } r = r_2$$

$$\epsilon_1 E_{r1} = \epsilon_2 E_{r2} \qquad \text{at } r = r_2 \text{ as } \nabla\cdot\mathbf{D} = 0$$

Assuming $\sigma = 0$ and the attenuation constants $\alpha_1 = \alpha_2 = 0$ for both the media and putting

$$\epsilon = \epsilon_1/\epsilon_2,\ \beta = \beta_1 - \beta_2,\ \gamma_1 = j\beta_1 \text{ and } \gamma_2 = j\beta_2$$

the following equations are obtained from (11), (12) and (13).

$$\begin{aligned}
&A_1J_m(k_1r_1) + B_1Y_m(k_1r_1) = 0 \\
&[A_1\epsilon k_1J'_m(k_1r_2) + B_1\epsilon k_1Y'_m(k_1r_2)]\,e^{-j\beta z} - A_2k_2J'_m(k_2r_2) = 0 \qquad (14) \\
&[A_1k_1J'_m(k_1r_2)\,\epsilon\beta_1 + B_1k_1Y'_m(k_1r_2)\,\epsilon\beta_1]\,e^{-j\beta z} - A_2k_2\beta_2J'_m(k_2r_2) = 0
\end{aligned}$$

In order that A_1, B_1, and A_2 are not zero the determinant of their coefficients in eq. (14) must vanish.

Substituting $k' = k_2/k_1$ and $\beta' = \beta_2/\beta_1$ in eq. (14) the following two equations of A_1 and B_1 in terms of A_2 are obtained.

$$A_1 [J'_m (k_1r_2) Y_m (k_1r_1) - Y'_m (k_1r_2) J_m (k_1r_2)]$$

$$= A_2 \frac{k'}{\epsilon} J'_m (k_2r_2) Y_m (k_1r_1) e^{j\beta z} \quad (15)$$

$$B_1 [Y'_m (k_1r_2) J_m (k_1r_1) - J'_m (k_1r_2) Y_m (k_1r_1)]$$

$$= A_2 \frac{k' \beta'}{\epsilon} J'_m (k_2r_2) J_m (k_1r_1) e^{j\beta z}$$

The equations (15) can be written as

$$A_1 = A_2A_3e^{j\beta z} \text{ and } B_1 = A_2A_4e^{j\beta z} \quad (15\,a)$$

A_2 can be evaluated from the expression for the power flowing through the guide.

The average power flowing through the guide per square meter of surface normal to the axis of the tube is given by the sum of

$$p_{z1} = \tfrac{1}{2} [E_{r1}H^*_{\theta 1} - E_{\theta 1}H^*_{r1}] \text{ in the medium 1}$$

and $$p_{z2} = \tfrac{1}{2} [E_{r2}H^*_{\theta 2} - E_{\theta 2}H^*_{r2}] \text{ in the medium 2.}$$

The total average power transmitted through the tube

$$P_z = \tfrac{1}{2} \iint_{\substack{\text{surface}\\\text{normal}\\\text{to the}\\\text{axis of}\\\text{the tube}}} p_{z1}ds + \tfrac{1}{2} \iint_{\substack{\text{surface}\\\text{normal}\\\text{to the}\\\text{axis of}\\\text{the tube}}} p_{z2}ds$$

The peak power flowing through the tube

$$\hat{P}_z = \int_{r=r_2}^{r_1} \int_{\theta=0}^{2\pi} [E_{r1}H^*_{\theta 1} - E_{\theta 1}H^*_{r1}] r d\theta dr$$

$$+ \int_{r=0}^{r_2} \int_{\theta=0}^{2\pi} [E_{r2}H^*_{\theta 2} - E_{\theta 2}H^*_{r2}] r d\theta dr \quad (16)$$

In the above expression for the peak power flow the values for k_1 and k_2 are

$$k_1^2 = \gamma_1^2 + \omega^2\mu_1\epsilon_1$$

$$k_2^2 = \gamma_2^2 + \omega^2\mu_2\epsilon_2$$

Or, expressed in terms of the phase velocities v_{p1} and v_{p2} and phase constants β_1 and β_2 in the two media

$$k_1^2 = \beta_1^2 \left(\frac{v_{p1}^2}{c_1^2} - 1\right)$$

$$k_2^2 = \beta_2^2 \left(\frac{v_{p2}^2}{c_2^2} - 1\right)$$

where $c_1^2 = 1/\mu_1\epsilon_1$ and $c_2^2 = 1/\mu_2\epsilon_2$

The values for k_1 and k_2 may be real or imaginary depending on the values of v_{p1} and v_{p2} with respect to c_1 and c_2 respectively. However, the method of analysis remains unaltered irrespective of the character of k_1 and k_2. Let us consider that the constants of the two media are such that k_1 is real and k_2 is imaginary. Let $k_2 = jk'_2$. The field components in the second medium, therefore, involve Bessel functions of complex argument and hence it is convenient to replace J's for the second medium by the modified Bessel functions I's.

From eqs. (11), (12), (15 *a*) and (16)

$$\hat{P} = 2\pi A_2^2 e^{-2\gamma_1 z} \frac{\gamma_1 (j\omega\epsilon_1 + \sigma)^2}{j\omega e_1} e^{2j\beta z} [k_1^2 \{ \int_{r_2}^{r_1} A_3^2 (J'_m (k_1 r))^2 r dr$$

$$+ \int_{r_2}^{r_1} A_4^2 (Y'_m (k_1 r))^2 r dr + 2 \int_{r_2}^{r_1} A_3 A_4 J'_m (k_1 r) Y'_m (k_1 r) r dr\}$$

$$- m^2 \{ \int_{r_2}^{r_1} A_3^2 J_m^2 (k_1 r) \frac{1}{r} dr + \int_{r_2}^{r_1} A_4^2 Y_m^2 (k_1 r) \frac{1}{r} dr$$

$$+ 2 \int_{r_2}^{r_1} A_3 A_4 J_m (k_1 r) Y_m (k_1 r) \frac{1}{r} dr\}]$$

$$+ 2\pi A_2^2 e^{-2\gamma_2 z} \frac{\gamma_2 (j\omega\epsilon_2 + \sigma)^2}{j\omega\epsilon_2} [k_2^2 \{ \int_{r=0}^{r_2} j^{2m+2} (I'_m (k'_2 r))^2 r dr\}$$

$$- m^2 \int_{r=0}^{r_2} j^{2m} \frac{1}{r} I^2_m (k'_2 r) dr]$$

as $J_m (k_2 r) = j^m I_m (k'_2 r)$ and $J'_m (k_2 r) = - j^{m+1} I'_m (k'_2 r)$

When $\sigma = 0$, $\gamma_2 = j\beta_2$, $\gamma_1 = j\beta_1$ the expression for $\hat{P}$ shows that it is proportional to some function of $\beta_1, \beta_2, \epsilon_1, \epsilon_2, r_1, r_2$ and k_1, k'_2. So the expression for the peak power flow is

$$\hat{P} = 2\pi A_2^2 \omega [F (\beta_1, \epsilon_1, r_2, r_1, k_1) + F' (\beta_2, \epsilon_2, r_2, k'_2)] \tag{17}$$

So,

$$A_2 = P^{\frac{1}{2}} \tag{18}$$

where

$$P = \left\{\frac{\hat{P}}{2\pi\omega [F + F']}\right\} \tag{18 a}$$

The field components in the two media for the TM mode can then be expressed from (11), (12) and (18) as follows :

Medium 2: $0 \leqslant r \leqslant r_2$

$$H_{z2} = 0$$

$$H_{r2} = -j^{m+1}(j\omega\epsilon_2 + \sigma)\frac{m}{r} P^{\frac{1}{2}} I_m(k'_2 r) \substack{\cos \\ \sin} m\theta e^{-j\beta_2 z}$$

$$H_{\theta 2} = j^{m+1}(j\omega\epsilon_2 + \sigma) k_2 P^{\frac{1}{2}} I'_m(k'_2 r) \substack{\cos \\ \sin} m\theta e^{-j\beta_2 z}$$

$$E_{r2} = j^{m+2}\beta_2 \frac{(j\omega\epsilon_2 + \sigma)}{j\omega\epsilon_2} k_2 P^{\frac{1}{2}} I'_m(k'_2 r) \substack{\cos \\ \sin} m\theta e^{-j\beta_2 z} \tag{19}$$

$$E_{\theta 2} = -j^m \frac{\beta_2(j\omega\epsilon_2 + \sigma)}{j\omega\epsilon_2}\frac{m}{r} P^{\frac{1}{2}} I_m(k'_2 r) \substack{\sin \\ \cos} m\theta e^{-j\beta_2 z}$$

$$E_{z2} = -\frac{j\omega\epsilon_2 + \sigma}{j\omega\epsilon_2} P^{\frac{1}{2}} [\{k_2^2 j^{m+2} I''_m(k'_2 r)\}$$
$$- \frac{1}{r}\{k_2 j^{m+1} I'_m(k'_2 r)\} - \frac{m^2}{r^2}\{j^m I_m(k'_2 r)\}] \substack{\cos \\ \sin} m\theta e^{-j\beta_2 z}$$

Medium 1: $r_2 \leqslant r \leqslant r_1$

$$H_{z1} = 0$$

$$H_{r1} = -j(j\omega\epsilon_1 + \sigma)\frac{m}{r} P^{\frac{1}{2}} [A_3 J_m(k_1 r) + A_4 Y_m(k_1 r)] \substack{\sin \\ \cos} m\theta e^{-j\beta_2 z}$$

$$H_{\theta 1} = -(j\omega\epsilon_1 + \sigma) P^{\frac{1}{2}} [A_3 k_1 J'_m(k_1 r) + A_4 k_1 Y'_m(k_1 r)] \substack{\cos \\ \sin} m\theta e^{-j\beta_2 z}$$

$$E_{r1} = -\frac{j\beta_1(j\omega\epsilon_1 + \sigma)}{j\omega\epsilon_1} P^{\frac{1}{2}} [A_3 k_1 J'_m(k_1 r) + A_4 k_1 Y'_m(k_1 r)] \substack{\cos \\ \sin} m\theta e^{-j\beta_2 z}$$

$$E_{\theta 1} = -\frac{\beta_1(j\omega\epsilon_1 + \sigma)}{j\omega\epsilon_1}\frac{m}{r} P^{\frac{1}{2}} [A_3 J_m(k_1 r) + A_4 Y_m(k_1 r)] \substack{\sin \\ \cos} m\theta e^{-j\beta_2 z} \tag{19a}$$

$$E_{z1} = -\frac{(j\omega\epsilon_1 + \sigma)}{j\omega\epsilon_1} P^{\frac{1}{2}} [\{A_3 k_1^2 J''_m(k_1 r) + A_4 k_1^2 Y''_m(k_1 r)\}$$
$$+ \frac{1}{r}\{A_3 k_1 J'_m(k_1 r) + A_4 k_1 Y'_m(k_1 r)\}$$
$$- \frac{m^2}{r^2}\{A_3 J_m(k_1 r) + A_4 Y_m(k_1 r)\}] \substack{\cos \\ \sin} m\theta e^{-j\beta_2 z}$$

TM_0 mode.—In this case $m = 0$ and the mode is circularly symmetrical. The field components for the TM_0 mode in the two media derived from equations (19) and (19 *a*) and expressed in real parts are:

Medium 2:

$$H_{z2} = H_{r2} = E_{\theta 2} = 0$$

$$H_{\theta 2} = -\ \omega \epsilon_2 k_2 P^{\frac{1}{2}}\, I'_0\,(k'_2 r) \cos \beta_2 z$$

$$E_{r2} = -\ \beta_2 k_2 P^{\frac{1}{2}}\, I'_0\,(k'_2 r) \cos \beta_2 z$$

$$E_{z2} = -\ P^{\frac{1}{2}} \left[\frac{k'_2}{r} I'_0\,(k'_2 r) - k_1^2\, I''_0 (k'_2 r)\right] \cos \beta_2 z \qquad (20)$$

Medium 1:

$$H_{z1} = H_{r1} = H_{\theta 1} = E_{\theta 1} = 0$$

This shows that there is no wave propagation in the first medium. The wave is entirely concentrated in the second medium. It is also evident from (19) and (19 *a*) that in the case of any other TM mode the wave travels in both the media. It may therefore be concluded that it is only in the case of the circularly symmetrical mode TM_0 that the wave travels in the inner dielectric but in the case of any other mode the wave is hybrid.

Hollow Wave Guide

When $\epsilon_2 = \epsilon_1 = \epsilon_{air}$ and $r_2 = r_1$ the problem reduces to the case of a hollow wave guide. In this case

$$k_1 = k_2,\ k' = 1,\ \epsilon = 1,\ \beta' = 1,\ \beta_2 = \beta_1,\ \beta = 0.$$

From (15) the values of A_1 and B_1 can be modified as follows:

$$A_1 = -\ A_2 k_1 r_1\, J'_m\,(k_1 r_1)\, Y_m\,(k_1 r_1)$$

$$B_1 = A_2 k_1 r_1\, J'_m\,(k_1 r_1)\, J_m\,(k_1 r_1) \qquad (21)$$

as $$J'_m\,(k_1 r_1)\, Y_m\,(k_1 r_1) - Y'_m\,(k_1 r_1)\, J_m\,(k_1 r_1) = -\ \frac{1}{k_1 r_1}$$

In the case of the hollow wave guide all Y_m's are eliminated from the solution as Y_m's $\rightarrow -\infty$ as the axis of the guide is approached. Substituting (21) in (12) the E_z component can be written as

$$E_z = -\ \frac{A_1}{r^2} [k_1^2 r^2\, J''_m\,(k_1 r) + k_1 r\, J'_m\,(k_1 r) - m^2\, J_m\,(k_1 r)]$$

$$= A_1 k_1^2\, J_m\,(k_1 r)$$

as (Gray and Mathews, 1922; Dwight, 1949)

$$k_1^2 r^2 J''_m(k_1 r) = (m^2 - m - k_1^2 r^2) J_m(k_1 r) + k_1 r J_{m+1}(k_1 r)$$

and

$$k_1 r J'_m(k_1 r) = m J_m(k_1 r) - k_1 r J_{m+1}(k_1 r)$$

The components for the TM mode in the hollow wave guide are

$$H_z = 0$$

$$H_r = A\omega\epsilon_1 \frac{m}{r} J_m(k_1 r) \cos m\theta \cos \beta_1 z$$

$$H_\theta = A\omega\epsilon_1 k_1 J'_m(k_1 r) \cos m\theta \sin \beta_1 z$$

$$E_r = A k_1 \beta_1 J'_m(k_1 r) \cos m\theta \sin \beta_1 z$$

$$E_\theta = - A\beta_1 \frac{m}{r} J_m(k_1 r) \cos m\theta \cos \beta_1 z$$

$$E_z = - A k_1^2 J_m(k_1 r) \cos m\theta \cos \beta_1 z \tag{22}$$

where

$$A = A_3 P^{\frac{1}{2}}$$

So the hollow wave guide can be treated as a special case of the guide filled coaxially with two dielectrics.

PROPAGATION CHARACTERISTICS

From (12), (13) and (15 *a*)

$$A_1 J_m(k_1 r_1) + B_1 Y_m(k_1 r_1) = 0$$

$$\frac{A_1}{B_1} = - \frac{Y_m(k_1 r_1)}{J_m(k_1 r_1)} = \frac{A_3}{A_4} \tag{23}$$

But J_m and Y_m can be expressed as

$$J_m(k_1 r_1) = \left[\frac{2}{\pi k_1 r_1}\right]^{\frac{1}{2}} \left[P_m(k_1 r_1) \cos\left(k_1 r_1 - \frac{m\pi}{2} - \frac{\pi}{4}\right) - Q_m(k_1 r_1) \sin\left(k_1 r_1 - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right] \tag{23 a}$$

$$Y_m(k_1 r_1) = \left[\frac{2}{\pi k_1 r_1}\right]^{\frac{1}{2}} \left[P_m(k_1 r_1) \sin\left(k_1 r_1 - \frac{m\pi}{2} - \frac{\pi}{4}\right) + Q_m(k_1 r_1) \cos\left(k_1 r_1 - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right] \tag{23 b}$$

where,

$$P_m(k_1r_1) \approx 1 - \frac{(4m^2-1^2)(4m^2-3^2)}{2!\,(8\,k_1r_1)^2} + \ldots.$$

$$Q_m(k_1r_1) \approx \frac{4m^2-1^2}{1!\;8\;k_1r_1} - \ldots.$$

For large $k_1 r_1$, $P_m(k_1r_1) \approx 1$ and $Q_m(k_1r_1) \approx 0$. (23 *c*)

From (23) to (23 *c*) $k_1 = \frac{1}{r_1}\left[\frac{m\pi}{2} + \frac{\pi}{4} + \tan^{-1}\delta\right]$

$$\text{So, } \gamma_1 = \left[\frac{1}{r_1^2}\left\{\frac{m\pi}{2} + \frac{\pi}{4} + \tan^{-1}\delta\right\}^2 - \omega^2\mu_1\epsilon_1\right]^{\frac{1}{2}} \tag{24}$$

where $\delta = A_3/A_4$.

In order that propagation may take place γ_1 must be imaginary.

So, the following condition must be fulfilled.

$$\omega^2\mu_1\epsilon_1 > \frac{1}{r_1^2}\left[\frac{m\pi}{2} + \frac{\pi}{4} + \tan^{-1}\delta\right]^2 \tag{24 a}$$

The expression for γ_1 in (24) may be written as

$$\gamma_1 = \alpha_1 + j\beta_1 = j\sqrt{\omega^2\mu_1\epsilon_1 - \frac{1}{r_1^2}\left[\frac{m\pi}{2} + \frac{\pi}{4} + \tan^{-1}\delta\right]^2}$$

$$\alpha_1 = 0$$

$$\beta_1 = \sqrt{\omega^2\mu_1\epsilon_1 - \frac{1}{r_1^2}\left[\frac{m\pi}{2} + \frac{\pi}{4} + \tan^{-1}\delta\right]^2} \tag{24 b}$$

In order that propagation may take place β_1 must be real and the condition (24 *a*) must be fulfilled. This indicates that there must be a minimum frequency f_c below which propagation will not take place. The critical frequency is given by

$$f_c = \frac{1}{2\pi r_1\sqrt{\mu_1\epsilon_1}}\left[\frac{m\pi}{2} + \frac{\pi}{4} + \tan^{-1}\delta\right] \tag{24 c}$$

For frequencies greater than f_c there will be propagation of the wave down the tube.

A cut off wavelength λ_c may be defined as $\lambda_c = c_1/f_c$, *i.e.*,

$$\lambda_c = 2\pi r_1 \Big/ \left[\frac{m\pi}{2} + \frac{\pi}{4} + \tan^{-1}\delta\right] \tag{24 d}$$

where c_1 is the velocity of a free wave in a medium of constants μ_1 and ϵ_1. The phase velocity of the wave is given by

$$c_{p1} = \omega/\beta_1 = \omega\Big/\left[\omega^2\mu_1\epsilon_1 - \frac{1}{r_1^2}\left\{\frac{m\pi}{2} + \frac{\pi}{4}\tan^{-1}\delta\right\}^2\right]^{\frac{1}{2}} \tag{24 e}$$

The group velocity c_{g1} is

$$c_{g1} \doteqdot 1\Big/\frac{\partial\beta_1}{\partial\omega} = \left[\omega^2\mu_1\epsilon_1 - \frac{1}{r_1^2}\left\{\frac{m\pi}{2} + \frac{\pi}{4} + \tan^{-1}\delta\right\}^2\right]^{\frac{1}{2}}\Big/\omega\mu_1\epsilon_1 \tag{24 f}$$

The wavelength λ_{g1} in the guide is

$$\lambda_{g1} = 2\pi/\beta_1 = 2\pi\Big/\left[\omega^2\mu_1\epsilon_1 - \frac{1}{r_1^2}\left\{\frac{m\pi}{2} + \frac{\pi}{4} + \tan^{-1}\delta\right\}^2\right]^{\frac{1}{2}} \tag{24 g}$$

For all the TM modes except TM_0 the wave travels in both the media. The propagation characteristics in the second medium will be different from those in the first medium. The relation between the propagation characteristics in the second medium with respect to the first are found as follows. From (13), (19) and (19 *a*) the following relation between k_2 and k_1 is obtained

$$\frac{k_2}{k_1} = -\frac{A_3 J'_m(k_1r_2) + A_4 Y'_m(k_1r_2)}{J'_m(k_2r_2)}\epsilon$$

Or,

$$\left(\frac{k_2}{k_1}\right)^{\frac{1}{2}} = 2A_4\frac{\cos(k_1r_1 - k_1r_2)\,\epsilon}{[\pm\sin(k_1r_1 + k_2r_2) - \cos(k_1r_1 - k_2r_2)]} \tag{25}$$

as

$$A_3 = -A_4\frac{Y_m(k_1r_1)}{J_m(k_1r_1)}$$

and for the large arguments

$$J_m(k_1r_1) \approx \left(\frac{2}{\pi k_1r_1}\right)^{\frac{1}{2}}\left[\cos\left(k_1r_1 - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$Y_m(k_1r_1) \approx \left(\frac{2}{\pi k_1r_1}\right)^{\frac{1}{2}}\left[\sin\left(k_1r_1 - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$J'_m(k_1r_2) \approx -\left(\frac{2}{\pi k_1r_2}\right)^{\frac{1}{2}}\left[\sin\left(k_1r_2 - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$J'_m(k_2r_2) \approx -\left(\frac{2}{\pi k_2r_2}\right)^{\frac{1}{2}}\left[\sin\left(k_2r_2 - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$Y'_m(k_1r_2) \approx \left(\frac{2}{\pi k_1r_2}\right)^{\frac{1}{2}}\left[\cos\left(k_1r_2 - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]$$

The equation (25) may further be reduced to

$$\left(\frac{k_2}{k_1}\right)^{\frac{1}{2}} = \frac{2\epsilon A_4}{\pm \sin (k_1 r_1 + k_2 r_2) - 1} \tag{25 a}$$

The above expression shows that the limiting values of k_2 will depend on $\sin (k_1 r_1 + k_2 r_2)$ the limiting value of which is determined from $k_1 r_1 + k_2 r_2 = n\pi/2$, where n is an integer. k_2 in (22 *a*) will be maximum when n is even and will be minimum when n is odd provided the + ve and the − ve signs are chosen for odd and even values of $\sin (k_1 r_1 + k_2 r_2)$ respectively. So, the limiting values of k_2 will lie between $k_1 \epsilon^2 A_4^2$ and $4 k_1 \epsilon^2 A_4^2$. So, γ_2 will lie between $[k_1^2 \epsilon^4 A_4^4 - \omega^2 \mu_2 \epsilon_2]^{\frac{1}{2}}$ and $[16 k_1^2 \epsilon^4 A_4^4 - \omega^2 \mu_2 \epsilon_2]^{\frac{1}{2}}$. Hence the limiting values of β_2 are $[\omega^2 \mu_2 \epsilon_2 - k_1^2 \epsilon^4 A_4^4]^{\frac{1}{2}}$ and $[\omega^2 \mu_2 \epsilon_2 - 16 k_1^2 \epsilon^4 A_4^4]^{\frac{1}{2}}$. In order that propagation may take place through the second medium β_2 must be real and hence $\omega^2 \mu_2 \epsilon_2$ should be $> 16 k_1^2 \epsilon^4 A_4^4$. So, the critical frequency and critical wavelengths are given by

$$f_{c2} = 4 k_1 \epsilon^2 A_4^2 / 2\pi \sqrt{\mu_2 \epsilon_2} \text{ and } \lambda_{c2} = 2\pi / 4 \epsilon^2 k_1 A_4^2 \text{ respectively.}$$

The phase c_{p2} and the group c_{g2} velocities lie between

$$c_{p2} \rightarrow \omega/[\omega^2 \mu_2 \epsilon_2 - k_1^2 \epsilon^4 A_4^4]^{\frac{1}{2}} \text{ and } \omega/[\omega^2 \mu_2 \epsilon_2 - 16 k_1^2 \epsilon^4 A_4^4]^{\frac{1}{2}}$$

$$c_{g2} \rightarrow \frac{[\omega^2 \mu_2 \epsilon_2 - k_1^2 \epsilon^4 A_4^4]^{\frac{1}{2}}}{[\omega \mu_2 \epsilon_2 - \omega \mu_1 \epsilon_1 \epsilon^4 A_4^4]} \text{ and } \frac{[\omega^2 \mu_2 \epsilon_2 - 16 k_1^2 \epsilon^4 A_4^4]^{\frac{1}{2}}}{[\omega \mu_2 \epsilon_2 - 16 \epsilon^4 A_4^4 \omega \mu_1 \epsilon_1]}$$

The guide wavelength λ_{g2} lies between

$$2\pi / [\omega^2 \mu_2 \epsilon_2 - k_1^2 \epsilon^4 A_4^4]^{\frac{1}{2}} \text{ and } 2\pi / [\omega^2 \mu_2 \epsilon_2 - 16 k_1^2 \epsilon^4 A_4^4]^{\frac{1}{2}}$$

The propagation characteristics of the TM wave in both the media are collected in the table for convenience of reference.

Propagation characteristics of the TM mode in the two media

	Medium 1	Medium 2
k	$\frac{1}{r_1}\left[\frac{m\pi}{2}+\frac{\pi}{4}+\tan^{-1}\delta\right]$	$k_1\epsilon^2A_4^{\,2}$ and $4\,\epsilon^2k_1A_4^{\,2}$
γ	$\left[\frac{1}{r_1^{\,2}}\left\{\frac{m\pi}{2}+\frac{\pi}{4}+\tan^{-1}\delta\right\}-\omega^2\mu_1\epsilon_1\right]^{\frac{1}{2}}$	$[k_1^{\,2}\epsilon^4A_4^{\,4}-\omega^2\mu_2\epsilon_2]^{\frac{1}{2}}$ and $[16\,k_1^{\,2}\epsilon^4A_4^{\,4}-\omega^2\mu_2\epsilon_2]^{\frac{1}{2}}$
β	$\left[\omega^2\mu_1\epsilon_1-\frac{1}{r_1^{\,2}}\left\{\frac{m\pi}{2}+\frac{\pi}{4}+\tan^{-1}\delta\right\}^2\right]^{\frac{1}{2}}$	$[\omega^2\mu_2\epsilon_2-k_1^{\,2}\epsilon^4A_4^{\,4}]^{\frac{1}{2}}$ and $[\omega^2\mu_2\epsilon_2-16\,k_1^{\,2}\epsilon^4A_4^{\,4}]^{\frac{1}{2}}$
f_c	$\frac{1}{2\pi r_1\sqrt{\mu_1\epsilon_1}}\left[\frac{m\pi}{2}+\frac{\pi}{4}+\tan^{-1}\delta\right]$	$4\,k_1\epsilon_1A_4^{\,2}/2\pi\sqrt{\mu_2\epsilon_2}$
λ_c	$2\pi r_1\Big/\left[\frac{m\pi}{2}+\frac{\pi}{4}+\tan^{-1}\delta\right]$	$2\pi/4k_1\epsilon^2A_4^{\,2}$
c_p	$\omega\Big/\left[\omega^2\mu_1\epsilon_1-\frac{1}{r_1^{\,2}}\left\{\frac{m\pi}{2}+\frac{\pi}{4}+\tan^{-1}\delta\right\}^2\right]$	$\omega/[\omega^2\mu_2\epsilon_2-k_1^{\,2}\epsilon^4A_4^{\,4}]^{\frac{1}{2}}$ and $\omega/[\omega^2\mu_2\epsilon_2-16\,k_1^{\,2}\epsilon^4A_4^{\,4}]^{\frac{1}{2}}$
c_g	$\dfrac{\left[\omega^2\mu_1\epsilon_1-\frac{1}{r_1^{\,2}}\left\{\frac{m\pi}{2}+\frac{\pi}{4}+\tan^{-1}\delta\right\}^2\right]^{\frac{1}{2}}}{\omega\,\mu_1\epsilon_1}$	$\dfrac{[\omega^2\mu_2\epsilon_2-k_1^{\,2}A_4^{\,4}\epsilon^4]^{\frac{1}{2}}}{[\omega\mu_2\epsilon_2-\omega\mu_1\epsilon_1A_4^{\,4}\epsilon^4]}$ and $\dfrac{[\omega^2\mu_2\epsilon_2-16\,A_4^{\,4}\epsilon^4k_1^{\,2}]^{\frac{1}{2}}}{[\omega\mu_2\epsilon_2-16\,A_4^{\,4}\epsilon^4\omega\mu_1\epsilon_1]}$
λ_g	$2\pi\Big/\left[\omega^2\mu_1\epsilon_1-\frac{1}{r_1^{\,2}}\left\{\frac{m\pi}{2}+\frac{\pi}{4}+\tan^{-1}\delta\right\}^2\right]^{\frac{1}{2}}$	$2\pi/[\omega^2\mu_2\epsilon_2-k_1^{\,2}\epsilon^4A_4^{\,4}]^{\frac{1}{2}}$ and $2\pi/[\omega^2\mu_2\epsilon_2-16\,k_1^{\,2}\epsilon^4A_4^{\,4}]^{\frac{1}{2}}$

(*to be continued*)

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