SOME ASPECTS OF THE SPECTRUM OF MODULATED PULSES

By V. Narayana Rao, M.Sc. (Tech.)

(Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore-3)

Summary

A modified method of deriving the spectrum of modulated pulses is given here. According to this method, it is shown that there are two distinct types among the basic pulse frequency and pulse phase modulation systems and these types are compared with continuous wave frequency and phase modulation systems. The distortion that arises with the different methods of demodulation are examined and curves have been drawn for harmonic distortion in the case of pulse width modulation.

1. Introduction

One of the early methods of analysis of the spectrum of modulated pulses was to set up an equation for a train of pulses and then to modulate any of the required parameters such as the duration, phase or the repetition frequency.[1] The validity of such a procedure has been doubted and an improved method of analysis has been developed by Fitch.[3] In this method an expression is set up for a finite number N of positive steps located at time intervals of T_r and an equal number of negative steps displaced by a time interval l, corresponding to the width of the pulses. The expressions for the positive and negative steps when added result in an expression for a train of N pulses. If N is made to increase indefinitely, the following expression for an infinite train of pulses will be obtained.

$$F(t) = \frac{A}{2\pi j} \sum_{k=-\infty}^{\infty} \frac{1}{k} \left(e^{j\pi k l f_r} - e^{-j\pi k l f_r}\right) e^{j2\pi k f_r t} \qquad (1)$$

where A is the amplitude of the pulses and f_r the pulse repetition frequency. This expression clearly shows the individual effects of the leading and the trailing edges each of which can be modulated in any desired manner. The analysis carried out by Fitch has been confined mainly to the determination of the harmonic distortion produced by the lower sidebands of the pulse repetition frequency harmonics when the demodulation is carried out by means of a low pass filter.

2. Modified Method of Analysis (Pulse Frequency Modulation)

It is intended in the following analysis to take up a detailed consideration of the spectrum of modulated pulses and to compare them with the spectrum of amplitude and angular c.w. systems. For the purpose of obtaining a clear understanding of the relationship between the modulating wave and the modulated pulse train the following picture of the modulation process may be assumed (see Fig. 1).

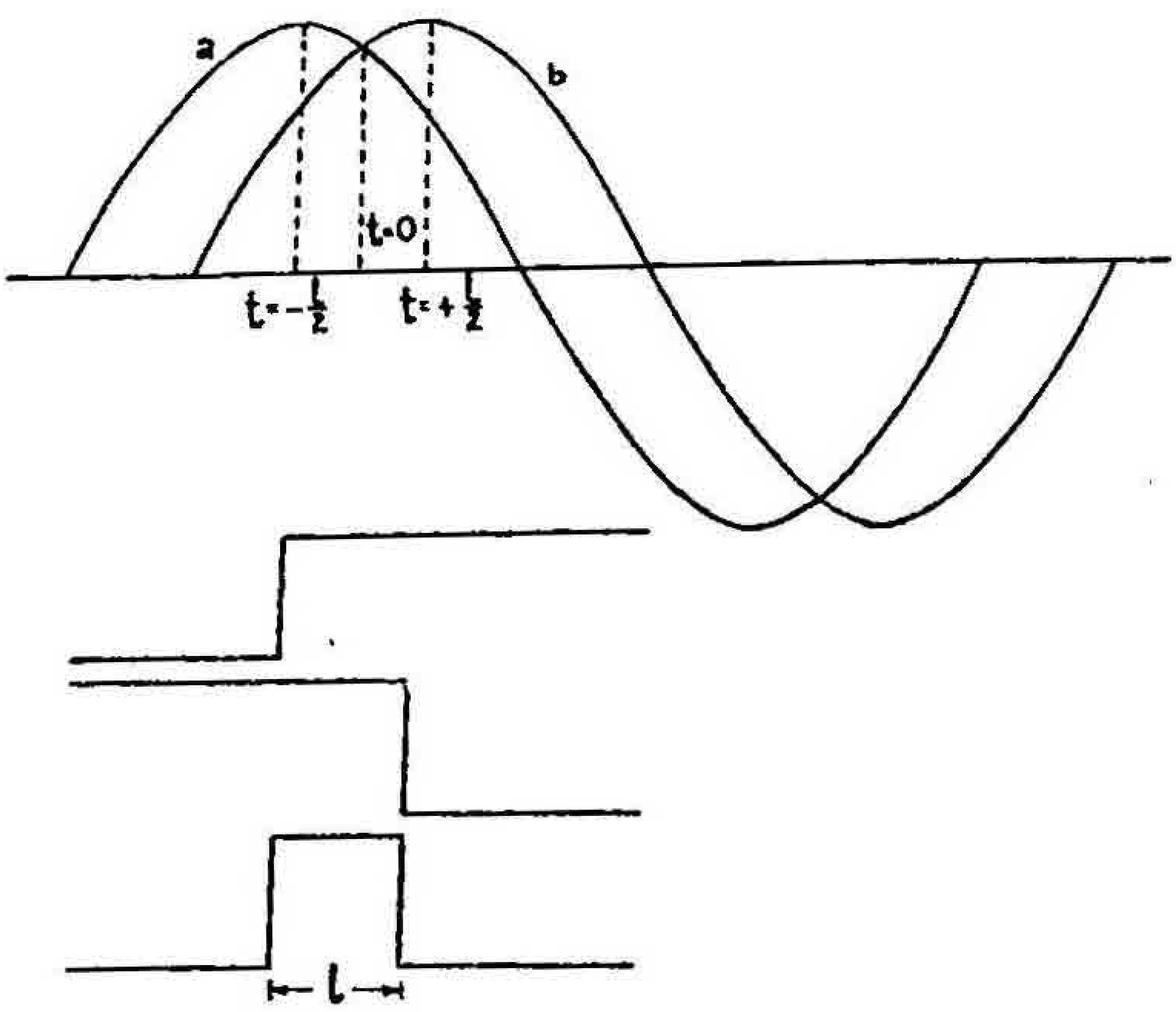

FIG. 1

Here (a) and (b) are two sinusoidal waves of frequency f_r displaced relative to one another by an amount l, the width of the unmodulated pulse. The positive and the negative steps which give rise to the pulse train, are assumed to occur at the peaks of the waveforms (a) and (b) respectively. For pulse frequency modulation the two waveforms (a) and (b) are regarded as being frequency modulated. Thus the expressions $E \cos 2\pi f_r (t + l/2)$ and $E \cos 2\pi f_r (t - l/2)$ for (a) and (b) will change over on modulation by a single tone of frequency, f_m, to the following expressions:

$$f(t) = E \cos \left[2\pi f_r (t + l/2) + \frac{f_d}{f_m} \sin (2\pi f_m t + \phi) \right] \text{ for } (a) \tag{2}$$

$$f(t) = E \cos \left[2\pi f_r (t - l/2) + \frac{f_d}{f_m} \sin (2\pi f_m t + \phi) \right] \text{ for } (b) \tag{3}$$

In these equations f_d is the frequency deviation and ϕ is the phase angle of the modulating waveform. Without any modulation, the time of occurrence of the positive and the negative steps will be given by

$$2\pi f_r (t + l/2) = 2n\pi \tag{4}$$

$$2\pi f_r (t - l/2) = 2n\pi \tag{5}$$

With modulation these instants will change over to the values given by

$$2\pi f_r (t + l/2) + \frac{f_d}{f_m} \sin (2\pi f_m t + \phi) = 2n\pi \tag{6}$$

$$2\pi f_r (t - l/2) + \frac{f_d}{f_m} \sin (2\pi f_m t + \phi) = 2n\pi \tag{7}$$

There are really two distinct types of frequency modulated pulse waveforms. In the first type, the displacement of the waveform (*b*) of Fig. 1 from its unmodulated position at any instant of time t, is determined by the value of the modulating voltage at that instant. In the second type, this displacement is determined by the value of the modulating voltage at time, $t - l$. It is clear that in the first type there is bound to be a variation in the width of the pulses along the modulation cycle as well as a positional displacement. In the second type the width of the pulses will remain constant and there will be only positional displacement. Considering first type for which the equations (6) and (7) apply, it is seen from these equations that modulation can be taken into account by substituting

$$l + \frac{f_d}{\pi f_r f_m} \sin (2\pi f_m t + \phi) \text{ and } l - \frac{f_d}{\pi f_r f_m} \sin (2\pi f_m t + \phi)$$

for l in the expressions for the leading and trailing edges in the equation (1). Then the expression for the modulation pulse train becomes

$$F(t) = \frac{A}{2\pi j} \sum_{k=-\infty}^{\infty} \frac{1}{k} \left\{ e^{j[\pi k l f_r + \frac{k f_d}{f_m} \sin (2\pi f_m t + \phi)]} - e^{-j[\pi k l f_r - \frac{k f_d}{f_m} \sin (2\pi f_m t + \phi)]} \right\} e^{j2\pi f_r k t} \tag{8}$$

Using the formula, $e^{jz \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(z) e^{jn\theta}$, where $J_n(z)$ is the Bessel function of the first kind and of order n, and after subsequent simplification, the following expression for the spectrum will be obtained:

$$F(t) = Alf_r + 2Alf_r \sum_{k=1}^{\infty} \frac{\sin \pi l k f_r}{\pi l k f_r} \left[J_0\left(\frac{k f_d}{f_m}\right) \cos 2\pi k f_r t + \sum_{n=1}^{\infty} J_n\left(\frac{k f_d}{f_m}\right) \left\{ \cos \overline{2\pi (k f_r + n f_m) t + n\phi} + (-1)^n \times \cos \overline{2\pi (k f_r - n f_m) t - n\phi} \right\} \right] \tag{9}$$

This expression may be compared with that for the spectrum of a frequency modulated continuous wave given by

$$F(t) = A\, J_0\left(\frac{f_d}{f_m}\right) \cos 2\pi f_r t + A \sum_{n=1}^{\infty} J_n\left(\frac{f_d}{f_m}\right) \Big\{ \cos\, 2\pi(f_r + nf_m)\, t + \phi n$$
$$+ (-1)^n \cos 2\pi(f_r - nf_m)\, t - n\phi \Big\} \tag{9 a}$$

Comparison of (9) and (9 a) leads to the following conclusions: (i) With this type of pulse frequency modulation, there will be no sideband accompanying the zero or the " d.c. " component of the pulse spectrum. Hence a low pass filter used for the demodulation will not yield any modulating frequency component. (ii) The kth harmonic of the pulse repetition frequency is frequency modulated, the modulation index being kf_d/f_m. (iii) As the order of the pulse repetition frequency increases, the amplitude of the carrier and sidebands will diminish as shown by the factor $\sin \pi lkf_r/k$. It is clear that pulses frequency modulated in this manner can be demodulated by means of a band pass filter which will extract one of the repetition frequency harmonics along with its sidebands and by applying it to a frequency discriminator circuit. It is worth noting that there will be no harmonic distortion accompanying this method of demodulation.

The other type of pulse frequency modulation will now be considered. In this type the displacement of the waveform (b) of Fig. 1 from its unmodulated position at any instant of time t will depend on the value of the modulating voltage at time $t - l$. Hence the equations which will determine the position of the trailing and leading edges of the pulses are:

$$2\pi f_r (t + l/2) + \frac{f_d}{f_m} \sin (2\pi f_m t + \phi) = 2n\pi \tag{10}$$

$$2\pi f_r (t - l/2) + \frac{f_d}{f_m} \sin (2\pi f_m \overline{t - l} + \phi) = 2n\pi \tag{11}$$

On comparing these with the equations (4) and (5), it is found that modulation can be taken into account by substituting $l + \dfrac{f_d}{\pi f_r f_m} \sin (2\pi f_m t + \phi)$ and $l - \dfrac{f_d}{\pi f_r f_m} \sin (2\pi f_m \overline{t - l} + \phi)$ for l in the expressions for the leading and trailing edges respectively in the equation (1). The spectrum is thus given by

$$F(t) = \frac{A}{2\pi j} \sum_{k=1}^{\infty} \frac{1}{k} \Big\{ e^{j[\pi k l f_r + \frac{kf_d}{f_m} \sin (2\pi f_m t + \phi)]}$$
$$- e^{-j[\pi k l f_r - \frac{kf_d}{f_m} \sin (2\pi f_m \overline{t - l} + \phi)]} \Big\} e^{j 2\pi k f_r t} \tag{12}$$

After simplification this becomes,

$$F(t) = Alf_r + Af_d \frac{\sin \pi l f_m}{\pi f_m} \cos(2\pi f_m t + \phi - \pi l f_m) + 2Alf_r \sum_{k=1}^{\infty}$$

$$\left\{ J_0\left(\frac{kf_d}{f_m}\right) \frac{\sin \pi l k f_r}{\pi l k f_r} \cos 2\pi k f_r t + \sum_{n=1}^{\infty} J_n\left(\frac{kf_d}{f_m}\right) \left[\frac{\sin \pi l (kf_r + nf_m)}{\pi l k f_r} \right.\right.$$

$$\cos\left(2\pi\,[kf_r + nf_m]\,t + n\phi - n\pi l f_m\right) + (-1)^n \frac{\sin \pi l (kf_r - nf_m)}{\pi l k f_r}$$

$$\left.\left. \cos\left(2\pi\,[kf_r - nf_m]\,t - n\phi + n\pi l f_m\right) \right] \right\} \quad (13)$$

It will be interesting to compare this with the equations (9) and (9 *a*) which are the expressions for the frequency modulated pulses of the first type and frequency modulated continuous wave respectively. This comparison leads to the following conclusions:

(i) The zero or the " d.c." component of the pulse spectrum has a sideband of the modulating frequency of amplitude $Af_d (\sin \pi l f_m / \pi f_m)$. Modulation can therefore be recovered by means of a low pass filter and there will be no harmonic distortion as there are no harmonic terms in the zero order pulse spectrum. There will however be some distortion due to the lower sidebands of the harmonics of the pulse repetition frequency.

(ii) If the sidebands of the kth pulse repetition frequency harmonic are considered, it is seen that the upper and lower sidebands of the same order are not equal in amplitude whereas for pure frequency modulation they should be equal, as can be seen from equation (9 *a*). This situation is shown in Fig. 2 where the amplitudes of the kth pulse repetition frequency harmonic and its first three upper and lower sidebands are plotted as functions of k. It is seen that as k is increased, the upper and lower sidebands of a given order tend to assume equal values. It may therefore be concluded that each of the pulse repetition frequency harmonics has, in addition to frequency modulation, a certain degree of amplitude modulation which diminishes as the order of the harmonic increases. If the signal is demodulated by selecting any of the pulse repetition frequency harmonics and sidebands and applying it to a frequency discriminator, there is bound to be a certain measure of distortion.

3. Pulse Phase Modulation

The expression for a train of modulated pulses can be obtained by phase modulating the two waveforms (*a*) and (*b*) of Fig. 1. Here again there are

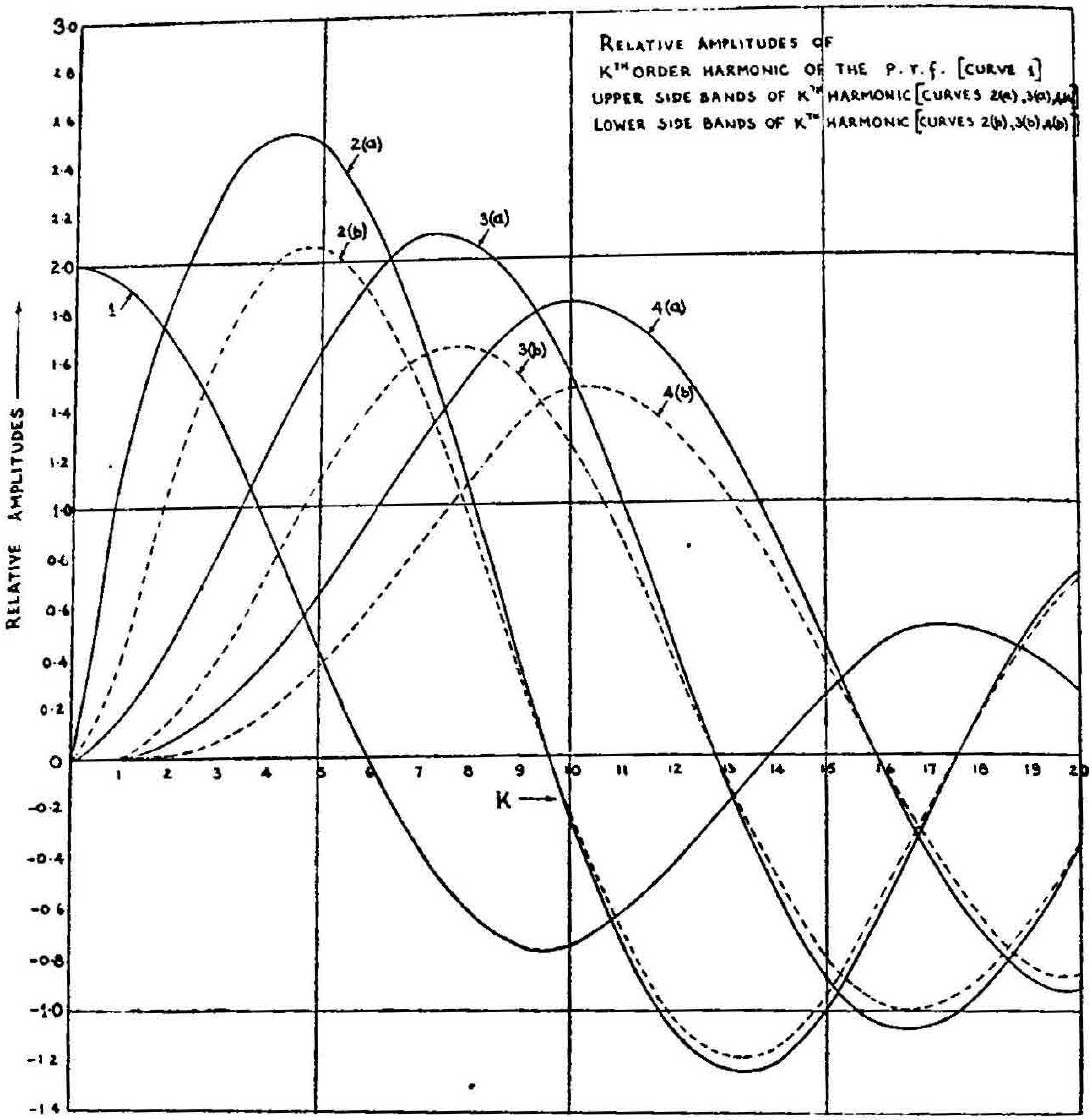

FIG. 2

two distinct types of phase modulated pulses. In one of them the phase of the waveform (*b*) of Fig. 1 at any instant of time is determined by the value of the modulating voltage at that instant. Therefore the displacement of the leading and trailing edges are not equal and hence there will be a variation in the width of the pulses along the modulation cycle as well as a positional displacement. Considering this type of modulation first, the expressions for the modulated waveforms of (*a*) and (*b*) of Fig. 1 are

$$f(t) = \mathrm{E} \cos 2\pi f_r \left[(t + l/2) + l_d \sin (2\pi f_m t + \phi)\right] \tag{14}$$

$$f(t) = \mathrm{E} \cos 2\pi f_r \left[(t - l/2) + l'_d \sin (2\pi f_m t + \phi)\right] \tag{15}$$

where $2\pi f_r l_d$ and $2\pi f_r l_d'$ are the phase deviations of the waveforms (*a*) and (*b*) respectively. The two deviations are assumed as being different to make this treatment applicable to the case of pulse width modulation. By comparing these equations with (2) and (3) it is seen that modulation can be taken into account by substituting $l + 2l_d \sin(2\pi f_m t + \phi)$ and $l - 2l_d' \sin(2\pi f_m t + \phi)$ for l in the expressions for the leading and trailing edges respectively in equation (1). The spectrum is therefore given by

$$F(t) = \frac{A}{2\pi j} \sum_{k=-\infty}^{\infty} \frac{1}{k} \Big\{ e^{j[\pi k l f_r + 2\pi k f_r l_d \sin(2\pi f_m t + \phi)]} - e^{-j[\pi k l f_r - 2\pi k f_r l_d' \sin(2\pi f_m t + \phi)]} \Big\} e^{j2\pi f_r k t} \tag{16}$$

For pulse phase modulation the two deviations are equal, *i.e.*, $l_d = l_d'$. The above equation can be simplified into

$$F(t) = Alf_r + 2Alf_r \sum_{k=1}^{\infty} \frac{\sin \pi k l f_r}{\pi k l f_r} \Big\{ J_0(2\pi k f_r l_d) \cos 2\pi k f_r t + \sum_{n=1}^{\infty} J_n(2\pi k f_r l_d) [\cos(2\pi \overline{k f_r + n f_m}\, t + n\phi) + (-1)^n \cos(2\pi \overline{k f_r - n f_m}\, t - n\phi)] \Big\} \tag{17}$$

This expression is quite similar to (9) with the difference that kf_d/f_m is replaced by $2\pi k f_r l_d$. Evidently each of the pulse repetition frequency harmonics is phase modulated, the maximum deviation being $2\pi k f_r l_d$. The amplitudes of the pulse repetition frequency harmonics and their sidebands decrease with k according to the term $\sin \pi l k f_r / \pi k$. Also there is no sideband accompanying the zero or the " d.c." component of the pulse spectrum and hence modulation cannot be recovered by means of a low pass filter.

The other type of pulse phase modulation will now be considered. Here the leading edge of the pulse is modulated so that its displacement at any instant t is determined by the value of the modulating voltage at that same instant, namely t, whereas the displacement of the trailing edge is determined by the value of the modulating voltage at time $(t - l)$. Evidently in this type of modulation, the displacement of the two edges are equal and hence the width of the pulses will remain the same along the modulation cycle and there will be only positional displacements of the pulses. From the considerations previously given, it is clear that spectrum for this type of modulation will be obtained by substituting $l + 2l_d \sin(2\pi f_m t + \phi)$ and $l - 2l_d' \sin(2\pi f_m \overline{t - l} + \phi)$ for l in the expressions for the leading and trailing edges respectively in the equation (1). This gives the expression

$$F(t) = \frac{A}{2\pi j}\sum_{k=-\infty}^{\infty}\frac{1}{k}\left\{e^{j\pi k f_r [l + 2l_d \sin(2\pi f_m t + \phi)]} - e^{-j\pi k f_r [l - 2l_d' \sin(2\pi f_m t - \overline{l} + \phi)]}\right\} e^{j2\pi k f_r t} \tag{18}$$

Setting $l_d = l_d'$ and simplifying, the following expression is obtained for the spectrum:

$$F(t) = Alf_r + 2Al_d f_r \sin(\pi l f_m) \cdot \cos(2\pi f_m t + \phi - \pi l f_m) + \sum_{k=1}^{\infty} Alf_r \left\{ J_0(2\pi k f_r l_d) \frac{\sin \pi l k f_r}{\pi l k f_r} 2\cos 2\pi k f_r t + \sum_{n=1}^{\infty} 2J_n(2\pi k f_r l_d) \times \left[\frac{\sin \pi l (k f_r + n f_m)}{\pi l k f_r} \cos\left(2\pi [k f_r + n f_m] t + n\phi - n\pi l f_m\right) + (-1)^n \frac{\sin \pi l (k f_r - n f_m)}{\pi l k f_r} \cos\left(2\pi [k f_r - n f_m] t - n\phi + n\pi l f_m\right)\right]\right\} \tag{19}$$

This equation is very similar to (13). The "d.c." component of the pulse spectrum is accompanied by the modulating frequency term of amplitude $2Al_d f_r \sin \pi l f_m$. This can be recovered by means of a low pass filter and a suitable equaliser to take into account the factor $\sin \pi l f_m$ which shows that the amplitude is not constant but is proportional to the modulating frequency. There will be no harmonic distortion as there are no harmonic terms in the zero order pulse spectrum. There will however be a certain measure of distortion due to the lower sidebands of the harmonics of the pulse spectrum. Another interesting feature of this spectral distribution is that the upper and lower sidebands of the same order accompanying any of the pulse repetition harmonics are not equal in amplitude, whereas according to true phase modulation they should be equal to one another. Therefore it may be concluded that there is some accompanying amplitude modulation. Demodulation by means of a band pass filter and discriminator is accompanied by some distortion. Fig. 2 which was drawn for the case of pulse frequency modulation can be suitably modified to represent the spectral distribution in this case also.

4. Pulse Width Modulation

(*a*) *Symmetrical double edge modulation.*—The spectrum for width modulated pulses can be obtained by considering the spectrum of phase modulated pulses given by equation (16). If the trailing edge instead of being displaced in the same direction as the leading edge is displaced in the opposite direction, then pulse width modulation will be produced. Considering the case of symmetrical double edge modulation, the equation becomes,

$$F(t) = \frac{A}{2\pi j} \sum_{n,\, k=-\infty}^{\infty} \frac{1}{k} \left\{ J_n (2\pi k f_r l_d)\, e^{j\pi k l f_r} - J_n(-2\pi k f_r l_d)\, e^{-j\pi k l f_r} \right\} e^{j(2\pi [k f_r + n f_m] t + n\phi)} \quad (19\,a)$$

which simplifies into

$$F(t) = Alf_r + Alf_r m \sin(2\pi f_m t + \phi) + Alf_r \sum_{k=1}^{\infty} \left\{ 2J_0(\pi k f_r m l) \times \frac{\sin \pi k f_r l}{\pi k f_r l} \cos 2\pi k f_r t + 4J_1(\pi k f_r m l) \frac{\cos \pi k f_r l}{\pi k f_r l} \cos(2\pi k f_r t) \times \sin(2\pi f_m t + \phi) + 4J_2(\pi k f_r m l) \frac{\sin \pi k f_r l}{\pi k f_r l} \cos(2\pi k f_r t) \times \cos(2\pi 2 f_m t + 2\phi) + \text{etc.} \right\} \quad (20)$$

In the above equation the term l_d has been replaced by $\frac{1}{2}ml$, where m is the modulation index. Thus if $m = 1$, then the maximum and minimum values of the pulse width will be $2l$ and 0, which corresponds to the case of 100% modulation. From the above equation it is seen that the zero order of the pulse spectrum has a modulating frequency term of amplitude $Alf_r m$ and no harmonics. Therefore modulation can be recovered by means of a low pass filter and distortion will be only due to the lower sidebands of the pulse repetition frequency harmonics. The exact amount of this distortion for varying conditions has been worked out by Fitch.[3]

A very interesting feature of the equation (20) is that it shows that each of the pulse repetition frequency harmonics is amplitude modulated. It will be shown later that the distortion components of the sidebands are small. It is therefore possible to select any of the pulse repetition frequency harmonics along with the sidebands and apply the resulting waveform to a linear detector which would give the demodulated signal. The author has shown[4,5] that this method has a 3 db. improvement in signal to noise ratio over the usual system of demodulation which uses a low pass filter. It has also been shown that if N pulse repetition frequency harmonics are separately extracted and the output added after detection by means of a linear detector, the resulting improvement in signal to noise ratio will be 10 log 2N decibels. Thus if N is equal to 4 the improvement is 9 dbs. In view of this improvement in signal to noise ratio, it is worthwhile to examine in detail the distortion that would arise in such a system of demodulation. Considering the kth harmonic of the pulse repetition frequency, the expression for this term is

$$2\,\mathrm{A}lf_T \mathrm{J}_0(\pi k f_T ml)\frac{\sin \pi k f_T l}{\pi k f_T l}\cos 2\pi k f_T t\left\{1+\frac{2\mathrm{J}_1(\pi k f_T ml)}{\mathrm{J}_0(\pi k f_T ml)}\cot(\pi k f_T l)\right.$$
$$\times\sin(2\pi f_m t+\phi)+\frac{2\mathrm{J}_2(\pi k f_T ml)}{\mathrm{J}_0(\pi k f_T ml)}\cos(2\pi 2 f_m t+2\phi)+\frac{2\mathrm{J}_3(\pi k f_T ml)}{\mathrm{J}_0(\pi k f_T ml)}$$
$$\left.\times\cot(\pi k f_T l)\sin(2\pi 3 f_m t+3\phi)+\text{etc.}\right\} \quad (21)$$

It is therefore seen from an examination of this expression that the harmonic distortion accompanying this method of demodulation is given by

$$\text{Second harmonic distortion}=\frac{\mathrm{J}_2(\pi k f_T ml)}{\mathrm{J}_1(\pi k f_T ml)}\tan(\pi k f_T l) \quad (22)$$

$$\text{Third harmonic distortion}=\frac{\mathrm{J}_3(\pi k f_T ml)}{\mathrm{J}_1(\pi k f_T ml)} \quad . \quad (23)$$

It should be noted that these expressions for distortion refer to the case of a single pulse repetition frequency harmonic being extracted for detection. The results are shown graphically in Fig. 3 where the harmonic distortion values are plotted as a function of $k\pi lf_T$, which is $k\pi$ times the duty factor lf_T, for different values of the modulation index m.

If the first k_0 terms of the pulse spectrum are filtered out and detected separately and their outputs added in phase, the accompanying distortion will be given by

$$\text{Second harmonic distortion}=\frac{\sum_1^{k_0}\mathrm{J}_2(\pi k f_T ml)\sin(\pi k f_T l)/\pi k l f_T}{\sum_1^{k_0}\mathrm{J}_1(\pi k f_T ml)\cos(\pi k f_T l)/\pi k l f_T} \quad (24)$$

$$\text{Third harmonic distortion}=\frac{\sum_1^{k_0}\mathrm{J}_2(\pi k f_T ml)\cos(\pi k f_T l)/\pi k l f_T}{\sum_1^{k_0}\mathrm{J}_1(\pi k f_T ml)\cos(\pi k f_T l)/\pi k l f_T} \quad (25)$$

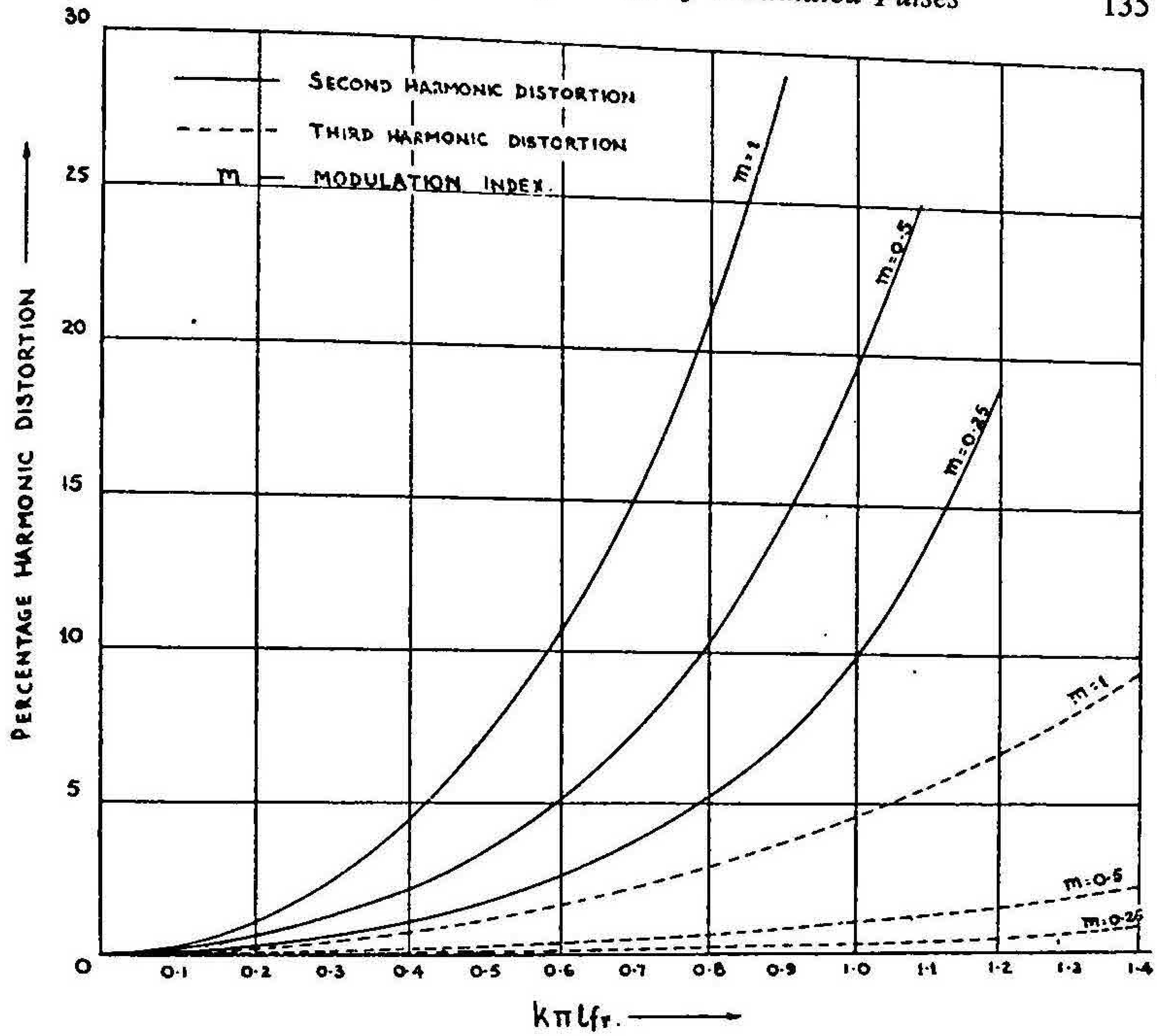

FIG. 3

(*b*) *Single edge width modulation.*—If only the leading edge is being modulated, the spectrum for this case can be obtained by setting $l_d' = 0$ in equation (16). The resulting expression on simplification becomes

$$F(t) = Alf_r + Alf_r m \sin(2\pi f_m t + \phi) + Alf_r \sum_{k=1}^{\infty} \left\{ \frac{\sin 2\pi k f_r (t - l/2)}{\pi k l f_r} \right.$$

$$+ \frac{J_0(2\pi k l f_r m)}{\pi k l f_r} \sin 2\pi k f_r (t + l/2) + \frac{2J_1(2\pi k l f_r m)}{\pi k l f_r} \cos 2\pi k f_r (t + l/2)$$

$$\sin(2\pi f_m t + \phi) + \frac{2J_2(2\pi k l f_r m)}{\pi k l f_r} \sin 2\pi k f_r (t + l/2)$$

$$\left. \cos(2\pi 2 f_m t + 2\phi) + \text{etc.} \right\} \quad (26)$$

In this case also the modulated signal can be extracted by means of a low pass filter and there will be distortion due only to the lower sidebands of the pulse repetititon frequency harmonics. Improved signal to noise ratio can be obtained by extracting the harmonics of the pulse spectrum along with their sidebands and adding in phase the detected outputs. The distortion arising in such a case can easily be determined from equation (26).

REFERENCES

1. Roberts, F. F. and Simmonds, J. C... " Multichannel Communication Systems," *Wireless Engineer*, 1945, 88, 538.
2. Cooke, D., Jelonek, Z., Oxford, A. J. and Fitch, E. " Pulse Communication," *Journal I. E. E.*, 1947, Part III A, No. 11, 8.
3. Fitch, E. .. " Spectrum of Modulated Pulses," *Ibid.*, 1947, 94, Part III A, No. 13, 556.
4. Cherry, E. C. and Rao, V. N. .. Discussion Note on "Pulse Communication," *Jour. I.E.E.*, 1947, 94, Part III A, No. 13, 586.
5. Narayana Rao, V. .. " An Improved Method of Demodulation of Width Modulated Pulses," *Electrotechnics*, March 1951, No. 23, 75.