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PROPAGATION OF MICROWAVES THROUGH A CYLINDRICAL METALLIC GUIDE FILLED COAXIALLY WITH TWO DIFFERENT DIELECTRICS—PART III

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SUMMARY

The propagation characteristics for the TE_{mn}, TM_{mn} and hence TE_{01} and TM_{11} modes in a cylindrical metallic guide having a coaxial structure with two different dielectrics have been derived. The result indicates that the phase velocity for a given mode can be adjusted to a preassigned value by a suitable choice of the dielectric constants and radii of the two media. The calculation of the power flowing through the guide in the case of the TE_{01} mode shows that most of the power is located in the medium having higher dielectric constant. The energy distribution in the radial direction in the case of the TE_{01} mode has been obtained by plotting the real part of the complex Poynting vector *vs.* radius.

INTRODUCTION

The present paper is a continuation of the previous one (Chatterjee, 1953) and deals with the theoretical investigations on the propagation characteristics of the TE mode. The object is also to make a comparative study of the propagation characteristics of the two TE_{01} and TM_{11} modes. It is also shown that the phase velocity can be adjusted to a preassigned value by a suitable choice of the dielectric constants and radii of the two dielectrics as pointed out by Frankel (1947), Bruck and Wicher (1947), Bānos, etc. (1951).

FIELD COMPONENTS OF THE TE MODE

It follows from Maxwell's equations that the field components E's and H's for the TE mode are related by the following equations

$$
\begin{aligned}
&E_z = 0 \\
&\frac{1}{r}\frac{\partial H_z}{\partial \theta} - \frac{\partial H_\theta}{\partial z} = j\omega\epsilon E_r \\
&\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = j\omega\epsilon E_\theta \\
&\frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) - \frac{1}{r}\frac{\partial H_r}{\partial \theta} = 0 \\
&-\frac{\partial E_\theta}{\partial z} = -j\omega\mu H_r \\
&\frac{\partial E_r}{\partial z} = -j\omega\mu H_\theta \\
&\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) - \frac{1}{r}\frac{\partial E_r}{\partial \theta} = -j\omega\mu H_z
\end{aligned}
\tag{1}
$$

From equation (1) the following differential equation in E_θ and E_r is obtained:

$$
\frac{\partial^2 E_\theta}{\partial z^2} + \omega^2\mu\epsilon E_\theta + \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta)\right] - \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial E_r}{\partial \theta}\right] = 0 \tag{2}
$$

The electric **E** and the magnetic **H** field intensities are expressed in terms of the Hertz vector Π^* as follows (Stratton, 1941):

$$
\mathbf{E} = -\mu\frac{\partial}{\partial t}\nabla \times \Pi^*
$$

$$
\mathbf{H} = \nabla \times \nabla \times \Pi^*
$$

Let us consider propagation in the z direction only. Then $\Pi_1 = 0$, $\Pi_2 = 0$, $\Pi_z \neq 0$ (*i.e.*,) if Π^* is directed along the z-axis, the components of **E** are

$$
\begin{aligned}
&E_r = -\frac{\mu}{r}\frac{\partial}{\partial t}\left[\frac{\partial \Pi^*_z}{\partial \theta}\right] \\
&E_\theta = \mu\frac{\partial}{\partial t}\left[\frac{\partial \Pi^*_z}{\partial r}\right] \\
&E_z = 0
\end{aligned}
\tag{3}
$$

Substituting (3) in (2) the following equation is obtained:

$$\mu \frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial z^2}\left(\frac{\partial \Pi^*_z}{\partial r}\right) + \omega^2\mu\epsilon\left(\frac{\partial \Pi^*_z}{\partial r}\right) + \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Pi^*_z}{\partial r}\right)\right\} + \frac{\partial}{\partial r}\left\{\frac{1}{r^2}\frac{\partial^2 \Pi^*_z}{\partial \theta^2}\right\}\right] = 0 \tag{4}$$

As the time variation is expressed by exp. $(j\omega t)$, the equation (4) reduces to

$$\frac{\partial^2 \Pi^*_z}{\partial z^2} + \omega^2\mu\epsilon\, \Pi^*_z + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Pi^*_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Pi^*_z}{\partial \theta} = \text{Constant} \tag{5}$$

as $j\omega\mu \neq 0$.

Let

$$\Pi^*_z = \mathrm{R}\Theta\mathrm{Z}, \tag{5 a}$$

where $\mathrm{R} = f(r)$, $\Theta = f(\theta)$ and $\mathrm{Z} = f(z)$ only.

Substituting (5 *a*) in (5), making the constants of integration equal to zero and dividing both sides by $\mathrm{R}\Theta\mathrm{Z}$, the following differential equation is obtained:

$$\frac{1}{\mathrm{Z}}\frac{d^2\mathrm{Z}}{dz^2} + \frac{1}{\mathrm{R}}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \mathrm{R}}{\partial r}\right) + \frac{1}{\Theta}\frac{1}{r^2}\frac{\partial^2\Theta}{\partial\theta^2} + \omega^2\mu\epsilon = 0 \tag{6}$$

Let

$$\frac{1}{\mathrm{Z}}\frac{d^2\mathrm{Z}}{dz^2} = \gamma^2 \tag{6 a}$$

and

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -m^2$$

where γ and m are constants independent of r, θ, z. The equations (6 *a*) when solved for Z and Θ respectively give

$$\mathrm{Z} = {}^{\cosh}_{\sinh}\, \gamma z \text{ and } \Theta = {}^{\cos}_{\sin}\, m\theta \tag{7}$$

From (6) and (6 *a*) the following equation in R is obtained.

$$\frac{d^2\mathrm{R}}{dr^2} + \frac{1}{r}\frac{d\mathrm{R}}{dr} + \mathrm{R}\left[(\omega^2\mu\epsilon + \gamma^2) - \frac{m^2}{r^2}\right] = 0 \tag{8}$$

which when solved gives

$$\mathrm{R} = \mathrm{A}\mathrm{J}_m(r\sqrt{\omega^2\mu\epsilon + \gamma^2}) + \mathrm{B}\mathrm{Y}_m(r\sqrt{\omega^2\mu\epsilon + \gamma^2}) \tag{8 a}$$

From (5 *a*), (7) and (8 *a*) Π^*_z is given as follows:

$$\Pi^*_z = [\mathrm{A}\mathrm{J}_m(r\sqrt{\omega^2\mu\epsilon+\gamma^2}) + \mathrm{B}\mathrm{Y}_m(r\sqrt{\omega^2\mu\epsilon+\gamma^2})] \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, m\theta\, \mathrm{Z}(z) \qquad (8\,b)$$

which can be written as

$$\Pi^*_z = [\mathrm{A}\mathrm{J}_m(kr) + \mathrm{B}\mathrm{Y}_m(kr)] \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, m\theta\, e^{-\gamma z} \qquad (9)$$

where

$$\mathrm{Z}(z) = e^{-\gamma z}$$

and

$$k = \sqrt{\omega^2\mu\epsilon+\gamma^2}\,.$$

Considering the time variation as given by exp. $(j\omega t)$, substituting (9) in (3) and omitting $e^{j\omega t}$ for convenience, the different components of E's are

$$\mathrm{E}_r = -j\omega\mu\frac{m}{r}[\mathrm{A}\mathrm{J}_m(kr) + \mathrm{B}\mathrm{Y}_m(kr)] \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\, m\theta\, e^{-\gamma z}$$

$$\mathrm{E}_\theta = j\omega\mu[k\mathrm{A}\,\mathrm{J}'_m(kr) + k\mathrm{B}\mathrm{Y}'_m(kr)] \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, m\theta\, e^{-\gamma z} \qquad (10)$$

$$\mathrm{E}_z = 0$$

From (1) and (10) the components H_r and H_θ are

$$\mathrm{H}_r = -\gamma[k\mathrm{A}\mathrm{J}'_m(kr) + k\mathrm{B}\mathrm{Y}'_m(kr)] \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, m\theta\, e^{-\gamma z}$$

$$\mathrm{H}_\theta = -\gamma\frac{m}{r}[\mathrm{A}\mathrm{J}_m(kr) + \mathrm{B}\mathrm{Y}_m(kr)] \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\, m\theta\, e^{-\gamma z} \qquad (10\,a)$$

From (1) and (3), the expression for H_z in terms of Π^*_z is obtained as follows:

$$-\mathrm{H}_z = \frac{\partial^2\Pi^*_z}{\partial r^2} + \frac{1}{r}\frac{\partial\Pi^*_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Pi^*_z}{\partial\theta^2} \qquad (10\,b)$$

From (10 *b*) and (9), the following expression for H_z is obtained:

$$\mathrm{H}_z = \left(k^2 - \frac{2m^2}{r^2}\right)\Big[\mathrm{A}\mathrm{J}_m(kr) + \mathrm{B}\mathrm{Y}_m(kr)\Big] \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\, m\theta\, e^{-\gamma z} \qquad (10\,c)$$

as (Gray and Mathews, 1922; Dwight, 1949)

$$k^2r^2\,\mathrm{J}''_m(kr) = (m^2 - m - k^2r^2)\,\mathrm{J}_m(kr) + kr\,\mathrm{J}_{m+1}(kr) \qquad (10\,d)$$

and

$$kr\,\mathrm{J}'_m(kr) = m\,\mathrm{J}_m(kr) - kr\,\mathrm{J}_{m+1}(kr)$$

The components of E's and H's for the TE mode collected together from (10), (10 *a*) and (10 *c*) are

$$E_r = -j\omega\mu \frac{m}{r} [AJ_m (kr) + BY_m (kr)] \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\theta\, e^{-\gamma z}$$

$$E_\theta = j\omega\mu\, [kA\, J'_m (kr) + kBY'_m (kr) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\theta\, e^{-\gamma z}$$

$$E_z = 0$$

$$H_r = -\gamma\, [kA\, J'_m (kr) + kBY'_m (kr)] \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\theta\, e^{-\gamma z} \tag{11}$$

$$H_\theta = -\gamma \frac{m}{r} [AJ_m (kr) + BY_m (kr)] \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\theta\, e^{-\gamma z}$$

$$H_z = \left(k^2 - \frac{2m^2}{r^2}\right) \Big[AJ_m (kr) + BY_m (kr) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\theta\, e^{-\gamma z}$$

From (11) the field components in the two media can be written as follows.
First medium :

$$r_2 \leqslant r \leqslant r_1$$

$$E_{r1} = -j\omega\mu_1 \frac{m}{r} [A_1 J_m (k_1 r) + B_1 Y_m (k_1 r)] \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} m\theta\, e^{-\gamma_1 z}$$

$$E_{\theta 1} = j\omega\mu_1 [k_1 A_1 J'_m (k_1 r) + k_1 B_1 Y'_m (k_1 r)] \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\theta\, e^{-\gamma_1 z}$$

$$E_{z1} = 0$$

$$H_{r1} = -\gamma_1 [k_1 A_1 J'_m (k_1 r) + k_1 B_1 Y'_m (k_1 r)] \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\theta\, e^{-\gamma_1 z} \tag{12}$$

$$H_{\theta 1} = -\gamma_1 \frac{m}{r} [A_1 J_m (k_1 r) + B_1 Y_m (k_1 r)] \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} m\theta\, e^{-\gamma_1 z}$$

$$H_{z1} = \left(k_1^2 - \frac{2m^2}{r^2}\right) \Big[A_1 J_m (k_1 r) + B_1 Y_m (k_1 r)\Big] \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\theta e^{-\gamma_1 z}$$

In the second medium Y_m's have infinite discontinuities in the axial region which is physically inadmissible. Hence Y_m's are omitted from the expressions for the field components in the second medium.

Second medium : $0 \leqslant r < r_2$

$$E_{r2} = -j\omega\mu_2 \frac{m}{r} [A_2 J_m (k_2 r)] \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} m\theta\, e^{-\gamma_2 z}$$

$$E_{\theta 2} = j\omega\mu_2 [k_2 A_2 J'_m (k_2 r)] \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} m\theta\, e^{-\gamma_2 z}$$

$$E_{z2} = 0$$

$$H_{r2} = -\gamma_2 \left[k_2 A_2 J'_m(k_2 r)\right] {}^{\cos}_{\sin} m\theta \, e^{-\gamma_2 z} \qquad (12\,a)$$

$$H_{\theta 2} = -\gamma_2 \frac{m}{r} \left[A_2 J_m(k_2 r)\right] {}^{\sin}_{\cos} m\theta \, e^{-\gamma_2 z}$$

$$H_{z2} = \left(k_2^2 - \frac{2m^2}{r^2}\right)\left[A_2 J_m(k_2 r)\right] {}^{\cos}_{\sin} m\theta \, e^{-\gamma_2 z}$$

EVALUATION OF A_1 AND B_1 IN TERMS OF A_2

The boundary conditions are

$$\begin{aligned} H_{z1} &= H_{z2} && \text{at } r = r_2 \\ E_{\theta 1} &= 0 && \text{at } r = r_1 \\ \mu_1 H_{r1} &= \mu_2 H_{r2} && \text{at } r = r_2 \text{ as } \nabla \cdot \mathbf{B} = 0 \end{aligned} \qquad (13)$$

Applying the boundary conditions, and assuming the dielectric to be lossless, *i.e.*, $\gamma_1 = \alpha_1 + j\beta_1 \doteqdot j\beta_1$ and $\gamma_2 = \alpha_2 + j\beta_2 \doteqdot j\beta_2$ the following equations are obtained from (12) and (12 *a*):

$$\beta' A_1 J'_m(k_1 r_2) e^{j\beta z} + \beta' B_1 Y'_m(k_1 r_2) e^{j\beta z} - \mu' k' A_2 J'_m(k_2 r_2) = 0$$

$$A_1 J'_m(k_1 r_1) + B_1 Y'_m(k_1 r_1) = 0 \qquad (13\,a)$$

$$\left(k_1^2 - \frac{2m^2}{r_2^2}\right) A_1 J_m(k_1 r_2) e^{j\beta z} + \left(k_1^2 - \frac{2m^2}{r_2^2}\right) B_1 Y_m(k_1 r_2) e^{j\beta z} -$$

$$-\left(k_2^2 - \frac{2m^2}{r_2^2}\right) A_2 J_m(k_2 r_2) = 0$$

where

$$\beta_2 - \beta_1 = \beta,\ k' = k_2/k_1,\ \beta' = \beta_1/\beta_2,\ \mu' = \mu_2/\mu_1.$$

In order that A_1, B_1 and A_2 be non-vanishing, the determinant of their coefficients must vanish. From (13 *a*) A_1 and B_1 can be expressed in terms of A_2 as follows

$$A_1 = A_2 \cdot A' \text{ and } B_1 = A_2 \cdot B' \qquad (13\,b)$$

where

$$A' = \frac{\mu' k'}{\beta'} \frac{J'_m(k_2 r_2) Y'_m(k_1 r_1)}{J'_m(k_1 r_2) Y'_m(k_1 r_1) - J'_m(k_1 r_1) Y'_m(k_1 r_2)} e^{-j\beta z} \qquad (13\,c)$$

$$B' = \frac{\mu' k'}{\beta'} \frac{J'_m(k_2 r_2) J_m(k_1 r_1)}{Y'_m(k_1 r_2) J'_m(k_1 r_1) - Y'_m(k_1 r_1) J'_m(k_1 r_2)} e^{-j\beta z}$$

EVALUATION OF A_2

A_2 can be evaluated from the expression for the peak power flowing through the guide which is

$$\hat{P}_z = \int_{r=r_2}^{r_1} \int_{\theta=0}^{2\pi} [E_{r1} H^*_{\theta 1} - E_{\theta 1} H^*_{r1}]\, r d\theta dr$$

$$+ \int_{r=0}^{r_2} \int_{\theta=0}^{2\pi} [E_{r2} H^*_{\theta 2} - E_{\theta 2} H^*_{r2}]\, r d\theta dr \quad (14)$$

The equation (14) when evaluated with the help of (12), (12 *a*) shows that $\hat{P}_z$ is some function of μ_1, μ_2, β_1, β_2, k_1, k_2 and r_2, r_1. So, $\hat{P}_z$ can be written as

$$\hat{P} = 2\pi\omega A_2^2 [F(\mu_1, \beta_1, k_1, r_1, r_2) + F'(\mu_2, \beta_2, k_2, r_2)] \quad (14\,a)$$

which gives the value of A_2 as

$$A_2 = (P')^{\frac{1}{2}} \quad (14\,b)$$

where

$$P' = \left[\frac{\hat{P}}{2\pi\omega (F + F')}\right] \quad (14\,c)$$

The field components of the TE mode in both the media can be expressed in real parts from (12), (12 *a*), (13 *b*) and (14 *b*) as follows:

First medium:

$$\left.\begin{aligned}
E_{r1} &= -\mu_1\omega (P')^{\frac{1}{2}} \frac{m}{r} [A'J_m(k_1 r) + B'Y_m(k_1 r)] \cos m\theta \sin \beta_1 z \\
E_{\theta 1} &= \mu_1\omega (P')^{\frac{1}{2}} k_1 [A'J'_m(k_1 r) + B'Y'_m(k_1 r)] \cos m\theta \sin \beta_1 z \\
E_{z1} &= 0 \\
H_{r1} &= -\beta_1 k_1 (P')^{\frac{1}{2}} [A'J'_m(k_1 r) + B'Y'_m(k_1 r)] \cos m\theta \sin \beta_1 z \\
H_{\theta 1} &= -\beta_1 \frac{m}{r} (P')^{\frac{1}{2}} [A'J_m(k_1 r) + B'Y_m(k_1 r)] \cos m\theta \sin \beta_1 z \\
H_{z1} &= \pm \left(k_1^2 - \frac{2m^2}{r^2}\right) (P')^{\frac{1}{2}} [A'J_m(k_1 r) + B'Y_m(k_1 r)] \cos m\theta \cos \beta_1 z
\end{aligned}\right\} \quad (15)$$

Second medium:

$$E_{r2} = -\mu_2\omega \frac{m}{r} (P')^{\frac{1}{2}} [J_m (k_2r)] \cos m\theta \sin \beta_2 z$$

$$E_{\theta 2} = \mu_2\omega\, k_2 (P')^{\frac{1}{2}} [J'_m (k_2r)] \cos m\theta \sin \beta_2 z$$

$$E_{z2} = 0$$

$$H_{r2} = -\beta_2 k_2 (P')^{\frac{1}{2}} [J'_m (k_2r)] \cos m\theta \sin \beta_2 z$$

$$H_{\theta 2} = -\beta_2 \frac{m}{r} (P')^{\frac{1}{2}} [J_m (k_2r)] \cos m\theta \sin \beta_2 z \tag{15 a}$$

$$H_{z2} = \pm \left(k_2^2 - \frac{2m^2}{r^2}\right) (P')^{\frac{1}{2}} [J_m (k_2r) \cos m\theta \cos \beta_2 z$$

PROPAGATION CHARACTERISTICS (TE_{mn} MODE)

Applying the boundary condition $\mu_1 H_{r1} = \mu_2 H_{r2}$ at $r = r_2$ the following equation is obtained from (15) and (15 *a*):—

$$-\mu_1\beta_1 k_1 (P')^{\frac{1}{2}} [A' J'_m (k_1r_2) + B'Y'_m (k_1r_2)] \cos m\theta \sin \beta_1 z$$

$$= -\mu_2\beta_2 k_2 (P')^{\frac{1}{2}} [J'_m (k_2r_2)] \cos m\theta \sin \beta_2 z \tag{16}$$

Substituting the values of A′ and B′ from (13 *c*), the equation (16) reduces to

$$\frac{J'_m (k_2r_2)\, Y'_m (k_1r_1)\, J'_m (k_1r_2) - J'_m (k_2r_2)\, J'_m (k_1r_1)\, Y'_m (k_1r_2)}{J'_m (k_1r_2)\, Y'_m (k_1r_1) - J'_m (k_1r_1)\, Y'_m (k_1r_2)}$$

$$= J'_m (k_2r_2) \frac{\sin \beta_2 z}{\sin \beta_1 z} e^{j\beta z} \tag{16 a}$$

which can be reduced to

$$\cos (\beta_2 - \beta_1) z + j \sin (\beta_2 - \beta_1) z = \frac{\sin \beta_1 z}{\sin \beta_2 z} \tag{16 b}$$

Separating the real and imaginary parts the following expressions are obtained from (16 *b*).:

$$\cos (\beta_2 - \beta_1) z = \frac{\sin \beta_1 z}{\sin \beta_2 z}$$

$$\sin (\beta_2 - \beta_1) z = 0$$

or

$$\beta_2 = \beta_1 + \frac{n\pi}{z} \tag{16 c}$$

The equation (16 *c*) gives a relation between the phase constants of the two media. Applying the boundary condition $H_{z1} = H_{z2}$ at $r = r_2$, utilising the relation

$$-\frac{A'}{B'} = \frac{Y'_m(k_1 r_1)}{J'_m(k_1 r_1)},$$

substituting A′ from (13 *c*) and using the relation (16 *c*), the following equation is obtained from (15) and (15 *a*):

$$\frac{J_m(k_1 r_2)\, Y'_m(k_1 r_1) - J'_m(k_1 r_1)\, Y_m(k_1 r_2)}{J'_m(k_1 r_2)\, Y'_m(k_1 r_1) - J'_m(k_1 r_1)\, Y'_m(k_1 r_2)}$$

$$= \frac{\beta'}{\mu' k'} \frac{k_2^2 - \frac{2m^2}{r_2^2}}{k_1^2 - \frac{2m^2}{r_2^2}} \frac{J_m(k_2 r_2)}{J'_m(k_2 r_2)} e^{j\beta z} \tag{17}$$

For large arguments J's and Y's can be written as follows (Dwight, *loc. cit.*)

$$J_m(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

$$J'_m(x) = -\left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \tag{17 a}$$

$$Y_m(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

$$Y'_m(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

Applying the relations (17 *a*) in (17), the following relation between k_1 and k_2 is obtained.

$$\frac{1}{k_1}\left[k_1^2 - \frac{2m^2}{r_2^2}\right] \cot k_1 (r_1 - r_2) = -\frac{\beta'}{\mu'} \frac{1}{k_2}\left[k_2^2 - \frac{2m^2}{r_2^2}\right] \cot\left(k_2 r_2 - \frac{m\pi}{2} - \frac{\pi}{4}\right) e^{j\beta z} \tag{17 b}$$

Separating the real and imaginary parts the following relations are obtained:

$$\frac{1}{k_1}\left[k_1{}^2 - \frac{2m^2}{r_2{}^2}\right] \cot k_1 (r_1 - r_2) = -\frac{\beta'}{\mu'}\frac{1}{k_2}\left[k_2{}^2 - \frac{2m^2}{r_2{}^2}\right] \cot\left(k_2 r_2 - \frac{m\pi}{2} - \frac{\pi}{4}\right) \cos \beta z \tag{17 c}$$

$$\frac{\beta'}{\mu'}\frac{1}{k_2}\left[k_2{}^2 - \frac{2m^2}{r_2{}^2}\right] \cot\left(k_2 r_2 - \frac{m\pi}{2} - \frac{\pi}{4}\right) \sin \beta z = 0 \tag{17 d}$$

From (17 *d*) it is evident that either $\sin \beta z = 0$ or $\cot\left(k_2 r_2 - \frac{m\pi}{2} - \frac{\pi}{4}\right)$ vanishes. Considering the second relation the value of k_2 is given by

$$k_2 = \frac{1}{r_2}\left[(m + 2n)\frac{\pi}{2} - \frac{\pi}{4}\right] \tag{18}$$

As $k_2 = [\omega^2 \mu_2 \epsilon_2 + \gamma_2{}^2]^{\frac{1}{2}}$, so the value of γ_2 is given by the following expression

$$\gamma_2 = \left[\frac{1}{r_2{}^2}\left\{(m + 2n)\frac{\pi}{2} - \frac{\pi}{4}\right\}^2 - \omega^2 \mu_2 \epsilon_2\right]^{\frac{1}{2}} \tag{18 a}$$

In order that propagation may take place through the second medium γ_2 must be imaginary (*i.e.*)

$$\omega^2 \mu_2 \epsilon_2 > \frac{1}{r_2{}^2}\left[(m + 2n)\frac{\pi}{2} - \frac{\pi}{4}\right]^2 \tag{18 b}$$

The equation (18 *a*) can be written as

$$\gamma_2 = \alpha_2 + j\beta_2 = j\left[\omega^2 \mu_2 \epsilon_2 - \frac{1}{r_2{}^2}\left\{(m + 2n)\frac{\pi}{2} - \frac{\pi}{4}\right\}^2\right]^{\frac{1}{2}}$$

So, the attenuation constant $\alpha_2 = 0$ and the phase constant β_2 in the second medium is given by

$$\beta_2 = \left[\omega^2 \mu_2 \epsilon_2 - \frac{1}{r_2{}^2}\left\{(m + 2n)\frac{\pi}{2} - \frac{\pi}{4}\right\}^2\right] \tag{18 c}$$

For propagation to take place β_2 must be real and the condition (18 *b*) must be fulfilled. The equations (18 *b*) and (18 *c*) indicate that there must be a cut-off frequency f_c given as follows below which there will be no propagation through the second medium

$$f_{c2} = \frac{c_2}{2\pi r_2}(m + 2n)\frac{\pi}{2} - \frac{\pi}{4}\Big] \tag{18 d}$$

where $c_2 = 1/\sqrt{\mu_2\epsilon_2}$ is the free wave velocity in the second medium having constants μ_2 and ϵ_2. The cut-off wavelength λ_{c2} corresponding to f_{c2} is

$$\lambda_{c2} = \frac{2\pi r_2}{\left[(m+2n)\frac{\pi}{2} - \frac{\pi}{4}\right]} \tag{18 e}$$

The phase velocity c_{p2} of the wave in the second medium is

$$c_{p2} = \frac{\omega}{\beta_2} = \frac{\omega}{\left[\omega^2\mu_2\epsilon_2 - \frac{1}{r_2^2}\left\{(m+2n)\frac{\pi}{2} - \frac{\pi}{4}\right\}^2\right]^{\frac{1}{2}}} \tag{18 f}$$

The group velocity c_{g2} in the second medium is

$$c_{g2} \doteqdot 1\Big/\frac{\partial\beta_2}{\partial\omega} = \frac{\left[\omega^2\mu_2\epsilon_2 - \frac{1}{r_2^2}\left\{(m+2n)\frac{\pi}{2} - \frac{\pi}{4}\right\}^2\right]^{\frac{1}{2}}}{\omega\,\mu_2\epsilon_2} \tag{18 g}$$

The guide wavelength λ_{g2} in the second medium is

$$\lambda_{g2} = \frac{2\pi}{\beta_2} = \frac{2\pi}{\left[\omega^2\mu_2\epsilon_2 - \frac{1}{r_2^2}\left\{(m+2n)\frac{\pi}{2} - \frac{\pi}{4}\right\}^2\right]^{\frac{1}{2}}} \tag{18 h}$$

The value of k_1 can be found from (17 *c*) and (18) as follows:

$$\frac{2m^2}{r_2^2}\frac{\cot k_1(r_1 - r_2)}{k_1} - k_1 \cot k_1(r_1 - r_2) = \frac{\beta'}{\mu'}\frac{b}{ar_2}\cos\beta z \cot(2n-1)\frac{\pi}{2} \tag{19}$$

where

$$(m+2n)\frac{\pi}{2} - \frac{\pi}{4} = a$$

$$\left[(m+2n)\frac{\pi}{2} - \frac{\pi}{4}\right]^2 - 2m^2 = b \tag{19 a}$$

and

$$n = 0,\ 1,\ 2,\ 3,\ 4\ldots\ldots$$

As $\cot(2n-1)\frac{\pi}{2} = 0$, k_1 is obtained from (19) as

$$k_1 = \pm\sqrt{2}\,\frac{m}{r_2} \tag{19 b}$$

As $k_1^2 = \omega^2\mu_1\epsilon_1 + \gamma_1^2$, the value of the propagation constant γ_1 in the first medium is

$$\gamma_1 = \left[\frac{2m^2}{r_2^2} - \omega^2\mu_1\epsilon_1\right]^{\frac{1}{2}} \tag{19 c}$$

In order that propagation may take place through the first medium γ_1 must be imaginary and

$$\omega^2\mu_1\epsilon_1 \text{ must be } > \frac{2m^2}{r_2^2} \tag{19 d}$$

Using the same arguments as followed above in the case of the second medium, the following constants for the first medium are obtained:

$$\beta_1 = \left[\omega^2\mu_1\epsilon_1 - \frac{2m^2}{r_2^2}\right]^{\frac{1}{2}} \tag{19 e}$$

$$f_{c1} = \pm\frac{c_1 m}{\sqrt{2}\pi r_2} \tag{19 f}$$

where

$$c_1 = 1/\sqrt{\mu_1\epsilon_1}$$

$$\lambda_{c1} = \pm\frac{\sqrt{2}\pi r_2}{m} \tag{19 g}$$

$$c_{p1} = \omega\bigg/\left[\omega^2\mu_1\epsilon_1 - \frac{2m^2}{r_2^2}\right]^{\frac{1}{2}} \tag{19 h}$$

$$c_{g1} \doteqdot \left[\omega^2\mu_1\epsilon_1 - \frac{2m^2}{r_2^2}\right]^{\frac{1}{2}}\bigg/\omega\mu_1\epsilon_1 \tag{19 i}$$

$$\lambda_{g1} = 2\pi\bigg/\left[\omega^2\mu_1\epsilon_1 - \frac{2m^2}{r_2^2}\right]^{\frac{1}{2}} \tag{19 j}$$

PROPAGATION CHARACTERISTICS OF THE TM_{mn} MODE

In the case of the TM mode the propagation characteristics in the second medium have been given (Chatterjee, 1953) in terms of A_4 which contains k_1. In the present paper k_1 and k_2 are separated and their values are given in a much simpler form.

Applying the boundary condition $H_{\theta 1} = H_{\theta 2}$ at $r = r_2$, utilising the relation

$$-\frac{A_3}{A_4} = \frac{Y_m(k_1 r_1)}{J_m(k_1 r_1)} \tag{20}$$

and substituting

$$A_4 = \frac{k'\beta'}{\epsilon} \frac{J'_m(k_2r_2)\, J_m(k_1r_1)}{Y'_m(k_1r_2)\, J_m(k_1r_1) - J'_m(k_1r_2)\, Y_m(k_1r_1)} \tag{20 a}$$

the following equation is obtained from (19) and (19 *a*) (Chatterjee, *loc. cit.*, Part II)

$$-\beta' J'_m(k_2r_2) = j^{m+1}\, I'_m(k'_2\, r_2) \tag{20 b}$$

which gives

$$\beta_2 = \beta_1 \tag{20 c}$$

Applying the boundary condition $E_{z1} = E_{z2}$ at $r = r_2$ and utilising the relation given in (20), we obtain the following relation from (19) and (19 *a*), (Chatterjee, *loc. cit.*, Part II).

$$A_4 \left[\frac{Y_m(k_1r_1)\, J_m(k_1r_2) - Y_m(k_1r_2)\, J_m(k_1r_1)}{J_m(k_1r_1)} \right]$$

$$= \left[k_2^2\, J''_m(k_2r_2) + \frac{k_2}{r_2} J'_m(k_2r_2) - \frac{m^2}{r_2^2} J_m(k_2r_2) \right] e^{j\beta z} \tag{20 d}$$

Utilising the relations given in (10 *d*) and substituting A_4 from (20 *a*), the equation (20 *d*) reduces to

$$\frac{J'_m(k_2r_2)}{J_m(k_2r_2)} \left[\frac{Y_m(k_1r_1) J_m(k_1r_2) - Y_m(k_1r_2)\, J_m(k_1r_1)}{Y'_m(k_1r_2)\, J_m(k_1r_1) - J'_m(k_1r_2)\, Y_m(k_1r_1)} \right]$$

$$= -\frac{k_2^2 \epsilon}{k'\beta'} e^{j\beta z} \tag{20 e}$$

Using the values of J_m's and Y_m's for large arguments as given in (17 *a*), the equation (20 *e*) can be reduced to the following:

$$\frac{1}{k_1} \tan k_1 (r_1 - r_2) = \frac{\epsilon}{\beta'} k_2 \cot \left(k_2 r_2 - \frac{m\pi}{2} - \frac{\pi}{4} \right) e^{j\beta z} \tag{20 f}$$

Separating the real and imaginary parts, the following relations are obtained from (20 *f*):

$$\frac{1}{k_1} \tan k_1 (r_1 - r_2) = \frac{\epsilon}{\beta'} k_2 \cot \left(k_2 r_2 - \frac{m\pi}{2} - \frac{\pi}{4} \right) \cos \beta z \tag{20 g}$$

$$\frac{\epsilon}{\beta'} k_2 \cot \left(k_2 r_2 - \frac{m\pi}{2} - \frac{\pi}{4} \right) \sin \beta z = 0 \tag{20 h}$$

From (20 h) it is evident that either

$$\sin \beta z = 0 \text{ or } \cot\left(k_2 r_2 - \frac{m\pi}{2} - \frac{\pi}{4}\right) = 0 \tag{20 i}$$

Considering the latter relation, the following value of k_2 is obtained:

$$k_2 = \frac{1}{r_2}\left[\frac{m\pi}{2} + (2n+1)\frac{\pi}{2} + \frac{\pi}{4}\right] \tag{21}$$

Following the same arguments as in the case of the TE mode, the following constants for the second medium are obtained:

$$\gamma_2 = \left[\frac{1}{r_2^2}\left\{\frac{m\pi}{2} + (2n+1)\frac{\pi}{2} + \frac{\pi}{4}\right\}^2 - \omega^2\mu_2\epsilon_2\right]^{\frac{1}{2}} \tag{21 a}$$

$$\beta_2 = \left[\omega^2\mu_2\epsilon_2 - \frac{1}{r_2^2}\left\{\frac{m\pi}{2} + (2n+1)\frac{\pi}{2} + \frac{\pi}{4}\right\}^2\right]^{\frac{1}{2}} \tag{21 b}$$

$$f_{c2} = \frac{c_2}{2\pi r_2}\left[\frac{m\pi}{2} + (2n+1)\frac{\pi}{2} + \frac{\pi}{4}\right] \tag{21 c}$$

$$\lambda_{c2} = \frac{2\pi r_2}{\left[\frac{m\pi}{2} + (2n+1)\frac{\pi}{2} + \frac{\pi}{4}\right]} \tag{21 d}$$

$$c_{p2} = \omega\Big/\left[\omega^2\mu_2\epsilon_2 - \frac{1}{r_2^2}\left\{\frac{m\pi}{2} + (2n+1)\frac{\pi}{2} + \frac{\pi}{4}\right\}^2\right]^{\frac{1}{2}} \tag{21 e}$$

$$c_{g2} \doteqdot \left[\omega^2\mu_2\epsilon_2 - \frac{1}{r_2^2}\left\{\frac{m\pi}{2} + (2n+1)\frac{\pi}{2} + \frac{\pi}{4}\right\}^2\right]^{\frac{1}{2}}\Big/\omega\mu_2\epsilon_2 \tag{21 f}$$

$$\lambda_{g2} = 2\pi\Big/\left[\omega^2\mu_2\epsilon_2 - \frac{1}{r_2^2}\left\{\frac{m\pi}{2} + (2n+1)\frac{\pi}{2} + \frac{\pi}{4}\right\}^2\right]^{\frac{1}{2}} \tag{21 g}$$

In order to find the constants in the first medium we proceed as follows: From (20 g) and (21) the following equation is obtained:

$$\frac{1}{k_1}\tan k_1(r_1 - r_2) = \frac{\epsilon}{\beta'}\frac{1}{r_2}\left[\frac{m\pi}{2} + (2n+1)\frac{\pi}{2} + \frac{\pi}{4}\right]\cot\left[(2n+1)\frac{\pi}{2}\right]\cos\beta z \tag{22}$$

For

$$n = 0,\ 1,\ 2 \ldots\ldots,\ \frac{1}{k_1}\tan k_1(r_1 - r_2) = 0 \tag{22 a}$$

which gives

$$k_1 = \frac{n\pi}{r_1 - r_2} \tag{22 b}$$

The other constants of the first medium are

$$\gamma_1 = \left[\left(\frac{n\pi}{r_1 - r_2}\right)^2 - \omega^2\mu_1\epsilon_1\right]^{\frac{1}{2}} \tag{22 c}$$

$$\beta_1 = \left[\omega^2\mu_1\epsilon_1 - \left(\frac{n\pi}{r_1 - r_2}\right)^2\right]^{\frac{1}{2}} \tag{22 d}$$

$$f_{c1} = \frac{c_1}{2}\frac{n}{r_1 - r_2} \tag{22 e}$$

$$\lambda_{c1} = \frac{2(r_1 - r_2)}{n} \tag{22 f}$$

$$\lambda_{g1} = 2\pi\Big/\left[\omega^2\mu_1\epsilon_1 - \left(\frac{\pi n}{r_1 - r_2}\right)^2\right]^{\frac{1}{2}} \tag{22 g}$$

$$c_{p1} = \omega\Big/\left[\omega^2\mu_1\epsilon_1 - \left(\frac{n\pi}{r_1 - r_2}\right)^2\right]^{\frac{1}{2}} \tag{22 h}$$

$$c_{g1} \doteq \frac{\left[\omega^2\mu_1\epsilon_1 - \left(\frac{n\pi}{r_1 - r_2}\right)^2\right]^{\frac{1}{2}}}{\omega\mu_1\epsilon_1} \tag{22 i}$$

For microwave propagation generally the lower modes TM_{01} ($m = 0$, $n = 1$), TE_{01} ($m = 0$, $n = 1$) and TM_{11} ($m = 1$, $n = 1$) are used. The propagation characteristics for the TE_{mn} and the TM_{mn} modes are collected together in Table I for convenience of reference. The propagation characteristics for the TE_{01} and the TM_{11} modes have been deduced from Table I by using proper values for the mode subscripts and are given in Table II. The following conclusions can be drawn from the tables.

1. *Phase Velocity*.—In the case of both the TE and the TM wave the phase velocities can be adjusted by adjusting the values of ϵ_1 and ϵ_2. For example, by increasing ϵ_2 compared to ϵ_1, the velocity of the wave in the second medium may be considerably lowered. This method of slowing down the wave may find application in the electronic devices, such as travelling-wave tube, linear accelerator, etc., where it is necessary to slow down

the wave along the axis in order that efficient interaction may occur between the wave and the electrons.

In the case of the TE_{01} wave the ratio of the phase velocities c_{p1}/c_{p2} in the two media is given from Table II as

$$\left(\frac{c_{p1}}{c_{p2}}\right)_{TE_{01}} = \sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}}\left[1 - \frac{9\pi^2}{16\, r_2^2\, \omega^2 \mu_2\epsilon_2}\right]^{\frac{1}{2}}$$

$$\doteqdot \sqrt{\frac{\epsilon_2}{\epsilon_1}}\left[1 - \frac{9\pi^2}{32\, \omega^2 r_2^2\, \epsilon_2\mu_2}\right] \tag{23}$$

TABLE I

Propagation Characteristics of TE_{mn} and TM_{mn} Modes

	TE_{mn}	
	First Medium	Second Medium
k	$\sqrt{2}\,\frac{m}{r_2}$	$\frac{1}{r_2}\left[(m+2n)\frac{\pi}{2} - \frac{\pi}{4}\right]$
γ	$\left[\frac{2m^2}{r_2^2} - \omega^2\mu_1\epsilon_1\right]^{\frac{1}{2}}$	$\left[\frac{1}{r_2^2}\left\{(m+2n)\frac{\pi}{2} - \frac{\pi}{4}\right\}^2 - \omega^2\mu_2\epsilon_2\right]^{\frac{1}{2}}$
β	$\left[\omega^2\mu_1\epsilon_1 - \frac{2m^2}{r_2^2}\right]^{\frac{1}{2}}$	$\left[\omega^2\mu_2\epsilon_2 - \frac{1}{r_2^2}\left\{(m+2n)\frac{\pi}{2} - \frac{\pi}{4}\right\}^2\right]^{\frac{1}{2}}$
f_c	$\frac{c_1 m}{\sqrt{2}\,\pi r_2}$	$\frac{c_2}{2\pi r_2}\left[(m+2n)\frac{\pi}{2} - \frac{\pi}{4}\right]$
λ_c	$\frac{\sqrt{2}\,\pi r_2}{m}$	$\frac{2\pi r_2}{\left[(m+2n)\frac{\pi}{2} - \frac{\pi}{4}\right]}$
c_p	$\frac{\omega}{\left[\omega^2\mu_1\epsilon_1 - \frac{2m^2}{r_2^2}\right]^{\frac{1}{2}}}$	$\frac{\omega}{\left[\omega^2\mu_2\epsilon_2 - \frac{1}{r_2^2}\left\{(m+2n)\frac{\pi}{2} - \frac{\pi}{4}\right\}^2\right]^{\frac{1}{2}}}$
c_g	$\frac{\left[\omega^2\mu_1\epsilon_1 - \frac{2m^2}{r_2^2}\right]^{\frac{1}{2}}}{\omega\mu_1\epsilon_1}$	$\frac{\left[\omega^2\mu_2\epsilon_2 - \frac{1}{r_2^2}\left\{(m+2n)\frac{\pi}{2} - \frac{\pi}{4}\right\}^2\right]^{\frac{1}{2}}}{\omega\mu_2\epsilon_2}$
λ_g	$\frac{2\pi}{\left[\omega^2\mu_1\epsilon_1 - \frac{2m^2}{r_2^2}\right]^{\frac{1}{2}}}$	$\frac{2\pi}{\left[\omega^2\mu_2\epsilon_2 - \frac{1}{r_2^2}\left\{(m+2n)\frac{\pi}{2} - \frac{\pi}{4}\right\}^2\right]^{\frac{1}{2}}}$

TABLE I (*Contd.*)

	TM_{mn}	
	First Medium	Second Medium
k	$\dfrac{n\pi}{r_1 - r_2}$	$\dfrac{1}{r_2}\left[\dfrac{m\pi}{2} + (2n+1)\dfrac{\pi}{2} + \dfrac{\pi}{4}\right]$
γ	$\left[\left(\dfrac{n\pi}{r_1 - r_2}\right)^2 - \omega^2\mu_1\epsilon_1\right]^{\frac{1}{2}}$	$\left[\dfrac{1}{r_2^2}\left\{\dfrac{m\pi}{2} + (2n+1)\dfrac{\pi}{2} + \dfrac{\pi}{4}\right\}^2 - \omega^2\mu_2\epsilon_2\right]^{\frac{1}{2}}$
β	$\left[\omega^2\mu_1\epsilon_1 - \left(\dfrac{n\pi}{r_1 - r_2}\right)^2\right]^{\frac{1}{2}}$	$\left[\omega^2\mu_2\epsilon_2 - \dfrac{1}{r_2^2}\left\{\dfrac{m\pi}{2} + (2n+1)\dfrac{\pi}{2} + \dfrac{\pi}{4}\right\}^2\right]^{\frac{1}{2}}$
f_c	$\dfrac{c_1 n}{2(r_1 - r_2)}$	$\dfrac{c_2}{2\pi r_2}\left[\dfrac{m\pi}{2} + (2n+1)\dfrac{\pi}{2} + \dfrac{\pi}{4}\right]$
λ_c	$\dfrac{2(r_1 - r_2)}{n}$	$\dfrac{2\pi r_2}{\left[\dfrac{m\pi}{2} + (2n+1)\dfrac{\pi}{2} + \dfrac{\pi}{4}\right]}$
c_p	$\dfrac{\omega}{\left[\omega^2\mu_1\epsilon_1 - \left(\dfrac{n\pi}{r_1 - r_2}\right)^2\right]^{\frac{1}{2}}}$	$\dfrac{\omega}{\left[\omega^2\mu_2\epsilon_2 - \dfrac{1}{r_2^2}\left\{\dfrac{m\pi}{2} + (2n+1)\dfrac{\pi}{2} + \dfrac{\pi}{4}\right\}^2\right]^{\frac{1}{2}}}$
c_g	$\dfrac{\left[\omega^2\mu_1\epsilon_1 - \left(\dfrac{n\pi}{r_1 - r_2}\right)^2\right]^{\frac{1}{2}}}{\omega\mu_1\epsilon_1}$	$\dfrac{\left[\omega^2\mu_2\epsilon_2 - \dfrac{1}{r_2^2}\left\{\dfrac{m\pi}{2} + (2n+1)\dfrac{\pi}{2} + \dfrac{\pi}{4}\right\}^2\right]^{\frac{1}{2}}}{\omega\mu_2\epsilon_2}$
λ_g	$\dfrac{2\pi}{\left[\omega^2\mu_1\epsilon_1 - \left(\dfrac{n\pi}{r_1 - r_2}\right)^2\right]^{\frac{1}{2}}}$	$\dfrac{2\pi}{\left[\omega^2\mu_2\epsilon_2 - \dfrac{1}{r_2^2}\left\{\dfrac{m\pi}{2} + (2n+1)\dfrac{\pi}{2} + \dfrac{\pi}{4}\right\}^2\right]^{\frac{1}{2}}}$

At microwave frequencies

$$c_{p1}/c_{p2} \doteqdot \sqrt{\epsilon_2/\epsilon_1} \tag{23 a}$$

In the caso of the TM_{11} wave the ratio of the phase velocities in the two media is given from Table II as

$$\left(\frac{c_{p1}}{c_{p2}}\right)_{TM_{11}} \doteqdot \sqrt{\frac{\epsilon_2}{\epsilon_1}}\, \frac{\left[1 - \dfrac{81\pi^2}{32\,\omega^2 r_2^2 \mu_2\epsilon_2}\right]}{\left[1 - \dfrac{\pi^2}{2\omega^2\mu_1\epsilon_1(r_1 - r_2)^2}\right]} \tag{24}$$

At microwave frequencies this reduces to the same relation as (23 *a*).

TABLE II

Propagation Characteristics of TE_{01} and TM_{11} Modes

	TE_{01}		TM_{11}	
	First Medium	Second Medium	First Medium	Second Medium
k	0	$3\pi/4r_2$	$\pi/(r_2 - r_2)$	$9\pi/4r_2$
γ	$j\omega\sqrt{\mu_1\epsilon_1}$	$\left[\frac{1}{r_2^2}\frac{9\pi^2}{16} - \omega^2\mu_2\epsilon_2\right]^{\frac{1}{2}}$	$\left[\left(\frac{\pi}{r_1-r_2}\right)^2 - \omega^2\mu_1\epsilon_1\right]^{\frac{1}{2}}$	$\left[\frac{81\pi^2}{16r_2^2} - \omega^2\mu_2\epsilon_2\right]^{\frac{1}{2}}$
β	$\omega\sqrt{\mu_1\epsilon_1}$	$\left[\omega^2\mu_2\epsilon_2 - \frac{9\pi^2}{16r_2^2}\right]^{\frac{1}{2}}$	$\left[\omega^2\mu_1\epsilon_1 - \left(\frac{\pi}{r_1-r_2}\right)^2\right]^{\frac{1}{2}}$	$\left[\omega^2\mu_2\epsilon_2 - \frac{81\pi^2}{16r_2^2}\right]^{\frac{1}{2}}$
f_c	0	$3c_2/8r_2$	$c_1/2(r_1 - r_2)$	$9c_2/8r_2$
λ_c	∞	$8r_2/3$	$2(r_1 - r_2)$	$8r_2/9$
c_p	c_1	$\frac{\omega}{\left[\omega^2\mu_2\epsilon_2 - \frac{9\pi^2}{16r_2^2}\right]^{\frac{1}{2}}}$	$\frac{\omega}{\left[\omega^2\mu_1\epsilon_1 - \left(\frac{\pi}{r_1-r_2}\right)^2\right]^{\frac{1}{2}}}$	$\frac{\omega}{\left[\omega^2\mu_2\epsilon_2 - \frac{81\pi^2}{16r_2^2}\right]^{\frac{1}{2}}}$
c_g	c_1	$\frac{\left[\omega^2\mu_2\epsilon_2 - \frac{9\pi^2}{16r_2^2}\right]^{\frac{1}{2}}}{\omega\mu_2\epsilon_2}$	$\frac{\left[\omega^2\mu_1\epsilon_1 - \left(\frac{\pi}{r_1-r_2}\right)^2\right]^{\frac{1}{2}}}{\omega\mu_1\epsilon_1}$	$\frac{\left[\omega^2\mu_2\epsilon_2 - \frac{81\pi^2}{16r_2^2}\right]^{\frac{1}{2}}}{\omega\mu_2\epsilon_2}$
λ_g	$\frac{2\pi c_1}{\omega}$	$\frac{2\pi}{\left[\omega^2\mu_2\epsilon_2 - \frac{9\pi^2}{16r_2^2}\right]^{\frac{1}{2}}}$	$\frac{2\pi}{\left[\omega^2\mu_1\epsilon_1 - \left(\frac{\pi}{r_1-r_2}\right)^2\right]^{\frac{1}{2}}}$	$\frac{2\pi}{\left[\omega^2\mu_2\epsilon_2 - \frac{81\pi^2}{16r_2^2}\right]^{\frac{1}{2}}}$

In the case of TE_{01} wave c_{p1}/c_{p2} is independent of the radius of the outer dielectric and depends only on the radius of the inner dielectric. But in the case of the TM_{11} wave the ratio c_{p1}/c_{p2} depends on r_2 as well as r_1. Let us consider the following example. Suppose a TE_{01} wave is to be set up at $\omega = 3 \cdot 10^{10}$ per second travelling at one-tenth the velocity of light in vacuum so that $c_{p2} = 3 \cdot 10^7$ metres per second. The value of r_2 is obtained as $r_2 = 2$ mm., if the following values for the constants of the two media are used

$$\epsilon_1 = 10^{-9}/36\pi \quad \text{Farad/meter}$$

$$\epsilon_2 = 10^{-7}/18\pi \quad \text{Farad/meter (Relative dielectric constant} = 200)$$

$$\mu_1 = \mu_2 = 4\pi \cdot 10^{-7} \quad \text{Henry/meter.}$$

This shows that a dielectric rod of diameter 14 mm. and having a relative dielectric constant 200 placed at the centre of a cylindrical guide filled with air will slow down a TE_{01} wave to one-tenth the speed of light in vacuum. It is to be noticed that the wave will be slowed down to one-tenth of its free space value irrespective of the diameter of the external dielectric. In the case of the TM wave the radius of the external dielectric comes into play. Using the above constants for the media and $r_2 = 7$ mm., in the case of the TM wave, it is found from the equation (24) that a radius of the outer dielectric necessary to reduce the speed of the TM wave to one-tenth of its free space value is 49 mm.

2. *Phase Constant.*—The phase constant β_2 in the second medium can be equal to the phase constant β_1 in the first medium (equation 20 *c*) in the case of the TM_{mn} mode only when the following relation between ϵ_1 and ϵ_2 is fulfilled.

$$\epsilon_1 - \epsilon_2 - = \frac{1}{\mu\omega^2}\left[\left(\frac{n\pi}{r_1 - r_2}\right)^2 - \frac{1}{r_2{}^2}\left\{\frac{m\pi}{2} + (2n+1)\frac{\pi}{2} + \frac{\pi}{4}\right\}^2\right]$$

For TM_{11} mode the above condition reduces to

$$\epsilon_1 - \epsilon_2 = \frac{\pi^2}{\mu\omega^2}\left[\frac{1}{(r_1 - r_2)^2} - \frac{81}{16 r_2{}^2}\right] \tag{25}$$

When $\omega = 3 \cdot 10^{10}$ per second, $r_1 = 0 \cdot 049$ meter, $r_2 = 0 \cdot 007$ meter, $\mu = 4\pi . 10^{-7}$ Henry per meter, the dielectric constants for the two media are related by the following relation

$$\epsilon_2 = \epsilon_1 + 14 \cdot 618 \times 10^{-9}$$

This shows that ϵ_2 has to be always greater than ϵ_1 in order that the wave may be slowed down in the second medium.

Power Flow in the Two Media

The peak power flowing through the two media is given by the following expression

$$\hat{P} = \hat{P}_1 + \hat{P}_2 = \int_{r=r_2}^{r_1}\int_{\theta=0}^{2\pi} (E_{r1}\, H^*{}_{\theta 1} - E_{\theta 1}\, H^*{}_{r1})\, r\, d\theta dr + \int_{r=0}^{r_2}\int_{\theta=0}^{2\pi}$$

$$(E_{r2}\, H^*{}_{\theta 2} - E_{\theta 2}\, H^*{}_{r2})\, r d\theta dr \tag{25 a}$$

The field components for the TE_{01} mode are given from (15) and (15 *a*) as follows

$$
\begin{aligned}
E_{z1} &= E_{r1} = H_{\theta 1} = 0 \\
E_{\theta 1} &= \mu_1 \omega (P')^{\frac{1}{2}} k_1 [A'J'_0(k_1 r) + B'Y'_0(k_1 r)] \sin \beta_1 z \\
H_{r1} &= - \beta_1 k_1 (P')^{\frac{1}{2}} [A'J'_0(k_1 r) + B'Y'_0(k_1 r)] \sin \beta_1 z \\
H_{z1} &= k_1^2 (P')^{\frac{1}{2}} [A' J_0(k_1 r) + B'Y_0(k_1 r)] \cos \beta_1 z \\
E_{\theta 2} &= \mu_2 \omega k_2 (P')^{\frac{1}{2}} J'_0(k_2 r) \sin \beta_2 z \\
H_{r2} &= - \beta_2 k_2 (P')^{\frac{1}{2}} J'_0(k_2 r) \sin \beta_2 z \\
H_{z2} &= k_2^2 (P')^{\frac{1}{2}} J_0(k_2 r) \cos \beta_2 z
\end{aligned}
\tag{26}
$$

For the TE_{01} mode $k_1 = 0$, so there is no power flow through the first medium. This means that the power flow is concentrated only in the second medium. So the total peak power flow is given from (26) and (25) as

$$
\hat{P} = \hat{P}_2 = A \sin^2 \beta_2 z \left[\frac{1}{2} r^2 \left\{ J_0^2(k_2 r) + J_1^2(k_2 r) - \frac{2J_0(k_2 r) J_1(k_2 r)}{k_2 r} \right\} \right]_0^{r_2}
\tag{27}
$$

where $A = - 2\pi\mu_2\omega\beta_2 k_2^2 P'$ may be considered as the amplitude term. The factor inside the bracket in (27) when plotted *vs.* r from $r = 0$ to $r = r_2$ gives the power distribution along the radial line in the second medium. It is evident that the power flow takes place throughout the volume of the second medium with a node along the axis and that the power flow is an increasing function with the radius of the dielectric. From equation (27) it is obvious that the power flow varies with z and the power flow can be considered as consisting of a term independent of the distance and a term varying consinusoidally with distance.

FIELD PATTERN FOR THE TE_{01} MODE

The field pattern for the TE_{01} mode in the second medium can be found from (26) by introducing $\exp(j\omega t)$. This gives the following expression for H_z and H_r

$$
\begin{aligned}
H_{z2} &= k_2^2 (P')^{\frac{1}{2}} J_0(k_2 r) \cos \beta_2 z \, e^{j\omega t} \\
H_{r2} &= - \beta_2 k_2 (P')^{\frac{1}{2}} J'_0(k_2 r) \sin \beta_2 z \, e^{j\omega t}
\end{aligned}
$$

A plot of the magnetic field at any instant of time may be obtained by putting t = constant = 0 and forming the differential equation for the lines of force (Barrow, 1936).

$$\frac{d(\beta z)}{dr} = \frac{|H_z|}{|H_r|}$$

which gives the following differential equation for the second medium.

$$\frac{|H_{z2}|}{|H_{r2}|} = -\frac{k_2 J_0(k_2 r)}{\beta_2 J'_0(k_2 r)} \cot \beta_2 z = \frac{k_2 J_0(k_2 r)}{\beta_2 J_1(k_2 r)} \cot \beta_2 z \qquad (28)$$

The above equation when solved graphically gives the field distribution at any instant of time. The wave propagation may be considered to be a movement of the field structure down the tube with the phase velocity given in Table II and without any alteration of shape or of magnitude. The field pattern for the TM_{11} mode can be found similarly.

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(to be continued)