REVIEWS

# On the Burnside Algebra of a Finite Group*

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Abstract | In this article first we shall prove the classical theorem of Burnside which asserts that the canonical Burnside mark homomorphism of the Burnside algebra $\mathrm{B}(G)$ of a finite group $G$ into the product $\mathbb{Z}$-algebra of rank $\#\mathfrak{C}_G$ is injective, where $\mathfrak{C}_G$ denote the set of conjugacy classes of the subgroups of $G$. We further prove that for any finite group $G$ the canonical $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}^{\mathfrak{C}_{\mathbf{Z}_G}} \to \mathbb{Z}^{\mathfrak{C}_G}$ maps the Burnside algebra $\mathrm{B}(\mathbf{Z}_G)$ of a finite cyclic group $\mathbf{Z}_G$ of order $\#G$ into the Burnside algebra $\mathrm{B}(G)$. We deduce quite a few elementary, but important results in finite group theory by using this canonical algebra homomorphism. Finally we describe the prime spectrum $\operatorname{Spec}\mathrm{B}(G)$ and maximal spectrum $\operatorname{Spm}\mathrm{B}(G)$ of $\mathrm{B}(G)$.

## Introduction

It is well known that the concept of group action plays fundamental role in almost all parts of mathematics and has many applications in physical sciences. In this (expository) article, we discuss basic properties of the *Burnside ring* $\mathrm{B}(G)$ of a finite group $G$. It is an algebraic construction that encodes the different ways $G$ can act on finite sets and is an axiomatic generalisation of Burnside's ideas from nineteenth century and techniques in [1], but the ring structure is a more recent development and appears in an article of Solomon [10]

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This article grew out of lectures on Group actions and their applications by Prof. D. P. Patil, Department of Mathematics/

Computer Science and Automation. The second author gratefully acknowledges wonderful support provided by Prof. D. P. Patil at every stage during the preparation of this article.

The Burnside ring $\mathrm{B}(G)$ of the finite group $G$ is the fundamental representation ring of $G$, namely the ring of *permutation* representations. It is the universal object in the category of finite $G$-sets and is an analogue of the ring $\mathbb{Z}$ of integers in this category. The Burnside ring is the natural framework to study the *invariants* attached to structured $G$-sets and it can be studied from different points of view.

In Section 1 we collect some preliminaries on group actions. In Section 2 we give construction of the Burnside ring of a finite group and prove the classical result of Burnside which asserts that the Burnside mark homomorphism is injective. In Section 3 we shall prove that the canonical $\mathbb{Z}$-algebra homomorphism $\mathbb{Z}^{\mathfrak{C}_{\mathbf{Z}_G}} \to \mathbb{Z}^{\mathfrak{C}_G}$ maps the Burnside algebra $\mathrm{B}(\mathbf{Z}_G)$ of a finite cyclic group $\mathbf{Z}_G$ of order $\#G$ into the Burnside algebra $\mathrm{B}(G)$ of $G$.

Important results in group theory can be proved by comparing group theoretic invariants of a finite group $G$ with the same invariants of the cyclic group $\mathbf{Z}_G$ of the same order as that of $G$. This precise conceptual interpretation is based on the various ideas and tricks introduced by Frobenious and Wielandt. Using the canonical homomorphism we deduce quite a few elementary, but important results in finite group theory. Finally, in Section 4 we describe the prime spectrum $\operatorname{Spec} \mathrm{B}(G)$ and maximal spectrum $\operatorname{Spm} \mathrm{B}(G)$ of $\mathrm{B}(G)$.

## § 1 Preliminaries – Group Actions

In this section let us recall the basic concepts which use *group actions* and set up some notation which will be used throughout this article. For concept of a group action and basic results concerning them, we refer the reader to see the article by Patil and Storch [8].

**1.1 Group Actions** Let us recall the important concept of operation of an (arbitrary) group on an (arbitrary) set. Let $G$ be a (multiplicatively written) group with the identity element $1 := 1_G$. An (left) operation or an action of $G$ (as a group) on a set $X$ is a map $G \times X \to X, (g,x) \mapsto gx$, with the following properties:

(i) $(gh)x = g(hx)$ for all $g, h \in G$ and all $x \in X$.
(ii) $1x = x$ for all $x \in X$,

or, equivalently, the map $\vartheta : G \to \mathfrak{S}(X), \vartheta(g) := \vartheta_g : x \mapsto gx$, is a group homomorphism. Conversely, if $\vartheta : G \to \mathfrak{S}(X)$ is a group homomorphism, then the map $G \times X \to X$ defined by $(g,x) \mapsto gx := \vartheta(g)(x)$ is an operation of $G$ on $X$.

A set $X$ with an operation of a group $G$ (from the left) is called a $G$-set or a $G$-space and the group homomorphism $\vartheta : G \to \mathfrak{S}(X)$ belonging to it is called the action homomorphism of the $G$-set $X$.

The kernel of $\vartheta$, i. e. the set of $g \in G$ with $\vartheta_g = \mathrm{id}_X$ or with $gx = x$ for all $x \in X$, is called the kernel of the operation. If this kernel is trivial, then the action is called faithful or effective. If this kernel is the whole group $G$, i. e. if $gx = x$ for all $g \in G$ and all $x \in X$, then the action is the so called trivial action.

The operation of $G$ on $X$ defines in a natural way an equivalence relation $\sim_G$ on $X$. Elements $x, y \in X$ are related if $y$ is obtained from $x$ by the operation $\vartheta_g$ of a suitable element $g \in G$, i. e.

$$x \sim_G y \iff \text{there exists } g \in G \text{ with } y = gx.$$

The equivalence class $Gx := \{gx \mid g \in G\}$ of an element $x \in X$ is called the $(G\text{-})$orbit of $x$. The orbit space of $X$, i. e. the set of all orbits $Gx, x \in X$, is denoted by $X \backslash G$. The isotropy group or the stabilizer of $x \in X$ is the subgroup $G_x := \{g \in G \mid gx = x\}$ of $G$. The point $x \in X$ is a fixed point of the operation if and only if $G_x = G$. The set of all fixed points is denoted by $\mathrm{Fix}_G X$ (or $X^G$). Therefore we have:

Orbit-Stabilizer Theorem: *Let $X$ be a $G$-set. The cardinality $\#Gx$ of the orbit $Gx$ of $x$ is the index $[G : G_x] := \#(G/G_x)$ of the stabilizer $G_x$ of $x$ in $G$,* i. e. $\#Gx = [G : G_x]$. *In particular, if $G$ is finite, then the cardinality $\#Gx$ of $Gx$ divides the order $\#G$ of $G$.*

*Furthermore, the stabilizers of the elements in the same orbit are conjugate subgroups, more precisely,* $G_{gx} = gG_xg^{-1}$, $g \in G, x \in X$.

If $X$ is finite and if we count the elements of $X$ with the help of orbits of a $G$-operation on $X$, then we get the class equation: *Let G be a finite group and let X be a finite G-set. Then*

$$\#X = \sum_{Gx \in X\backslash G} \#Gx = \sum_{Gx \in X\backslash G} [G : G_x] = \#\mathrm{Fix}_G X + \sum_{\substack{Gx \in X\backslash G,\\ Gx \neq \{x\}}} [G : G_x].$$

*In particular, if X is a finite set and if G is a finite p-group (p a prime number), then*

$$\#X \equiv \mathrm{Fix}_G X \pmod p.$$

A group operation $G \times X \to X$ is called transitive if it has exactly one orbit, i. e. if $X \neq \emptyset$ and if $Gx = X$ for one (and hence for all) $x \in X$, or equivalently, $G$ operates transitively on $X$ if $X \neq \emptyset$ and if for arbitrary $x, y \in X$ there exists $g \in G$ with $y = gx$. A $G$-space with transitive operation of the group $G$ is also called a homogeneous $G$-space. The last assertion in the Orbit-Stabilizer Theorem implies that *the stabilizers* $G_x$ *of the elements* $x \in X$ *with respect to a transitive operation of* $G$ *on* $X$ *form a full conjugacy class of subgroups of* $G$.

A group operation $G \times X \to X$ is called simply transitive if it is transitive and if one and hence all stabilizers $G_x$, $x \in X$, are trivial. Equivalently, $G$ operates simply transitively on $X$ if $X \neq \emptyset$ and if for arbitrary $x, y \in X$ there exists exactly one $g \in G$ with $y = gx$. If all isotropy groups are trivial, i. e. if $G$ operates simply transitively on each orbit of the operation, then the operation is called free.

More generally, for a subgroup $H \in \mathbf{Sub}(G)$ and a $G$-set $X$, let

$$\mathrm{Fix}_H X := \{x \in X \mid hx = x \text{ for all } h \in H\}$$

be the $H$-fixed point set of $X$. If $H' = gHg^{-1}$ for some $g \in G$, then the left translation $\mathrm{L}_g : X \to X$ induces a bijection $\mathrm{Fix}_H X \xrightarrow{\sim} \mathrm{Fix}_{H'} X$, in particular, $\#\,\mathrm{Fix}_H X = \#\,\mathrm{Fix}_{H'} X$.

**1.2 Examples** Let $G$ be a group. The set of all subgroups of $G$ is denoted by $\mathbf{Sub}(G)$.

(1) (Transitive $G$-sets) For a subgroup $H \in \mathbf{Sub}(G)$, the set $G/H$ of left cosets of $H$ in $G$ is a $G$-set with the induced action of from the (left) *regular action*[1] of $G$ on itself. Moreover, this $G$-set is transitive, the stabiliser of $H = e_G H$ is the subgroup $H$ itself and hence the stabilizer of $gH \in G/H$ is the conjugate subgroup $gHg^{-1}$, $g \in G$. This special homogeneous $G$-space is denoted by $X_H$. The conjugacy class $\{gHg^{-1} \mid g \in G\}$ of $H$ (in $G$) is called the isotropy class of $X_H$.

(2) (Conjugacy Classes of Subgroups) On the set $\mathbf{Sub}(G)$ $G$-operates by conjugation: $G \times \mathbf{Sub}(G) \to \mathbf{Sub}(G)$, $(g, H) \mapsto gHg^{-1}$. The orbit of $H \in \mathbf{Sub}(G)$ of this operation is denoted by $\mathfrak{h}$; it is precisely the conjugacy class $\{gHg^{-1} \mid g \in G\}$ and has the cardinality $\#\,\mathfrak{h} = [G : \mathrm{N}_G(H)]$, where $\mathrm{N}_G(H) := \{g \in G \mid gHg^{-1} = H\}$ is *normaliser of* $H$ in $G$. The corresponding quotient group $\mathrm{W}_G(H) := \mathrm{N}_G(H)/H$ is called the *Weyl group* of $H$. The orbit

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[1] The multipication $G \times G \to G$ in the group $G$ is the most natural simply transitive operation of $G$ onto itself. The corresponding action homomorphism $\vartheta : G \to \mathfrak{S}(G)$ maps $g$ to the left multiplication $\mathrm{L}_g : G \to G$, $x \mapsto gx$. This operation is called the (left) regular operation or the Cayley operation of $G$ onto itself.

space $\mathbf{Sub}(G)/G$ is denoted by $\mathfrak{C}_G$; its elements the conjugacy classes of subgroups in $G$ and are denoted by small gothic letters $\mathfrak{h}$. Therefore, if $H$ and $H'$ are subgroups of $G$, then $\mathfrak{h} = \mathfrak{h}'$ if and only if $H$ is conjugate to $H$ in $G$, i. e. $H' = gHG^{-1}$ for some $g \in G$.

(a) If $G = \mathbf{Z}_n$ is the cyclic group of order $n$, then for every divisor $d$ of $n$ there is a unique subgroup (also cyclic) $\mathbf{Z}_d)$ of order $d$. Further, since $G$ is abelian the conjugacy class, the normaliser and the Weyl group of $\mathbf{Z}_d$ (in $G$) are precisely $\mathfrak{z}_d = \{\mathbf{Z}_d\}$, $\mathrm{N}_{\mathbf{Z}_n}(\mathbf{Z}_d) = \mathbf{Z}_n$ and $\mathrm{W}_{\mathbf{Z}_n}(\mathbf{Z}_d) = \mathbf{Z}_n/\mathbf{Z}_d = \mathbf{Z}_{n/d}$, respectively. The map $\mathrm{Div}(n) := \{d \in \mathbb{N}^+ \mid d \text{ divides } n\} \to \mathfrak{C}_{\mathbf{Z}_n}$, $d \mapsto \mathbf{Z}_d$ is bijective. In particular, $\#\mathfrak{C}_{\mathbf{Z}_n} = \#\,\mathrm{Div}(n)$.

(b) If $G = \mathfrak{S}_3$ is the permutation group on $\{1,2,3\}$, then $\mathfrak{C}_{\mathfrak{S}_3}$ has precisely 4 elements, namely, the conjugacy classes of the subgroups $\{1\}, \mathbf{Z}_2, \mathbf{Z}_3$ and $\mathfrak{S}_3$.

(c) If $G = \mathfrak{A}_4$ is the alternating group on $\{1,2,3,4\}$, then $\mathfrak{C}_{\mathfrak{A}_4}$ has precisely 5 elements, namely, the conjugacy classes of the subgroups $\{1\}, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{D}_2$ and $\mathfrak{A}_4$. Moreover, their normalisers are

| $H$ | $\{1\}$ | $\mathbf{Z}_2$ | $\mathbf{Z}_3$ | $\mathbf{D}_2$ | $\mathfrak{A}_4$ |
|---|---|---|---|---|---|
| $\mathrm{N}_G(H)$ | $\mathfrak{A}_4$ | $\mathbf{D}_2$ | $\mathbf{Z}_3$ | $\mathfrak{A}_4$ | $\mathfrak{A}_4$ |

(d) If $G = \mathfrak{A}_5$ is the alternating group on $\{1,2,3,4,5\}$, then $\mathfrak{C}_{\mathfrak{A}_5}$ has precisely 9 elements, namely, the conjugacy classes of the subgroups $\{1\}, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_5, \mathbf{D}_2, \mathbf{D}_3, \mathbf{D}_5, \mathfrak{A}_4$ and $\mathfrak{A}_5$. Moreover, their normalisers are

| $H$ | $\{1\}$ | $\mathbf{Z}_2$ | $\mathbf{Z}_3$ | $\mathbf{Z}_5$ | $\mathbf{D}_2$ | $\mathbf{D}_3$ | $\mathbf{D}_5$ | $\mathfrak{A}_4$ | $\mathfrak{A}_5$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{N}_G(H)$ | $\mathfrak{A}_5$ | $\mathbf{D}_2$ | $\mathbf{D}_3$ | $\mathbf{D}_5$ | $\mathfrak{A}_4$ | $\mathbf{D}_3$ | $\mathbf{D}_5$ | $\mathfrak{A}_4$ | $\mathfrak{A}_5$ |

**1.3 The Category of $G$-sets** Let $G$ be a group. Let $X$ and $Y$ be $G$-sets with the operations $G \times X \to X$ and $G \times Y \to Y$. Then a $G$-homomorphism or just a $G$-morphism from $X$ to $Y$ is a map $f : X \to Y$ with $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$, i. e. the diagram

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ {\scriptstyle \mathrm{id} \times f}\downarrow & & \downarrow{\scriptstyle f} \\ G \times Y & \longrightarrow & Y \end{array}$$

is commutative. Such a $G$-morphism induces the map $\overline{f} : X/G \to Y/G$, $\overline{f}(Gx) = Gf(x)$, $x \in X$, of the corresponding orbit spaces. The set of $G$-homomorphisms from the $G$-set $X$ to the $G$-set $Y$ is denoted by $\mathrm{Hom}_G(X,Y)$. The identity map of a $G$-set and the composite of any two $G$-homomorphisms is again a $G$-homomorphism. (Left) $G$-sets and $G$-homomorphisms form the category of $G$-sets denoted by $G$-$\mathsf{SET}$. Isomorphisms in the category $G$-$\mathsf{SET}$ are called $G$-isomorphisms. A $G$-homomorphism $f : X \to Y$ is an $G$-isomorphism if and only if it is bijective. Two $G$-sets $X$ and $Y$ are called $G$-isomorphic if there exists a $G$-isomorphism $f : X \to Y$. The set of all $G$-isomorphisms of a $G$-set $X$ is denoted by $\mathrm{Aut}_G X$. It is clearly a group (under composition), in fact a subgroup of the permutation group $\mathfrak{S}(X)$ and hence operates canonically on $X$.

Isomorphism is an equivalence relation on $G$-$\mathsf{SET}$; its equivalences classes are denoted by small gothic letters. For example, the isomorphism class of a $G$-set $X$ is denoted by $\mathfrak{x}$. Therefore for $G$-sets $X$ and $Y$, we have $\mathfrak{x} = \mathfrak{y}$ if and only if there exists a $G$-isomorphism $X \to Y$.

With these notation and terminology we note the following observations (for their proof see [8, Example 1.10 and Theorem 1.11]) which will be used frequently in sequel:

**1.4 Proposition** *Let $X, Y$ be two finite G-sets and let $x \in X$, $y \in Y$.*

(1) *If $f : X \to Y$ is a G-homomorphism, then $G_x \subseteq G_{f(x)}$ and $f(\mathrm{Fix}_H X) \subseteq \mathrm{Fix}_H Y$ for every $H \in \mathbf{Sub}(G)$. In particular, if $f$ is an G-isomorphism, then $f$ induces a bijection $\mathrm{Fix}_H X \xrightarrow{\sim} \mathrm{Fix}_H Y$, in particular, $\#\,\mathrm{Fix}_H X = \#\,\mathrm{Fix}_H Y$. Moreover, $\#\,\mathrm{Fix}_H(X)$ depends only on the isomorphism class $\mathfrak{x}$ of $X$ and only on the conjugacy class $\mathfrak{h}$ of $H$ (in G) and hence it is denoted by $\#\,\mathrm{Fix}_{\mathfrak{h}}\,\mathfrak{x}$.*

(2) *If $X$ is transitive and if $G_x \subseteq G_y$, then there exists a unique G-homomorphism $f : X \to Y$ such that $y = f(x)$. Moreover, if $Y$ is also homogeneous, then the G-homomorphism $f$ is surjective. In particular, the map $\mathrm{Hom}_G(X,Y) \to \mathrm{Fix}_{G_x} Y$, $f \mapsto f(x)$ is a bijective.*

(3) *If $X$ is transitive, then the map $X_{G_x} = G/G_x \to X$, $hG_x \mapsto gx$ is well-defined and is an G-isomorphism. Further, $\mathrm{Aut}_G X \cong \mathrm{W}_G(H)$, where $H = G_{x_0}$ with arbitrary $x_0 \in X$.*

(4) *If $X$ is transitive, then the canonical operation of the automorphism group $\mathrm{Aut}_G X$ of $X$ on $X$ is free[2] on $X$. Moreover, if $H$ is a subgroup of G, then $\mathrm{Fix}_H X$ is invariant under the operation of $\mathrm{Aut}_G X$. In particular, $\mathrm{Fix}_H X$ is a union of $\mathrm{Aut}_G X$-orbits and $\#\,\mathrm{Fix}_H X = \#\,\mathrm{Aut}_G X \cdot \#\{\tilde{H} \in \mathbf{Sub}(G) \mid H \subseteq \tilde{H} = G_x \text{ for some } x \in X\}$. Furthermore, $\#\,\mathrm{Fix}_H X_H = \#\,\mathrm{Aut}_G X = [\mathrm{N}_G(H) : H]$.*

(5) *$X$ is G-isomorphic to the disjoint sum of its orbit.*

From this Proposition we easily deduce:

**1.5 Corollary** *Let $H$ and $H'$ be subgroups of G. Then*

(1) $\mathrm{Hom}_G(X_H, X_{H'}) \neq \emptyset$ *if and only if $H$ is a subconjugate of $H'$,* i. e. *$H$ is conjugate (in $G$) to a subgroup of $H'$. Moreover, every G-homomorphism $f : X_H \to X_{H'}$ is induced by a right translation $\mathrm{R}_g : G \to G$, $g \in G$ with $g^{-1}Hg \subseteq H'$.* i. e. *$f(aH) = agH'$ for all $aH \in X_H$. Further, $g$ and $g' \in G$ induces the same G-homomorphism $X_H \to X_{H'}$ if and only if $g^{-1}g' \in H'$.*

(2) *The transitive G-sets $X_H$ and $X_{H'}$ if and only if H and K are conjugates in G. In particular, the isomorphism class of the G-set $X_H$ depends only on the conjugacy class $\mathfrak{h}$ of the subgroup $H$ and hence it is denoted by $\mathfrak{x}_{\mathfrak{h}}$.*

**1.6 Corollary** *Let $X$ be a finite G-set. Then the isomorphism class $\mathfrak{x}$ of $X$ can be expressed as a $\mathbb{N}$-linear combination in the form: $\sum_{\mathfrak{h} \in \mathfrak{C}_G} \mu_{\mathfrak{h}}(\mathfrak{x}) \cdot \mathfrak{x}_{\mathfrak{h}}$ with uniquely determined natural numbers $\mu_{\mathfrak{h}}(\mathfrak{x}) \in \mathbb{N}$, $\mathfrak{h} \in \mathfrak{C}_G$.*

## § 2 Burnside Algebra of a Finite Group

Let $G$ be a finite group. The two natural binary operations on the set $\mathrm{B}^+(G)$ of isomorphism classes of finite $G$-sets, namely, addition and multiplication which are induced by the disjoint union and cartesian product induced by the diagonal operation, respectively. More precisely, for any two $G$-sets $X$ and $Y$, consider the $G$-sets $X \uplus Y$ (disjoint union of $X$ and $Y$ with the obvious $G$-operation defined by using the $G$-operations of $X$ and $Y$)

[2] A group $G$ operates freely on the set $X$ if isotropy subgroup $G_x$ at every $x \in X$ is trivial. In this case the cardinality of the each orbit $Gx$ of $x$ is the order of $G$, since the map $G \to Gx$, $g \mapsto gx$ is injective.

and $X \times Y$ the cartesian product of $X$ and $Y$ with the diagonal action $(g,(x,y)) \mapsto (gx,gy)$. Then we define the addition and multiplication on $\mathrm{B}^+(G)$ by:

$$\mathfrak{x} + \mathfrak{y} := [X \uplus Y] \ \text{ and } \ \mathfrak{x} \cdot \mathfrak{y} := [X \times Y]\,.$$

It is easy to check that both these binary operations are associative and have neutral elements $0 := 0_{\mathrm{B}^+(G)} = [\emptyset]$, where the $\emptyset$ is the trivial $G$-set with the trivial $G$-operations and $1 := 1_{\mathrm{B}^+(G)} = [\{x\}]$, where $\{x\}$ is any singleton set with the unique $G$-operation, respectively for the addition $+$ and the multiplication $\cdot$ on $\mathrm{B}^+(G)$. The Burnside ring $\mathrm{B}(G)$ of $G$ is the *universal ring*[3] or *Grothendieck ring* of $(\mathrm{B}^+(G),+,\cdot)$ (or of the category $G$-**sets** of $G$-sets); elements of $\mathrm{B}(G)$ are also called virtual $G$-sets. This means that $\mathrm{B}(G)$ is the free $\mathbb{Z}$-module with basis $\mathrm{T}(G) := \{\mathfrak{x}_{\mathfrak{h}} \mid \mathfrak{h} \in \mathfrak{C}_G\}$ (of isomorphism classes of transitive $G$-sets which is indexed by the set $\mathfrak{C}_G$ of conjugacy classes of subgroups of $G$, see Corollary 1.5 (2).) and hence the ring $\mathrm{B}(G)$ is a commutative ring and is a finite free $\mathbb{Z}$-module of the rank $\mathrm{Rank}_{\mathbb{Z}}\,\mathrm{B}(G) = \#\mathfrak{C}_G$, in particular, it is a noetherian ring. Further, every element $\mathfrak{x} \in \mathrm{B}(G)$ can be expressed as a $\mathbb{Z}$-linear combination in the form: $\sum_{\mathfrak{h}\in\mathfrak{C}_G} \mu_{\mathfrak{h}}(\mathfrak{x}) \cdot \mathfrak{x}_{\mathfrak{h}}$ with uniquely determined integer coefficients $\mu_{\mathfrak{h}}(\mathfrak{x}) \in \mathbb{Z}$, $\mathfrak{h} \in \mathfrak{C}_G$. The $\mathfrak{C}_G$-tuple $(\mu_{\mathfrak{h}}(\mathfrak{x}))_{\mathfrak{h}\in\mathfrak{C}_G}$ is called the Burnside type of $\mathfrak{x}$ and the function $\mathrm{B}(G) \to \mathbb{Z}^{\mathfrak{C}_G}$ defined by $\mathfrak{x} \mapsto (\mu_{\mathfrak{h}}(\mathfrak{x}))_{\mathfrak{h}\in\mathfrak{C}_G}$ is called the Burnside function of $G$.

For each subgroup $H \in \mathbf{Sub}(G)$ of $G$ and a $G$-set $X$, the fixed points $\mathrm{Fix}_H X$ of $X$ under $H$ defines a canonical ring homomorphism. More precisely, the number $\#\,\mathrm{Fix}_H(X)$ of invariant elements of $X$ under the subgroup $H$ is called the Burnside number of the $G$-set $X$ with respect to the subgroup $H$. This number depends only on the isomorphism class $\mathfrak{x}$ of $X$ and only on the conjugacy class $\mathfrak{h}$ of $H$ (in $G$), see 1.4 (1) and hence the assignment $\varphi^G_{\mathfrak{h}} : \mathrm{B}^+(G) \to \mathbb{Z}$ given by $\mathfrak{x} \mapsto \#\,\mathrm{Fix}_{\mathfrak{h}}\,\mathfrak{x} := \#\,\mathrm{Fix}_H X$, where $X \in \mathfrak{x}$ and $H \in \mathfrak{h}$, is well-defined. Further, it extends to a unique $\mathbb{Z}$-algebra homomorphism $\varphi^G_{\mathfrak{h}} : \mathrm{B}(G) \to \mathbb{Z}$ and is called the Burnside $H$-mark of $G$. Note that Burnside mark of the trivial subgroup $\{1\}$ of $G$ is the map $\varphi^G_1(\mathfrak{x}) = \#\,\mathfrak{x} = \#X$ for every $X \in G$-**sets**, $X \in \mathfrak{x}$.

The unique $\mathbb{Z}$-algebra homomorphism $\varphi^G : \mathrm{B}(G) \to \mathbb{Z}^{\mathfrak{C}_G}$ defined by $\mathfrak{x} \mapsto (\varphi^G_{\mathfrak{h}}(\mathfrak{x}))_{\mathfrak{h}\in\mathfrak{C}_G}$ called the Burnside character or mark homomorphism of $G$. The classical theorem of Burnside asserts that *the ring homomorphism $\varphi^G$ is injective*. We shall prove this in Theorem 2.8 below. Let us make the following convenient definiton:

**2.1 Definition-Lemma** The relation "$H$ is a subconjugate to $K$", i. e. $H$ is conjugate (in $G$) to a subgroup of $K$ on the set $\mathbf{Sub}(G)$ induces an order $\preccurlyeq$ on the set $\mathfrak{C}_G$ and hence on the subset $\mathrm{T}(G) := \{\mathfrak{x}_{\mathfrak{h}} \mid \mathfrak{h} \in \mathfrak{C}_G\}$ of $\mathrm{B}(G)$ consisting of isomorphism classes of finite transitive $G$-sets. Therefore, for every $\mathfrak{h},\mathfrak{k} \in \mathfrak{C}_G$ we have $\mathfrak{x}_{\mathfrak{h}} \preccurlyeq \mathfrak{x}_{\mathfrak{k}} \iff \mathfrak{h} \preccurlyeq \mathfrak{k}$, i. e. for each $H \in \mathfrak{h}$ and $K \in \mathfrak{h}'$ we have $H \subseteq gKg^{-1}$ for some $g \in G$. More precisely, we note (use 1.4 and 1.5) that: $\mathfrak{x}_{\mathfrak{h}} \preccurlyeq \mathfrak{x}_{\mathfrak{k}} \iff H \subseteq gKg^{-1}$ *for some* $g \in G \iff \varphi^G_{\mathfrak{h}}(\mathfrak{x}_{\mathfrak{k}}) = \#\,\mathrm{Fix}_H X_K \neq 0$. *Moreover, in this case* $\varphi^G_{\mathfrak{h}}(\mathfrak{x}_{\mathfrak{k}}) = [\mathrm{N}_G(K):K] \cdot \#\{K' \in \mathbf{Sub}(G) \mid H \subseteq K' = gKg^{-1}$ *for some* $g \in G\}$.

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[3] The Burnside ring is a commutative ring together with a homomorphism $\iota : \mathrm{B}^+(G) \to \mathrm{B}(G)$ such that for every homomorphism $\psi : \mathrm{B}^+ \to B$ of monoids (both additive and multiplicative) into a commutative ring $B$, there exists a unique ring homomorphism $\Psi : \mathrm{B}(G) \to B$ such that $\Psi \circ \iota = \varphi$. It is in general algebraic fact that such a universal ring exists.

**Proof:** For a proof of the last equivalence apply Proposition 1.4 (2) to transitive $G$-sets $X_H$ and $X_K$ with elements $x = H$ and $y = gK$ to conclude that $\mathrm{Fix}_H Y \neq \emptyset$. Therefore $\varphi_{\mathfrak{h}}^G(\mathfrak{x}_{\mathfrak{k}}) = \#\,\mathrm{Fix}_H(Y) \neq 0$. The last assertion follows easily from Proposition 1.4 (4). □

**2.2 Notation** In order to make our exposition understandable we use the following notation in sequel without giving the exact cross-reference: Let $G$ be a finite group.

- $\mathbf{Sub}(G) =$ the set of all subgroups of $G$; its elements are denoted by capital letters $H, H', K, K' \cdots$.
- $\mathfrak{C}_G$ the conjugacy classes of subgroups of $G$; its elements are denoted by small gothic letters $\mathfrak{h}, \mathfrak{h}', \mathfrak{k}, \mathfrak{k}' \cdots$.
- For $\mathfrak{h} \in \mathfrak{C}_G$ and $H \in \mathfrak{h}$, we put $|\,\mathfrak{h}\,| := \#H$, and $[\mathfrak{g} : \mathfrak{h}] := [G : H]$. Note that these equalities depends only on $\mathfrak{h}$ and not on $H \in \mathfrak{h}$.
- For $\mathfrak{h} \in \mathfrak{C}_G$ and $H \in \mathfrak{h}$, we have $\#\,\mathfrak{h} = [G : \mathrm{N}_G(H)]$ and hence $\#\,\mathrm{W}_{\mathfrak{g}}(\mathfrak{h}) := \#\,\mathrm{W}_G(H) := \mathrm{N}_G(H)/H$ depends only on $\mathfrak{h}$ and not on $H \in \mathfrak{h}$.
- $G$-SET $=$ the category of finite $G$-sets with $G$-morphisms; its elements are denoted by capital letters $X, X', Y, Y' \cdots$. For a $G$-set $X$, let $\mathrm{Aut}_G X$ denote the set of $G$-automorphisms of $X$.
- For a $G$-set $X$ and a subgroup $H \in \mathbf{Sub}(G)$, $\mathrm{Fix}_H X$ denote the $H$-fixed points of $X$. The cardinality $\#\,\mathrm{Fix}_H X$ depends only on the conjugacy class $\mathfrak{h}$ of $H$ and not on $H$.
- $\mathrm{B}^+(G) =$ the isomorphism classes of $G$-sets; its elements are denoted by small gothic letters $\mathfrak{x}, \mathfrak{x}', \mathfrak{y}, \mathfrak{y}' \cdots$. For $\mathfrak{x} \in \mathrm{B}^+(G)$, $X \in \mathfrak{x}$ and $\mathfrak{h} \in \mathfrak{C}_G$, $H \in \mathfrak{h}$, we put $\#\,\mathrm{Fix}_{\mathfrak{h}}\,\mathfrak{x} := \#\,\mathrm{Fix}_H X$ and $\#\,\mathrm{Aut}_{\mathfrak{g}}\,\mathfrak{x} := \#\,\mathrm{Aut}_G X$ which depends only on $\mathfrak{x}$ and on $\mathfrak{h}$.
- For $H \in \mathbf{Sub}(G)$, let $X_H :=$ the $G$-set of left cosets $G/H$ of $H$ in $G$. Every transitive $G$-set $X$ is $G$-isomorphic to some $X_H$, $H \in \mathbf{Sub}(G)$. Moreover, the conjugacy class $\mathfrak{h}$ of $H$ is uniquely determined by $X$.
- For $\mathfrak{h} \in \mathfrak{C}_G$ and $H \in \mathfrak{h}$, $\mathfrak{x}_{\mathfrak{h}} :=$ the image of $X_H$ in $\mathrm{B}^+(G)$ which depends only on $\mathfrak{h}$ and not on $H \in \mathfrak{h}$. Therefore $\mathfrak{x}_{\mathfrak{h}} = \mathfrak{x}_{\mathfrak{k}} \iff \mathfrak{h} = \mathfrak{k} \iff H$ and $K$ are conjugates in $G$ for every $H \in \mathfrak{h}$ and $K \in \mathfrak{k}$. We put $\mathrm{T}(G) := \{\mathfrak{x}_{\mathfrak{h}} \mid \mathfrak{h} \in \mathfrak{C}_G\}$.
- For $H \in \mathfrak{h}$, we put $\#\,\mathfrak{x}_{\mathfrak{h}} := [G : H]$ and $\#\,\mathrm{Aut}_{\mathfrak{g}}\,\mathfrak{x}_{\mathfrak{h}} = \#\,\mathrm{W}_G(H) = \#\,\mathrm{W}_{\mathfrak{g}}(\mathfrak{h})$.

**2.3 Corollary** *In the product of $\mathfrak{x}_{\mathfrak{h}}$ and $\mathfrak{x}_{\mathfrak{k}} \in \mathrm{T}(G)$ in the $\mathbb{Z}$-algebra $\mathrm{B}(G)$ only $\mathfrak{x}_{\mathfrak{l}}$ with $\mathfrak{l} \preccurlyeq \mathfrak{h}$ and $\mathfrak{l} \preccurlyeq \mathfrak{k}$ can occur with non-zero coefficients, In particular, $\mathfrak{x}_{\mathfrak{h}} \cdot \mathfrak{x}_{\mathfrak{k}} = \sum_{\mathfrak{l} \preccurlyeq \mathfrak{h}, \mathfrak{k}} \nu_{\mathfrak{l}} \mathfrak{x}_{\mathfrak{l}}$.*

**Proof:** Let $H \in \mathfrak{h}$ and $K \in \mathfrak{k}$. Suppose that $\mathfrak{x}_{\mathfrak{l}} \in \mathrm{T}(G)$, $\mathfrak{l} \in \mathfrak{C}_G$ occurs in $\mathfrak{x}_{\mathfrak{h}} \cdot \mathfrak{x}_{\mathfrak{k}}$ with non-zero coefficient $\nu_{\mathfrak{l}}$. Then for $L \in \mathfrak{l}$, $X_L$ is a orbit in the product $G$-set $X_H \times X_K$ and the projection maps $X_H \times X_K \to X_H$, $X_H \times X_K \to X_K$ are $G$-homomorphisms and hence induce $G$-homomorphisms $X_L \to X_H$ and $X_L \to X_K$. Therefore $\mathfrak{l} \preccurlyeq \mathfrak{h}$ and $\mathfrak{l} \preccurlyeq \mathfrak{k}$ by 1.5 and 2.1 □

**2.4 Corollary** *Let $H (\in \mathfrak{h})$ be a subgroup of a finite group $G$ and let $\mathfrak{x} = \sum_{\mathfrak{h} \in \mathfrak{C}_G} \mu_{\mathfrak{h}}(x) x_{\mathfrak{h}} \in$* $\mathrm{B}(G)$*. Then $H (\in \mathfrak{h})$ is a maximal subgroup with $\mu_{\mathfrak{h}}(\mathfrak{x}) \neq 0$ if and only if it is a maximal subgroup with $\varphi_{\mathfrak{h}}^G(\mathfrak{x}) \neq 0$. Moreover, in this case we have*

$$\varphi_{\mathfrak{h}}^G(\mathfrak{x}) = \mu_{\mathfrak{h}}(\mathfrak{x}) \cdot \varphi_{\mathfrak{h}}^G(\mathfrak{x}_{\mathfrak{h}}) = \mu_{\mathfrak{h}}(\mathfrak{x}) \cdot \#\,\mathrm{W}_{\mathfrak{g}}(\mathfrak{h})\,.$$

**Proof:** Suppose that $H \in \mathfrak{h}$ is a maximal subgroup with $\mu_{\mathfrak{h}}(\mathfrak{x}) \neq 0$. Then $\mathfrak{x} = \sum_{\mathfrak{k} \preccurlyeq \mathfrak{h}} \mu_{\mathfrak{k}}(\mathfrak{x})\, \mathfrak{x}_{\mathfrak{k}}$. Applying $\varphi_{\mathfrak{h}}$ and using 2.1 we get $\varphi_{\mathfrak{h}}(\mathfrak{x}) = \mu_{\mathfrak{h}}(\mathfrak{x}) \varphi_{\mathfrak{h}}(\mathfrak{x}_{\mathfrak{h}}) \neq 0$. The converse can be proved similarly. The last assertion follows immediately from the last part of 2.1. □

**2.5 Corollary** *Let p is a prime number, G be a finite p-group and let* $\mathfrak{x} = \sum_{\mathfrak{h} \in \mathfrak{C}_G} \mu_{\mathfrak{h}}(x) x_{\mathfrak{h}} \in \mathrm{B}(G)$. *Then*

$$\varphi_1^G(x) = \sum_{\mathfrak{h} \in \mathfrak{C}_G} \mu_{\mathfrak{h}}(x) \cdot \# \mathfrak{x}_{\mathfrak{h}} \ \equiv \ \mu_G(x) = \varphi_{\mathfrak{g}}^G(x) \pmod p .$$

**Proof:** Since $\varphi_1(\mathfrak{x}_{\mathfrak{h}}) = \# \mathfrak{x}_{\mathfrak{h}} \equiv 0$ ( mod $p$ for every $\mathfrak{h} \neq \mathfrak{g}$ by class equation, applying the Corollary 2.4 (to $\mathfrak{h} = \{1\}$) the assertion is immediate. □

**2.6 Corollary** *Let G be a finite and let p be a prime divisor of* $\# G$. *Let* $H (\in \mathfrak{h})$ *be a p-subgroup and* $H' (\in \mathfrak{h}')$ *be a subgroup of G with* $[G : H']$ *coprime to p. Then*

$$\varphi_{\mathfrak{h}}^G(\mathfrak{x}_{\mathfrak{h}'}) \equiv \varphi_1^G(\mathfrak{x}_{\mathfrak{h}'}) = [G : H'] \ \not\equiv \ 0 \pmod p .$$

*In particular, H is a conjugate (in G) to a subgroup of* $H'$.

**2.7 Corollary** (Sylow) *Let G be a finite and let p be a prime divisor of* $\# G$. *If* $G_p$ *is a Sylow p-subgroup of G and if* $H \subseteq G$ *be a subgroup G which is a p-group, then H is conjugate to a subgroup of* $G_p$. *In particular, all Sylow p-subgroups if G are conjugates in G.*

Now we shall prove the following classical theorem of Burnside:

**2.8 Theorem** (Burnside) *For a finite group G, the Burnside mark-homomorphism*

$$\varphi^G : \mathrm{B}(G) \to \mathbb{Z}^{\mathfrak{C}_G}, \qquad \mathfrak{x} \mapsto (\varphi_{\mathfrak{h}}^G(\mathfrak{x}))_{\mathfrak{h} \in \mathfrak{C}_G}$$

*is injective.*

**Proof:** First note that the order $\preccurlyeq$ on the finite set $\mathfrak{C}_G$ (defined in 2.1) can be refined to a total order which we again denote by the same symbol $\preccurlyeq$. The matrix of $\varphi^G$ with respect to bases $\{\mathfrak{x}_{\mathfrak{h}} \mid \mathfrak{h} \in \mathfrak{C}_G\}$ of $\mathrm{B}(G)$ with order $\preccurlyeq$ and the standard basis $\{\mathrm{e}_{\mathfrak{h}} \mid \mathfrak{h} \in \mathfrak{C}_G\}$ of $\mathbb{Z}^{\mathfrak{C}_G}$ is upper triangular, since the $(\mathfrak{h}, \mathfrak{h}')$-th is entry $\varphi_{\mathfrak{h}}^G(\mathfrak{x}_{\mathfrak{h}'}) = 0$ for all $\mathfrak{h}' \preccurlyeq \mathfrak{h}$ and $\mathfrak{h}' \neq \mathfrak{h}$ by 2.1. Further, the diagonal entries $\varphi_{\mathfrak{h}}^G(\mathfrak{x}_{\mathfrak{h}}) \neq 0$ again by 2.1 and hence the determinant of $\varphi^G$ is non-zero. Therefore $\varphi^G$ is injective by [9, Teil 1]. □

**2.9 Example** The Burnside's table of marks is the table formed by putting all possible marks together as follows: As in the proof of Theorem 2.8 we refine the order $\preccurlyeq$ on the finite set $\mathfrak{C}_G$ (defined in 2.1) refine to a total order which we again denote by the same symbol $\preccurlyeq$. Now, form the $\mathfrak{C}_G \times \mathfrak{C}_G$-table whose $(\mathfrak{k}, \mathfrak{h})$-th entry is $\# \mathrm{Fix}_{\mathfrak{h}} \mathfrak{x}_{\mathfrak{k}}$. The ring structure of the Burnside algebra $\mathrm{B}(G)$ can be deduced from the Burnside's table of marks. The generators of the $\mathbb{Z}$-module $\mathrm{B}(G)$ are the rows of the table and the product (componentwise multiplication of row vectors) of marks which can then be decomposed as a $\mathbb{Z}$-linear combination of all the rows. For example, for the Burnside's table of marks for the symmetric group $G = \mathfrak{S}_3$ is:

| | $\{1\}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathfrak{S}_3$ |
|---|---|---|---|---|
| $\mathfrak{S}_3/\{1\}$ | 6 | 0 | 0 | 0 |
| $\mathfrak{S}_3/\mathbf{Z}_2$ | 3 | 1 | 0 | 0 |
| $\mathfrak{S}_3/\mathbf{Z}_3$ | 2 | 0 | 2 | 0 |
| $\mathfrak{S}_3/\mathfrak{S}_3$ | 1 | 1 | 1 | 1 |

The product of the isomorphism classes $\mathfrak{x}_{\mathfrak{h}}$ and $\mathfrak{x}_{\mathfrak{k}}$ of the $G$-sets $X_H = \mathfrak{S}_3/H$ and $X_K = \mathfrak{S}_3/K$ coresponding to subgroups $H = \mathbf{Z}_2$ and $K = \mathbf{Z}_3$ is then $\mathfrak{x}_{\mathfrak{h}} \cdot \mathfrak{x}_{\mathfrak{k}} = \mathfrak{x}_1$, since $(3,1,0,0) \cdot (2,0,2,0) =$

$(6,0,0,0)$, where $\mathfrak{x}_1$ denote the isomorphism class of the $\mathfrak{S}_3$-set $\mathfrak{S}_3/\{1\}$ corresponding to the trivial subgroup $\{1\}$.

One can easily verify that the following table is indeed the Burnside's table of marks for the alternating group $G=\mathfrak{A}_5$:

| | $\{1\}$ | $\mathbf{Z}_2$ | $\mathbf{Z}_3$ | $\mathbf{D}_2$ | $\mathbf{Z}_5$ | $\mathbf{D}_3$ | $\mathbf{D}_5$ | $\mathfrak{A}_4$ | $\mathfrak{A}_5$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathfrak{A}_5/\{1\}$ | 60 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathfrak{A}_5/\mathbf{Z}_2$ | 30 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathfrak{A}_5/\mathbf{Z}_3$ | 20 | $\cdot$ | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathfrak{A}_5/\mathbf{D}_2$ | 15 | 3 | $\cdot$ | 3 | 0 | 0 | 0 | 0 | 0 |
| $\mathfrak{A}_5/\mathbf{Z}_5$ | 12 | $\cdot$ | $\cdot$ | $\cdot$ | 2 | 0 | 0 | 0 | 0 |
| $\mathfrak{A}_5/\mathbf{D}_3$ | 10 | 2 | 1 | $\cdot$ | $\cdot$ | 1 | 0 | 0 | 0 |
| $\mathfrak{A}_5/\mathbf{D}_5$ | 6 | 2 | $\cdot$ | $\cdot$ | 1 | $\cdot$ | 1 | 0 | 0 |
| $\mathfrak{A}_5/\mathfrak{A}_4$ | 5 | 1 | 2 | 1 | $\cdot$ | $\cdot$ | $\cdot$ | 1 | 0 |
| $\mathfrak{A}_5/\mathfrak{A}_5$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**2.10 Example** One can also make the multiplication table for the generating set consisting of isomorphism classes of transitive $G$-sets $\mathrm{T}(G)=\{\mathfrak{x}_{\mathfrak{h}}\mid \mathfrak{h}\in\mathfrak{C}_G\}$ in the Burnside algebra $\mathrm{B}(G)$. For example, for the alternating group $G=\mathfrak{A}_4$ one can easily verify the following multiplication table:

| | $\mathfrak{A}_4/\mathfrak{A}_4$ | $\mathfrak{A}_4/\mathbf{Z}_2$ | $\mathfrak{A}_4/\mathbf{Z}_3$ | $\mathfrak{A}_4/\mathbf{D}_2$ | $\mathfrak{A}_4/\{1\}$ |
|---|---|---|---|---|---|
| $\mathfrak{A}_4/\mathfrak{A}_4$ | $\mathfrak{A}_4/\mathfrak{A}_4$ | $\mathfrak{A}_4/\mathbf{D}_2$ | $\mathfrak{A}_4/\mathbf{Z}_3$ | $\mathfrak{A}_4/\mathbf{Z}_2$ | $\mathfrak{A}_4/\{1\}$ |
| $\mathfrak{A}_4/\mathbf{D}_2$ | $3\cdot\mathfrak{A}_4/\mathfrak{A}_4$ | $3\cdot\mathfrak{A}_4/\mathbf{Z}_2$ | $\mathfrak{A}_4/\mathfrak{A}_4$ | $3\cdot\mathfrak{A}_4/\mathbf{D}_2$ | 0 |
| $\mathfrak{A}_4/\mathbf{Z}_3$ | $4\cdot\mathfrak{A}_4/\mathfrak{A}_4$ | $2\cdot\mathfrak{A}_4/\mathfrak{A}_4$ | $\mathfrak{A}_4/\mathfrak{A}_4+\mathfrak{A}_4/\mathbf{Z}_3$ | 0 | 0 |
| $\mathfrak{A}_4/\mathbf{Z}_2$ | $6\cdot\mathfrak{A}_4/\mathfrak{A}_4$ | $2\cdot\mathfrak{A}_4/\mathfrak{A}_4+2\cdot\mathfrak{A}_4/\mathbf{Z}_2$ | 0 | 0 | 0 |
| $\mathfrak{A}_4/\mathfrak{A}_4$ | $12\cdot\mathfrak{A}_4/\mathfrak{A}_4$ | 0 | 0 | 0 | 0 |

**2.11 Example** Let $G$ and $H$ be a finite groups. If $X$ is a $G$-set and $Y$ is a $H$-set, then $X\times Y$ is a $G\times H$-set cannonically and hence the assignment $(X,Y)\mapsto X\times Y$ induces a canonical ring isomorphism $\mathrm{B}(G)\otimes_{\mathbb{Z}}\mathrm{B}(H)\xrightarrow{\sim}\mathrm{B}(G\times Y)$.

**2.12 Corollary** *Let G be a finite group. The cokernel of the Burnside mark homomorphism* $\varphi^G:\mathrm{B}(G)\to\mathbf{Z}^{\mathfrak{C}_G}$ *is isomorphic to the product* $\mathbb{Z}$*-algebra* $\prod_{\mathfrak{h}\in\mathfrak{C}_G}(\mathbb{Z}/\#\mathrm{W}_{\mathfrak{g}}(\mathfrak{h})\mathbb{Z})$.

In the sequel we shall identify the Burnside algebra $\mathrm{B}(G)$ with its image $\varphi^G(\mathrm{B}(G))$ in the product algebra $\mathbb{Z}^{\mathfrak{C}_G}$.

**2.13 Corollary** *Let G be a finite group. Then the product* $\mathbb{Z}$*-algebra* $\mathbb{Z}^{\mathfrak{C}_G}$ *is integral over the Burnside algebra* $\mathrm{B}(G)$ *of G. Moreover, if* $\boldsymbol{x}=\left(x_{\mathfrak{h}}\right)_{\mathfrak{h}\in\mathfrak{c}_G}\in\mathrm{B}(G)$ *and* $\boldsymbol{y}=\left(y_{\mathfrak{h}}\right)_{\mathfrak{h}\in\mathfrak{c}_G}\in\mathbb{Z}^{\mathfrak{C}_G}$ *with* $x_{\mathfrak{h}}\equiv y_{\mathfrak{h}}\pmod{\#G}$ *for all* $\mathfrak{h}\in\mathfrak{C}_G$*, then* $\boldsymbol{y}\in\mathrm{B}(G)$. *In particular,*

$\# G \in \mathrm{Ann}_{\mathrm{B}(G)} \left(\mathbb{Z}^{\mathfrak{C}_G}/\mathrm{B}(G)\right)$. i. e. *$\# G$ belong to the conductor*[4] *of the integral extension*[5] $\mathrm{B}(G) \subseteq \mathbb{Z}^{\mathfrak{C}_G}$.

**2.14 Remark** Let $G$ be a finite group. Then the total quotient ring of the Burside algebra $\mathrm{B}(G)$ is isomorphic to the ring $\mathrm{B}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$. The integral closure of $\mathrm{B}(G)$ in its total quotient ring is the product $\mathbb{Z}$-algebra $\mathbb{Z}^{\mathfrak{C}_G}$. Therefore the inclusion $\varphi : \mathrm{B}(G) \to \mathbb{Z}^{\mathfrak{C}_G}$ is determined by the ring-theortic properties of $\mathrm{B}(G)$ alone. Moreover, by Corollary 2.13 we have canonical isomorohisms $\mathrm{B}(G)[\frac{1}{\# G}] \xrightarrow{\sim} \mathbb{Z}^{\mathfrak{C}_G}[\frac{1}{\# G}]$ and $\mathrm{B}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathbb{Z}^{\mathfrak{C}_G} \otimes_{\mathbb{Z}} \mathbb{Q}$.

**2.15 Example** (Burnside numbers of the exterior powers of $G$-sets) Let $X$ be a finite $G$-set and let $q \in \mathbb{N}$. Then on the set $\mathfrak{P}_q(X) := \{Y \in \mathfrak{P}(X) \mid \# Y = q\}$ of all subsets of $X$ of cardinality $q$, $G$ induce the canonical action (from $X$) as: $G \times \mathfrak{P}_q(X) \to \mathfrak{P}_q(X)$, $Y \mapsto gY := \{gy \mid y \in Y\}$. This $G$-set $\mathfrak{P}_q(X)$ is called the $q$-exterior power of the $G$-set $X$.

The set $\mathfrak{P}_0(X)$ is $G$-isomorphic to $G/G$ and $\mathfrak{P}_1(X)$ is $G$-isomorphic to $X$. Moreover, the canonical map $\mathfrak{P}_m(X \uplus Y) \to \prod_{i+j=m} \mathfrak{P}_i(X) \times \mathfrak{P}_j(Y)$, $Z \mapsto (Z \cap X, Z \cap Y)$ is a $G$-isomorphism.

Let $X = G$ be the (left) regular $G$-set, i. e. with the $G$-operation: $G \times G \to G$, $(g,x) \mapsto gx$. Then for every subgroup $H$ of $G$, the number $\varphi_{\mathfrak{h}}(\mathfrak{P}_q(G)) = \# \,\mathrm{Fix}_H(\mathfrak{P}_q(G))$ of $H$-invariant subsets of cardinality $q$ in $G$ is called the Burnside number of $H$ in $G$. This number depends only on $q$, $\# G$ and $\# H$. More precisely, we prove:

**2.16 Lemma** *For every finite group G, every subgroup H of G and every natural number $q \in \mathbb{N}$, we have*:

$$\# \,\mathrm{Fix}_H(\mathfrak{P}_q(G)) = \begin{cases} 0, & \text{if } \# H \text{ does not divide } q\,, \\ \dbinom{[G:H]}{\frac{q}{\# H}}, & \text{if } \# H \text{ divide } q\,. \end{cases}$$

*In particular, if $\# H = q$, then $\varphi_{\mathfrak{h}}(\mathfrak{P}_q(G)) = [G:H]$ and $\mu_{\mathfrak{h}}(\mathfrak{P}_q(G)) = [G : \mathrm{N}_G(H)]$.*

**Proof:** Let $Y \in \mathfrak{P}_q(G)$. then $Y$ is $H$-invariant if and only if $Y$ is a union of (some) right cosets of $H$ in $G$. In particular, $\# H$ divides $\# Y = q$ and the fixed set $\mathrm{Fix}_H(\mathfrak{P}_q(G))$ of $H$-invariant subsets $Y \in \mathfrak{P}_q(G)$ corresponds to the set $\mathfrak{P}_{q/\# H}(G/H)$ of cardinality $q/\# H$. Therefore

$$\# \,\mathrm{Fix}_H(\mathfrak{P}_q(G)) = \# \mathfrak{P}_{q/\# H}(G/H) = \binom{[G:H]}{\frac{q}{\# H}}. \qquad \square$$

**2.17 Example** Let $n \in \mathbb{N}^+$ and let $\mathrm{D}_n := \{d \in \mathbb{N} \mid d \text{ divides } n\}$ be the set of divisors of $n$ (in $\mathbb{N}$). Then the family $\mathfrak{P}_d(\mathbf{Z}_n)$, $d \in \mathrm{D}_n$, of $\mathbf{Z}_n$-sets forms a $\mathbb{Z}$-basis for the Burnside algebra $\mathrm{B}(\mathbf{Z}_n))$ of the cyclic group $\mathbf{Z}_n$ of order $n$. In particular, $\mathrm{Rank}_{\mathbb{Z}}\, \mathrm{B}(\mathbf{Z}_n) = \# \mathrm{D}_n$.

**Proof:** For $d, d' \in \mathrm{D}_n$, we have $\mu_{d,d'} := \mu_{\mathbf{Z}_{d'}}(\mathfrak{P}_d(\mathbf{Z}_n)) \in \mathbb{Z}$ and the isomorphism class $[\mathfrak{P}_d(\mathbf{Z}_n)] \in \mathrm{B}(\mathbf{Z}_n)$ is

$$\mathfrak{P}_d(\mathbf{Z}_n) = \sum_{d' \in \mathrm{D}_n} \mu_{d,d'} \cdot \mathfrak{z}_{d'} \quad \text{in} \quad \mathrm{B}(\mathbb{Z}_n)\,,$$

where $\mathfrak{z}'_d$ denote the isomorphism class of the $\mathbf{Z}_n$-set $\mathbf{Z}_n/\mathbf{Z}_{d'}$. Therefore it is enough to prove that the matrix $\mathfrak{M} := \left(\mu_{d,d'}\right)_{d,d' \in \mathrm{D}_n} \in \mathrm{M}_{\# \mathrm{D}_n}(\mathbb{Z})$ is invertible, i. e. to show that the determinant

[4] Let $A \subseteq B$ be a ring extension. The ideal $\mathfrak{C}_{B|A} := \mathrm{Ann}_A B/A := \{a \in A \mid a \cdot B \subseteq A\}$ is called the conductor (ideal) of $B$ over $A$. It is the largest ideal in $A$ which is also an ideal in $B$. For a reduced ring $A$, the conductor ideal $\mathfrak{C}_A = \mathfrak{C}_{\overline{A}/A}$ of $\overline{A}$ over $A$ is called the conductor of $A$, where $\overline{A}$ denote the integral closure of $A$ in its total quotient ring.

[5] A ring extension $A \subseteq B$ is said to be integral if every element $b \in B$ satisfies a monic polynomial over $A$, i. e. if $b^n + a_1 b^{n-1} + \cdots + a_{n-1} b + a_n = 0$ for some $n \in \mathbb{N}^+$ and $a_1, \ldots, a_n \in A$. For example, if $B$ is a finite module over $A$, then $A \subseteq B$ is an integral extension.

$\operatorname{Det}\mathfrak{M} \in \mathbb{Z}^{\times} = \{\pm\}$. It follows by Lemma 2.16 that the matrix $\mathfrak{M}$ is upper triangular (with the order $\preccurlyeq$ on $\mathrm{D}_n$ defined by: $d \preccurlyeq d'$ if $d \le d'$). Moreover, we have $\mu_{d,d} = \mu_{\mathbf{Z}_d}(\mathfrak{P}_d(\mathbf{Z}_n)) = [\mathbf{Z}_n : \mathrm{N}_{\mathbf{Z}_n}(\mathbf{Z}_d)] = 1$ and hence all the diagonal entries of the matrix $\mathfrak{M}$ are 1. This proves that $\operatorname{Det}\mathfrak{M} = 1$. □

**2.18 Remark** Let $G$ be a finite group. On $G$-sets there are many constructions which provide the Burnside ring $\mathrm{B}(G)$ additional structures. For example, exterior powers (see Example 2.15 and also [13]) and symmetric powers[6] (see also [12]) of $G$-sets. The exterior powers and symmetric powers of $G$-sets will yield the (different) structure of a $\lambda$-ring[7] on $\mathrm{B}(G)$. This structure on $\mathrm{B}(G)$ is then used to derive classical results of elementary group theory.

## § 3 The Canonical Homomorphism

Important results in group theory can be proved by comparing group theoretic invariants of a finite group $G$ with the same invariants of the cyclic group $\mathbf{Z}_G$ of the same order as that of $G$. In this section we prove the following theorem which was proved by Dress in [3]:

**3.1 Theorem** (Dress) *Let $G$ be a finite group and let $\mathbf{Z}_G$ be the cyclic group of same order $n = \#G$. Further, let $\varphi^G : \mathrm{B}(G) \to \mathbb{Z}^{\mathfrak{C}_G}$ and $\varphi^{\mathbf{Z}_G} : \mathrm{B}(\mathbf{Z}_G) \to \mathbb{Z}^{\mathfrak{C}_{\mathbf{Z}_G}}$ be the Burnside mark homomorphisms of $G$ and $\mathbf{Z}_G$, respectively. Let $\alpha^* : \mathbb{Z}^{\mathfrak{C}_{\mathbf{Z}_G}} \to \mathbb{Z}^{\mathfrak{C}_G}$, $\nu \mapsto \nu \circ \alpha$ be the canonical ring homomorphism induced by the map $\alpha : \mathfrak{C}_G \to \mathfrak{C}_{\mathbf{Z}_G}$, $\mathfrak{h} \mapsto [\mathbf{Z}_H]$(= the conjugacy class of $\mathbf{Z}_H$ in $\mathbf{Z}_G$). Then the image of $\varphi^{\mathbf{Z}_G}(\mathrm{B}(\mathbf{Z}_G))$ is contained in $\varphi^G(\mathrm{B}(G))$,* i. e. *$\alpha^*$ induce an algebra homomorphism $\alpha^*_G : \mathrm{B}(Zb_G) \to \mathrm{B}(G)$ such that the diagram*

$$\begin{array}{ccc}
\mathrm{B}(\mathbf{Z}_G) & \xrightarrow{\ \alpha^*_G\ } & \mathrm{B}(G) \\
\Big\downarrow{\scriptstyle \varphi^{\mathbf{Z}_G}} & & \Big\downarrow{\scriptstyle \varphi^{G}} \\
\mathbb{Z}^{\mathfrak{C}_{\mathbf{Z}_G}} & \xrightarrow{\ \alpha^*\ } & \mathbb{Z}^{\mathfrak{C}_G}
\end{array}$$

*is commutative. In particular, $\alpha^*_G : \mathrm{B}(\mathbf{Z}_G) \to \mathrm{B}(G)$ is a ring homomorphism with the following property: for every subgroup $H$ of $G$ the diagram of rings and rings homomorphisms*

$$\begin{array}{ccc}
\mathrm{B}(\mathbf{Z}_G) & \xrightarrow{\ \alpha^*_G\ } & \mathrm{B}(G) \\
{\scriptstyle \varphi^{\mathbf{Z}_G}_{\mathfrak{z}_{\mathfrak{h}}}} \searrow & & \swarrow {\scriptstyle \varphi^{G}_{\mathfrak{h}}} \\
& \mathbb{Z} &
\end{array}$$

---

[6] Let $X$ be a finite $G$-set. Then the symmetric group $\mathfrak{S}_n$ acts on the $n$-fold cartesian product $X^n$ by permutation of components, moreover, this operation commutes with the $G$-action and hence it induces $G$-action on the orbit space $\mathrm{S}^n(X) := X^n/\mathfrak{S}_n$. This $G$-set is called the $n$-th symmetric power of $X$. For two $G$-sets $X$ and $Y$, there exists a canonical isomorphism of $G$-sets $\mathrm{S}^n(X \uplus Y) \xrightarrow{\ \sim\ } \uplus_{i+j=n} \mathrm{S}^i(X) \times \mathrm{S}^j(Y)$.

[7] Let $R$ be a commutative ring. The structure of a $\lambda$-ring on $R$ consists of a sequences $\lambda^n : R \to R$, $n \in \mathbb{N}$ with the properties: (i) $\lambda^0(a) = 1$ for all $a \in R$. (ii) $\lambda^1 = \mathrm{id}_R$. (iii) For every $n \in \mathbb{N}$, we have $\lambda^n(a+b) = \sum_{i+j=n} \lambda^i(a)\lambda^j(b)$ for all $a, b \in R$.

*is commutative, i. e.* $\varphi_{\mathfrak{h}}^{G}(\alpha_G^*(\mathfrak{x}')) = \#\,\mathrm{Fix}_H(\alpha_G^*(X')) = \#\,\mathrm{Fix}_{\mathbf{Z}_H}(X') = \varphi_{\mathfrak{z}_{\mathfrak{h}}}^{\mathbf{Z}_G}(\mathfrak{x}')$ *for every* $\mathbf{Z}_G$*-set* $X'$ *and for every subgroup* $H$ *of* $G$*, where* $\mathfrak{h}$*,* $\mathfrak{z}_{\mathfrak{h}}$ *and* $\mathfrak{x}'$ *are images of* $H$*,* $\mathbf{Z}_H$ *and* $X'$ *in* $\mathfrak{C}_G$*,* $\mathfrak{C}_{\mathbf{Z}_G}$ *and* $\mathrm{B}^+(\mathbf{Z}_G)$*, respectively.*

**Proof:** We shall identify (using 2.8) $\mathrm{B}(\mathbf{Z}_G)$ with its image $\varphi^{\mathbf{Z}_G}(\mathrm{B}(\mathbf{Z}_G)) \subseteq \mathbb{Z}^{\mathfrak{C}_{\mathbf{Z}_G}}$ and $\mathrm{B}(G)$ with its image $\varphi^{G}(\mathrm{B}(G)) \subseteq \mathbb{Z}^{\mathfrak{C}_G}$. Then $\alpha_G^*$ maps the $\mathbf{Z}_G$-set $\mathfrak{P}_q(\mathbf{Z}_G) \in \mathrm{B}(\mathbf{Z}_G) \subseteq \mathbf{Z}_G^{\mathfrak{C}_{\mathbf{Z}_G}}$ onto the $G$-set $\mathfrak{P}_q(G) \in \mathrm{B}(G) \subseteq \mathbb{Z}^{\mathfrak{C}_G}$. Now, the assertion is immediate from Example 2.17. □

**3.2 Remark** It is clear that the map $\alpha_G^*$ is injective if and only if for every divisor $m$ of $\#G$, there exists a subgroup $H$ of order $m$ in $G$. Further, the image $\mathrm{Im}\,\alpha_G^*$ is contained in the subgroup

$$\{\mathfrak{x} \in \mathrm{B}(G) \mid \varphi_{\mathfrak{h}}(\mathfrak{x}) = \varphi_{\mathfrak{k}}(\mathfrak{x}) \text{ for all } \mathfrak{h}, \mathfrak{k} \in \mathfrak{C}_G \text{ with } |\,\mathfrak{h}\,| = |\,\mathfrak{k}\,|\,\}\,.$$

In particular, *the map* $\alpha_G^* : \mathrm{B}(G) \to \mathrm{Im}\,\alpha_G^*$ *is an isomorphism if and only if* $G$ *is nilpotent.*[8]

Now we use the above Theorem 3.1 to deduce many classical results of Sylow and Frobenius in elementary group theory. First we note the following:

**3.3 Corollary** *Let* $G$ *be a finite group and let* $d$ *be a divisor of the order* $n := \#G$ *of* $G$*. Let* $X_{\mathbf{Z}_{n/d}}$ *be the transitive* $\mathbf{Z}_G$*-set corresponding to the (unique) cyclic subgroup* $\mathbf{Z}_{n/d}$ *of* $\mathbf{Z}_G$ *of order* $n/d$ *and let* $\mathfrak{z}_{n/d}$ *be its image in the Burnside algebra* $\mathrm{B}(\mathbf{Z}_G)$*. Further, let* $\mathfrak{x}_d := \alpha_G^*(\mathfrak{z}_{n/d})$*. Then:*

(1) $\varphi_{\mathfrak{h}}^{G}(\mathfrak{x}_d) = \begin{cases} d, & \text{if } \ d \ \text{divides } \#\,\mathfrak{x}_{\mathfrak{h}}, \\ 0, & \text{otherwise.} \end{cases}$

*In particular,* $\mu_{\mathfrak{h}}(\mathfrak{x}_d) = 0$ *if* $d$ *does not divide* $\#\,\mathfrak{x}_{\mathfrak{h}}$ *and* $\mu_{\mathfrak{h}}(\mathfrak{x}_d) = \#\,\mathfrak{h}$ *if* $d = \#\,\mathfrak{x}_{\mathfrak{h}}$*.*

(2) *If there exists a subgroup* $H$ *in* $G$ *of index* $d$*, then* $H$ *is a maximal subgroup with* $\mu_{\mathfrak{h}}(\mathfrak{x}_d) \neq 0$ *and* $\mu_{\mathfrak{h}}(\mathfrak{x}_d) \cdot \#\,\mathrm{W}_{\mathfrak{g}}(\mathfrak{h}) = \varphi_{\mathfrak{h}}(\mathfrak{x}_d) = d = \#\,\mathfrak{x}_{\mathfrak{h}}$*.*

**Proof:** (1) By the commutativity of the diagram in Theorem 3.1 we have

$$\varphi_{\mathfrak{h}}^{G}(\mathfrak{x}_d) = \varphi_{\mathfrak{z}_{\mathfrak{h}}}^{\mathbf{Z}_n}(\mathfrak{z}_{n/d}) = [\mathbf{Z}_n : \mathbf{Z}_{n/d}] = d \quad \text{if} \quad \mathbf{Z}_H \subseteq \mathbf{Z}_{n//d}\,,$$

i. e. if the index $d$ of $\mathbf{Z}_{n/d}$ in $\mathbf{Z}_n$ divides the index $[\mathbf{Z}_n : \mathbf{Z}_H] = [G:H] = \#\,\mathfrak{x}_{\mathfrak{h}}$ and $\varphi_{\mathfrak{h}}^{G}(\mathfrak{x}_d) = 0$ otherwise. Finally, if $\#\,\mathfrak{x}_{\mathfrak{h}} = [G:H] = d$, then $H$ is a maximal subgroup of $G$ with $\mu_{\mathfrak{h}}(\mathfrak{x}_d) \neq 0$ and hence

$$\mu_{\mathfrak{h}}(\mathfrak{x}_d) = \frac{\varphi_{\mathfrak{h}}^{G}(\mathfrak{x}_d)}{[\mathrm{N}_G(H):H]} = \frac{d}{[\mathrm{N}_G(H):H]} = \frac{[G:H]}{[\mathrm{N}_G(H):H]} = [G:\mathrm{N}_G(H)] = \#\,\mathfrak{h}$$

(2) Immediate from the part (1) and Corollary 2.4 □

**3.4 Corollary** (Sylow) *Let* $G$ *be a finite group and let* $d$ *be a divisor of* $n := \#G$*. Then* $d$ *is the greatest common divisor of those indices* $[G:H]$ *which are multiples of* $d$*,* i. e.

$$d = \mathrm{GCD}\{[G:H] \mid H \in \mathbf{Sub}(G) \text{ and } d \text{ divides } [G:H]\}\,.$$

[8] A (finite) group $G$ is called nilpotent if There exists $m \in \mathbb{N}$ such that $\mathrm{Z}^m(G) = \{1\}$, where $\mathrm{Z}^i(G)$ are defined inductively: $\mathrm{Z}^0(G) := G$ and $\mathrm{Z}^{i+1}(G) := [\mathrm{Z}^i(G), G] =$ the commutator of $\mathrm{Z}^i(G)$ and $G$ which is the subgroup generated by the commutators $xyx^{-1}y^{-1}, x \in \mathrm{Z}^i(G)\, y \in G$. *A finite group G is nilpotent if and only if every Sylow p-subgroup of G is normal in G, or equivalently, G is a product of* $p_i$*-groups,* $i = 1, \ldots, r$*, where* $\#G = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ *is the prime decomposition of* $\#G$*.*

**Proof:** Let $\mathfrak{x}_d \in \mathrm{B}(G)$ be as in Corollary 3.3. Applying the homomorphism $\varphi_1 : \mathrm{B}(G) \to \mathbb{Z}$ to the equation

$$\mathfrak{x}_d = \sum_{\mathfrak{h}\in\mathfrak{C}_G} \mu_{\mathfrak{h}}(\mathfrak{x}_d)\,\mathfrak{x}_{\mathfrak{h}} \;=^{3.2\ (2)} \sum_{\substack{\mathfrak{h},\\ d\,|\,[G:H]}} \mu_{\mathfrak{h}}(\mathfrak{x}_d)\,\mathfrak{x}_{\mathfrak{h}}$$

we get:

$$(3.4.1)\quad d = \varphi_1(\mathfrak{x}_d) = \sum_{\mathfrak{h},\,d\,|\,\#\,\mathfrak{x}_{\mathfrak{h}}} \mu_{\mathfrak{h}}(\mathfrak{x}_d)\varphi_1(\mathfrak{x}_{\mathfrak{h}}) = \sum_{\mathfrak{h},\,d\,|\,\#\,\mathfrak{x}_{\mathfrak{h}}} \mu_{\mathfrak{h}}(\mathfrak{x}_d)\cdot\#\,\mathfrak{x}_{\mathfrak{h}} \in \sum_{\mathfrak{h},\,d\,|\,\#\,\mathfrak{x}_{\mathfrak{h}}} \mathbb{Z}\cdot[G:H].$$

□

Further, if $\#G/d$ is a power of a prime number $p$, then we can even deduce the classical (first) theorem of Sylow on the existence of $p$-Sylow subgroups:

**3.5 Corollary** (Sylow-Frobenius) *Let $G$ be a finite group of order $\#G = d\cdot p^r$, where $p$ is a prime number. Then there exists a subgroup $H \subseteq G$ of order $\#H = p^r$. In particular, there exists a Sylow $p$-group $G_p$.* i. e. *a subgroup $G_p$ of $p$-power order such that its index $[G:G_p]$ is coprime to $p$.*

**3.6 Corollary** *Let $G$ be a finite and let $p^r$ be a prime-power divisor of $\#G$. Then the number of subgroups $H$ of $G$ of order $p^r$ is congruent modulo $p$.*

**Proof:** Let $d := \#G/p^r$. Dividing the equation (3.4.1) in Corollary 3.4 we get:

$$\begin{aligned} 1 = \sum_{\mathfrak{h}\ d\,|\,\#\,\mathfrak{x}_{\mathfrak{h}}} \mu_{\mathfrak{h}}(\mathfrak{x}_d)\left(\#\,\mathfrak{x}_{\mathfrak{h}}/d\right) &= \sum_{\mathfrak{h},\#|\,\mathfrak{h}\,|\,|\,p^r} \mu_{\mathfrak{h}}(\mathfrak{x}_{\mathfrak{h}})\,(p^r/|\,\mathfrak{h}\,|)\\ &\equiv \sum_{\mathfrak{h},\#|\,\mathfrak{h}\,|=p^r} \mu_{\mathfrak{h}}(\mathfrak{x}_d)\\ &= \sum_{\mathfrak{h},|\,\mathfrak{h}\,|=p^r} \#\,\mathfrak{h} = \#\{H\in\mathbf{Sub}(G)\mid \#H = p^r\}\ (\mathrm{mod}\ \ p)\,. \end{aligned}$$

□

We further prove the following result of Frobenius on the number of solutions of the pure equation $x^n = 1$ in a finite group

**3.7 Theorem** (Frobenius) *Let $G$ be a finite group and let $m$ be a divisor of its order $\#G$. Let $S_m := \{g\in G \mid g^m = 1\}$ be the set of those elements of $G$ whose order divides $m$. Then*

$$\#S_m \;\equiv\; 0 \ \ \mathrm{mod}\ \ m\,.$$

**Proof:** Let $\#G = d\cdot m$. Then by Corollary 3.3 we have

$$\sum_{g\in G}\varphi_{\mathfrak{h}_g}(\mathfrak{x}_d) \;=^{3.3} \sum_{\substack{g\in G\\ d \text{ divides } [G:\mathrm{H}(g)]}} d = \sum_{\substack{g\in G,\\ \mathrm{ord}\,g \text{ divides } m}} d = d\cdot\#S_m\,.$$

Now the assertion follows from the following counting Lemma of Burnside 3.8. □

**3.8 Lemma** (Burnside) *Let $G$ be a finite group and let $X$ be a finite $G$-set. Then:*

$$\#G\cdot\#X/G = \sum_{g\in G}\mathrm{Fix}_g X\,,\quad \textit{where}\quad \mathrm{Fix}_g X = \{x\in X \mid gx = x\}\,.$$

*In particular, for every* $\mathfrak{x} \in \mathrm{B}(G)$*, we have the congruence relation:*

$$\sum_{g\in G} \varphi_{\mathfrak{h}_g}(\mathfrak{x}) \;\equiv\; 0 \;\; (\mathrm{mod}\;\; \#G)\,,$$

*where* $\mathfrak{h}_g$ *denote the conjugacy class of the subgroup* $\mathrm{H}(g)$ *(of G) generated by g.*

**Proof:** Consider $Y := \{(g,x) \in G\times X \mid gx = x\}$ and compute the cardinality of $Y$ by using the fibres of the projection maps $Y \to G$, $(g,x) \mapsto g$ and $Y \to X/G$, $(g,x) \mapsto Gx$. ☐

**3.9 Corollary** *Let G be a finite group and let H be its subgroup. Then for every G-set X, we have*

$$\sum_{gH\in \mathrm{W}_G(H)} \varphi_{\langle g,\mathfrak{h}\rangle}(\mathfrak{x}) \;\equiv\; 0 \;\; (\mathrm{mod}\;\; \#\mathrm{W}_G(H))\,,$$

*where* $\langle g,\mathfrak{h}\rangle$ *denote the conjugacy class of the subgroup generated by g and H (of G).*

**Proof:** Let $gH \in \mathrm{W}_G(H)$ and denote $\langle g,H\rangle$ the subgroup generated by $g$ and $H$. Then $\mathrm{Fix}_{gH}(\mathrm{Fix}_H X) = \mathrm{Fix}_{\langle g,H\rangle} X$ and hence apply 3.8 to the $\mathrm{W}_G(H)$-set $\mathrm{Fix}_H X$ to obtain the required congruence. ☐

## § 4 The Spectrum of $\mathrm{B}(G)$

In this section we shall describe the prime spectrum of the Burnside algebra $\mathrm{B}(G)$ of a finite group $G$, especially, its closed points, irreducible and connected components and their relation to group theory.

We use the notations of Section 1 and Section 2, moreover, we put $\varphi_{\mathfrak{h}} := \varphi_{\mathfrak{h}}^G$ for $\mathfrak{h} \in \mathfrak{C}_G$ and $\varphi = \varphi^G$. With this first we prove the following:

**4.1 Lemma** *Let X be a G-set and* $H \in \mathbf{Sub}(G)$ *be a subgroup. Then in the product* $\mathfrak{x}\cdot\mathfrak{x}_{\mathfrak{h}}$ *in the* $\mathbb{Z}$*-algebra* $\mathrm{B}(G)$ *only* $\mathfrak{x}_{\mathfrak{l}}$ *with* $\mathfrak{l} \preccurlyeq \mathfrak{h}$ *can occur with non-zero coefficients,* i. e. $\mathfrak{x}\cdot\mathfrak{x}_{\mathfrak{k}} = \sum_{\mathfrak{l}\preccurlyeq\mathfrak{h}} \nu_{\mathfrak{l}}\,\mathfrak{x}_{\mathfrak{l}}$. *Moreover,* $\nu_{\mathfrak{h}} = \#\,\mathrm{Fix}_{\mathfrak{h}}\,\mathfrak{x}$.

**Proof:** The first assertion is immediate from Corollary 2.3. To compute the integers $\nu_{\mathfrak{h}}$, apply the ring homomorphism $\varphi_{\mathfrak{h}}$ and use $\mathrm{Fix}_H X_L =$ for every $L \in \mathfrak{l}$ with $\mathfrak{l} \preccurlyeq \mathfrak{h}$. ☐

**4.2 Lemma** *Let* $\varphi : \mathrm{B}(G) \to \mathbb{Z}$ *be a* $\mathbb{Z}$*-algebra homomorphism. Then there exists a unique* $\mathfrak{h} \in \mathfrak{C}_G$ *such that* $\varphi = \varphi_{\mathfrak{h}}$. *In particular,* $\#\,\mathrm{Hom}_{\mathbb{Z}\text{-}alg}(\mathrm{B}(G),\mathbb{Z}) = \#\,\mathfrak{C}_G$.

**Proof:** Clearly, $\varphi$ is surjective. Put $\mathfrak{p} := \mathrm{Ker}\,\varphi$. First note that the subset $\mathfrak{C}_G \setminus (\mathfrak{C}_G \cap \mathfrak{p}) = \{\mathfrak{h} \in \mathfrak{C}_G \mid \mathfrak{x}_{\mathfrak{h}} \notin \mathfrak{P}\}$ of the ordered set $(\mathfrak{C}_G, \preccurlyeq)$ (see 2.1) has a unique minimal element. For, if $\mathfrak{h}$ and $\mathfrak{k}$ are two minimal elements in $\mathfrak{C}_G \setminus (\mathfrak{C}_G \cap \mathfrak{p})$, then

$$\text{(4.2.1)} \qquad \sum_{\mathfrak{l}\preccurlyeq\mathfrak{h}\,,\,\mathfrak{l}\preccurlyeq\mathfrak{k}} \nu_{\mathfrak{l}}\,\mathfrak{x}_{\mathfrak{l}} = \mathfrak{x}_{\mathfrak{h}}\cdot\mathfrak{x}_{\mathfrak{k}} \notin \mathfrak{p}\,, \quad \text{where} \quad \nu_{\mathfrak{l}} \in \mathbb{Z}\,.$$

Therefore at least one $\mathfrak{x}_{\mathfrak{l}} \notin \mathfrak{p}$ (with $\nu_{\mathfrak{l}} \neq 0$) and so $\mathfrak{l} = \mathfrak{h} = \mathfrak{k}$. Furthermore, if $\mathfrak{h} := \min(\mathfrak{C}_G \setminus (\mathfrak{C}_G \cap \mathfrak{p}))$, then for every $G$-set $X \in \mathfrak{x} \in \mathrm{B}^+(G)$ we have

$$\text{(4.2.2)} \qquad \mathfrak{x}\cdot\mathfrak{x}_{\mathfrak{h}} = \nu_{\mathfrak{h}}\,\mathfrak{x}_{\mathfrak{h}} + \sum_{\mathfrak{k}\preccurlyeq\mathfrak{h}\,,\,\mathfrak{k}\neq\mathfrak{h}} \nu_{\mathfrak{k}}\,\mathfrak{x}_{\mathfrak{k}}\,, \quad \text{where} \quad \nu_{\mathfrak{k}} \in \mathbb{Z}\,.$$

Applying $\varphi$ to the equation (4.2.2) we get (since $x_{\mathfrak{k}} \in \mathfrak{p}$ for every $\mathfrak{k} \preccurlyeq \mathfrak{h}$, $\mathfrak{k} \neq \mathfrak{h}$)

(4.2.2) $$\varphi(x)\varphi(x_{\mathfrak{h}}) = \varphi(x \cdot x_{\mathfrak{h}}) = \nu_{\mathfrak{h}}\varphi(x_{\mathfrak{h}}).$$

and hence by Lemma 4.1 $\varphi(x) = \nu_{\mathfrak{h}} = \#\operatorname{Fix}_{\mathfrak{h}} x$ for every $x \in \mathrm{B}^+(G)$. This proves that $\varphi = \varphi_{\mathfrak{h}}$. □

For each subgroup $H \in \mathbf{Sub}(G)$ of $G$ and a prime number $p$, the contraction $(\varphi_H)^{-1}(\mathbb{Z}p) \subseteq \mathrm{B}(G)$ of the prime ideal $\mathbb{Z}p$ in $\mathbb{Z}$ is a prime ideal in $\mathrm{B}(G)$. It is clear that it depends only on the conjugacy class $\mathfrak{h}$ of $H$. This prime ideal in $\mathrm{B}(G)$ is denoted by $\mathfrak{p}(\mathfrak{h}, p)$.

**4.3 Corollary** *Let $\mathfrak{q}$ be a prime ideal in $\mathrm{B}(G)$. Then the subset $\mathrm{T}(\mathfrak{q}) := \{\mathfrak{h} \in \mathfrak{C}_G \mid x_{\mathfrak{h}} \notin \mathfrak{q}\}$ of $\mathfrak{C}_G$ contains exactly one minimal element $x_{\mathfrak{h}}$ with respect to the order $\preccurlyeq$ defined in 2.1. Moreover, $\mathfrak{q} = \mathfrak{p}(\mathfrak{h}, p)$, where $p$ is the characteristic of the integral domain $\mathrm{B}(G)/\mathfrak{q}$.*

**Proof:** Immediate from Lemma 4.2 and Lemma 4.1. □

**4.4 Remark** Different subgroups may define the same prime ideal, for example: *Let $H$, $H'$ be subgroups in $G$ such that $H$ is normal in $H'$ and the quotient group $H'/H$ is a $p$-group for some $p \in \mathbb{P}$. Then $\mathfrak{p}(\mathfrak{h}, p) = \mathfrak{p}(\mathfrak{h}', p)$.* **Proof:** For a $G$-set $X$, the set $\operatorname{Fix}_H \setminus \operatorname{Fix}_{H'} X$ is a $H'/H$-set and its orbits have cardinalities $p^r$, $r \geq 1$. Moreover, one can describe the defining set $\mathrm{D}(\mathfrak{p}, p) := \{\mathfrak{h} \in \mathfrak{C}_G \mid \mathfrak{p} = \mathfrak{p}(\mathfrak{h}, p)\}$ by using group theory, see 4.8. It is the set of conjugacy classes which lie in between (with respect to the order defined on $\mathfrak{C}_G$) the conjugacy classes $\mathfrak{h}^p$ and $\mathfrak{h}$, where $\mathfrak{h}^p$ is the conjugacy class of the subgroup $H^p$

For every prime ideal $\mathfrak{q} \in \operatorname{Spec}\mathrm{B}(G)$, the quotient ring $\mathrm{B}(G)/\mathfrak{q}$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/\mathbb{Z}p$ for some prime number $p \in \mathbb{P}$ (we use $\mathbb{P}$ to denote the set of all prime numbers). Therefore $\operatorname{Spec}\mathrm{B}(G) = \operatorname{Spec}_0\mathrm{B}(G) \bigcup \left(\bigcup_{p\in P}\operatorname{Spec}_p\mathrm{B}(G)\right)$, where

$$\operatorname{Spec}_0\mathrm{B}(G) := \{\mathfrak{q} \in \operatorname{Spec}\mathrm{B}(G) \mid \mathrm{B}(G)/\mathfrak{q} \cong \mathbb{Z}\} \cong \operatorname{Hom}_{\text{rings}}(\mathrm{B}(G), \mathbb{Z}) \quad \text{and}$$
$$\operatorname{Spec}_p\mathrm{B}(G) := \{\mathfrak{q} \in \operatorname{Spec}\mathrm{B}(G) \mid \mathrm{B}(G)/\mathfrak{q} \cong \mathbb{Z}/\mathbb{Z}p\} \cong \operatorname{Hom}_{\text{rings}}(\mathrm{B}(G), \mathbb{Z}/\mathbb{Z}p).$$

Moreover, $\operatorname{Spec}_0\mathrm{B}(G)$ is precisely the set of minimal prime ideals in $\mathrm{B}(G)$ and hence (since $\mathrm{B}(G)$ is noetherian) is a finite set and is the set $\operatorname{Ass}\mathrm{B}(G)$ of associated prime ideals in $\mathrm{B}(G)$. Further, the union $\bigcup_{p\in\mathbb{P}}\operatorname{Spec}_p\mathrm{B}(G)$ is the maximal spectrum $\operatorname{Spm}\mathrm{B}(G)$ of $\mathrm{B}(G)$.

**4.5 Remark** Since the map $\varphi : \mathrm{B}(G) \to \mathbb{Z}^{\mathfrak{C}_G}$ is injective and integral (see Corollary 2.13), the canonical map $\operatorname{Spec}\varphi : \operatorname{Spec}\mathbb{Z}^{\mathfrak{C}_G} \to \operatorname{Spec}\mathrm{B}(G)$ is surjective, see [7, Lemma 3.B.9 (3)]. One can also use this and the following explicit description of the prime spectrum of the product $\mathbb{Z}$-algebra $\mathbb{Z}^{\mathfrak{C}_G}$ to prove the above results on the spectrum $\operatorname{Spec}\mathrm{B}(G)$ of the Burnside algebra of $G$.

$$\operatorname{Spec}\mathbb{Z}^{\mathfrak{C}_G} = \operatorname{Spec}_0\mathbb{Z}^{\mathfrak{C}_G} \cup \operatorname{Spm}\mathbb{Z}^{\mathfrak{C}_G},$$

where $\operatorname{Spec}_0\mathbb{Z}^{\mathfrak{C}_G} = \{\pi_{\mathfrak{h}}^{-1}(0) \mid \mathfrak{h} \in \mathfrak{C}_G\}$, $\operatorname{Spec}_p\mathbb{Z}^{\mathfrak{C}_G} = \{\pi_{\mathfrak{h}}^{-1}(p) \mid \mathfrak{h} \in \mathfrak{C}_G\}, p \in \mathbb{P}$, $\pi_{\mathfrak{h}} : \mathbb{Z}^{\mathfrak{C}_G} \to \mathbb{Z}$, is the $\mathfrak{h}$-th projection and $\operatorname{Spm}\mathbb{Z}^{\mathfrak{C}_G} = \bigcup_{p\in\mathbb{P}}\operatorname{Spec}_p\mathbb{Z}^{\mathfrak{C}_G}$ is the maximal spectrum of $\mathbb{Z}^{\mathfrak{C}_G}$.

**4.6 Corollary** *Let $G$ be a finite group and let $\mathrm{B}(G)$ be the Burnside algebra of $G$. Then the Krull dimension of $\mathrm{B}(G)$ is $\dim\mathrm{B}(G) = 1$. Moreover, the irreducible components of the prime spectrum $\operatorname{Spec}\mathrm{B}(G)$ are precisely $\{\mathrm{V}(\mathfrak{p}(\mathfrak{h}, 0)) \mid \mathfrak{h} \in \mathfrak{C}_G\}$. In particular, $\operatorname{Spec}\mathrm{B}(G)$ is 1-pure dimensional.*

**4.7 Remark** One can use [7, Remark 3.A.19] to describe connected components of the prime spectrum $\operatorname{Spec}\mathrm{B}(G)$ as follows: Define a graph $\Gamma_G$ by taking the set of vertices $\mathfrak{C}_G$ and connect two vertices $\mathfrak{h}$ and $\mathfrak{k}$ by an edge if and only if the corresponding irreducible components $\mathrm{V}(\mathfrak{p}(\mathfrak{h}, 0))$ and $\mathrm{V}(\mathfrak{p}(\mathfrak{k}, 0))$ intersects, i. e. if there exists a prime number $p \in \mathbb{P}$ such that

$\mathfrak{p}(\mathfrak{h},0) \subseteq \mathfrak{p}(\mathfrak{h},p)$ and $\mathfrak{p}(\mathfrak{k},0) \subseteq \mathfrak{p}(\mathfrak{k},p)$. Then connected components of $\operatorname{Spec} \mathrm{B}(G)$ are the unions of the irreducible components corresponding to the vertices of the connected components of the graph $\Gamma_G$.

For more precise description of the prime and maximal spectrum of $\mathrm{B}(G)$, we need some definitions and results from group theory.

**4.8 Some Results from Group Theory** Let $G$ be a finite group.

(1) For a prime number $p \in \mathbb{P}$, let $G^p$ be the smallest normal subgroup of $H$ such that the quotient group $G/G^p$ is a $p$-group. It is clear that $G^p$ is a characteristic subgroup of $G$. Moreover, if $H \subseteq G$ is a normal subgroup of $G$ such that the quotient group $G/H$ is a $p$-group, then $H^p = G^p$. If $\mathrm{W}_G(H) \not\equiv 0 \bmod p$, then $H/H^p$ is a $p$-Sylow subgroup of $\mathrm{W}_G(H^p)$. A finite group $G$ is called $p$-p e r f e c t if $G^p = G$. For example, finite simple groups of order $\neq p$ are $p$-perfect.

(2) For a prime number $p \in \mathbb{P}$, let $H_p$ be the inverse image (in $\mathrm{N}_G(H^p)$) of any $p$-Sylow subgroup in $\mathrm{W}_G(H^p)$ under the canonical surjective map $\mathrm{N}_G(H^p) \to \mathrm{W}_G(H^p) = \mathrm{N}_G(H^p)/H^p$. Then $H^p = (H_p)^p$ is a characteristic subgroup of $H_p$ and hence $\mathrm{N}_G(H_p) \subseteq \mathrm{N}_G(H^p)$ and $p$ does not divide $[\mathrm{N}_G(H_p) : H_p] = \# \mathrm{W}_G(H_p)$.

(3) Let $G^{\mathrm{s}}$ denote the smallest normal subgroup of $G$ such that the quotient group $G/G^{\mathrm{s}}$ is solvable. It is easy to see that for a finite group $G$, $(G^{\mathrm{s}})^{\mathrm{s}} = G^{\mathrm{s}}$ and that $G$ is perfect[9] if and only if $G = G^{\mathrm{s}}$.

**4.9 Proposition** *Let G be a finite group. Then*

(1) *The map* $\mathfrak{C}_G \to \operatorname{Spec}_0 \mathrm{B}(G)$, $\mathfrak{h} \mapsto \mathfrak{p}(\mathfrak{h},0)$ *is bijective. Moreover,* (see Corollary 4.3) $\operatorname{Min} \mathrm{T}(\mathfrak{p}(\mathfrak{h},0)) = \mathfrak{x}_{\mathfrak{h}}$. *In particular,*
$$\# \operatorname{Spec}_0(\mathrm{B}(G)) = \# \mathfrak{C}_G .$$

(2) *For each* $p \in \mathbb{P}$, *let* $\mathfrak{C}_{G,p} := \{\mathfrak{h} \in \mathfrak{C}_G \mid \# \mathrm{Aut}_{\mathfrak{g}} \mathfrak{x}_{\mathfrak{h}} \not\equiv 0 \bmod p\}$. *Then the map*
$$\mathfrak{C}_{G,p} \to \operatorname{Spec}_p \mathrm{B}(G), \quad \mathfrak{h} \mapsto \mathfrak{p}(\mathfrak{h},p).$$
*is bijective with the inverse map* $\mathfrak{q} \mapsto \operatorname{Min}(\mathrm{T}(\mathfrak{q}), \preccurlyeq)$ (see Corollary 4.3). *In particular,*
$$\# \operatorname{Spec}_p(\mathrm{B}(G)) = \# \mathfrak{C}_{G,p} \leq \# \mathfrak{C}_G \quad \textit{and} \quad \# \operatorname{Spm}(\mathrm{B}(G)) = \sum_{p \in \mathbb{P}} \# \mathfrak{C}_{G,p} .$$

(3) *For a prime* $p \in \mathbb{P}$ *and* $\mathfrak{h}, \mathfrak{k} \in \mathfrak{C}_G$, *the equality* $\mathfrak{p}(\mathfrak{h},p) = \mathfrak{p}(\mathfrak{k},p)$ *holds if and only if* $\mathfrak{h}^p = \mathfrak{k}^p$, *where* $\mathfrak{h}^p$ *(respectively,* $\mathfrak{k}^p$*) is the conjugacy class of the subgroup* $H^p$ *(respectively,* $K^p$*). Moreover, in this case* $\operatorname{Min} \mathrm{T}(\mathfrak{p}(\mathfrak{h},p)) = \mathfrak{h}_p$ (see 4.3).

(4) *The irreducible components* $\mathrm{V}(\mathfrak{p}(\mathfrak{h},0))$ *and* $\mathrm{V}(\mathfrak{p}(\mathfrak{k},0))$ *of* $\operatorname{Spec} \mathrm{B}(G)$ *intersects if and only if* $\mathfrak{h}^p = \mathfrak{l}^p$ *for some prime number* $p \in \mathbb{P}$.

(5) *For* $\mathfrak{h} \in \mathfrak{C}_G$, *the prime ideals* $\mathfrak{p}(\mathfrak{h},p)$ *and* $\mathfrak{p}(\mathfrak{h}^{\mathrm{s}},0)$ *belong to the same connected component of* $\operatorname{Spec} \mathrm{B}(G)$.

**Proof:** (1) Note that $\mathfrak{p}(\mathfrak{h},0) = \{\mathfrak{x} \in \mathrm{B}(G) \mid \varphi_{\mathfrak{h}}(\mathfrak{x}) = 0\}$ and $\varphi_{\mathfrak{h}}$ induces an isomorphism $\mathrm{B}(G)/\mathfrak{p}(\mathfrak{h},0) \xrightarrow{\sim} \mathbb{Z}$. Therefore by Corollary 4.3 the map $\mathfrak{h} \mapsto \mathfrak{p}(\mathfrak{h},0)$ is surjective. If $\mathfrak{p}(\mathfrak{h},0) = \mathfrak{p}(\mathfrak{h}',0)$, then $\varphi_{\mathfrak{h}} = \varphi_{\mathfrak{h}'}$ and so $\mathfrak{h} = \mathfrak{h}'$.

[9] A group $G$ is called p e r f e c t if $G = [G,G]$(= the commutator subgroup of $G$).

(2) Let $p$ be the characteristic of $\mathrm{B}(G)/\mathfrak{q}$. The minimal element $\mathfrak{h}$ of $\mathrm{T}(\mathfrak{q})$ satisfies $\mathfrak{q} = \mathfrak{p}(\mathfrak{h}, p)$ and $\mathfrak{x}_{\mathfrak{h}} \notin \mathfrak{p}(\mathfrak{h}, p)$ and hence $\mathfrak{h} \in \mathfrak{C}_{G,p}$. This proves that the map $\mathfrak{h} \mapsto \mathfrak{p}(\mathfrak{h}, p)$ is surjective. If $\mathfrak{p}(\mathfrak{h}, p) = \mathfrak{p}(\mathfrak{h}', p)$, then $\varphi_{\mathfrak{h}} \equiv \varphi_{\mathfrak{h}'} \bmod p$ and hence $\mathfrak{h} = \mathfrak{h}'$, since $\varphi_{\mathfrak{h}'}(\mathfrak{x}_{\mathfrak{h}}) \equiv \varphi_{\mathfrak{h}}(\mathfrak{x}_{\mathfrak{h}}) \not\equiv 0 \bmod p$, i. e. $\mathfrak{h} \preccurlyeq \mathfrak{h}'$ and $\varphi_{\mathfrak{h}}(\mathfrak{x}_{\mathfrak{h}'}) \equiv \varphi_{\mathfrak{h}'}(\mathfrak{x}_{\mathfrak{h}'}) \not\equiv 0 \bmod p$, i. e. $\mathfrak{h}' \preccurlyeq \mathfrak{h}$.

(3) Suppose that $\mathfrak{p}(\mathfrak{h}, p) = \mathfrak{p}(\mathfrak{k}, p) =: \mathfrak{p}$ and $\mathfrak{l} := \mathrm{Min}(\mathrm{T}(\mathfrak{p}), \preccurlyeq)$ (see 4.3). Note that the isomorphism class $\mathfrak{x}_{\mathfrak{l}} \in \mathrm{T}(G)$ is uniquely determined by the congruences $\varphi_{\mathfrak{h}}(\mathfrak{x}) \equiv \varphi_{\mathfrak{l}}(\mathfrak{x}) \bmod p$ for all $\mathfrak{x} \in \mathrm{B}(G)$ and $\varphi_{\mathfrak{h}}(\mathfrak{x}_{\mathfrak{l}}) \equiv \varphi_{\mathfrak{l}}(\mathfrak{x}_{\mathfrak{l}}) = \#\,\mathrm{Aut}_{\mathfrak{g}}\,\mathfrak{x}_{\mathfrak{l}} \not\equiv 0 \bmod p$. But by 4.8 (2) this is just the case for the conjugacy class of the subgroup $H_p$. On the other hand we have $\varphi_{\mathfrak{h}}(\mathfrak{x}) \equiv \varphi_{\mathfrak{h}^p}(\mathfrak{x}) = \varphi_{\mathfrak{h}_p}(\mathfrak{x}) \bmod p$. This proves that $\mathrm{Min}\,\mathrm{T}(\mathfrak{p}) = \mathfrak{h}_p = \mathfrak{k}_p$, and hence $(\mathfrak{h}_p)^p = \mathfrak{h}^p = (\mathfrak{k}_p)^p = \mathfrak{k}^p$. Conversely, if $\mathfrak{h}^p = \mathfrak{k}^p$, then $\mathfrak{p}(\mathfrak{h}, p) = \mathfrak{p}(\mathfrak{k}, p)$ by the proof in Remark 4.4.

(4) Immediate from (3).

(5) Two prime ideals $\mathfrak{p}$ and $\mathfrak{q} \in \mathrm{Spec}\,\mathrm{B}(G)$ belong to the same connected component of $\mathrm{Spec}\,\mathrm{B}(G)$ if and only if there exists a sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of minimal prime ideals in $\mathrm{B}(G)$ such that $\mathfrak{p} \in \mathrm{V}(\mathfrak{p}_1)$, $\mathfrak{q} \in \mathrm{V}(\mathfrak{p}_n)$ and $\mathrm{V}(\mathfrak{p}_i) \cap \mathrm{V}(\mathfrak{p}_{i+1}) \neq \emptyset$ for every $i = 1, \ldots, n$. For every $H \in \mathfrak{h}$, since the quotient group $H/H^{\mathrm{s}}$ is solvable, by definition there exists a chain of subgroups $H^{\mathrm{s}} = H_n \subsetneq H_{n-1} \subsetneq \cdots \subsetneq H_1 \subsetneq H_0 = H$ such that $[H_{i-1} : H_i] = p_i$ is prime for every $i = 1, \ldots, n$. Then clearly $\mathfrak{p}(\mathfrak{h}, p) \in \mathrm{V}(\mathfrak{p}(\mathfrak{h}_0, 0))$, $\mathfrak{p}(\mathfrak{h}^{\mathrm{s}}, 0) \in \mathrm{V}(\mathfrak{p}(\mathfrak{h}_n, 0))$ and by 4.8 (1) and (3) above $\mathrm{V}(\mathfrak{p}(\mathfrak{h}_{i-1}, 0)) \cap \mathrm{V}(\mathfrak{p}(\mathfrak{h}_i, 0)) \neq \emptyset$ for every $i = 1, \ldots, n$. This proves the assertion. □

Finally, we end this section with the following interesting link between Group Theory and Algebraic Geometry:

**4.10 Theorem** (Dress) *Let $G$ be a finite group and let* $\mathrm{B}(G)$ *be the Burnside algebra of $G$. Then:*

(1) *Two prime ideals* $\mathfrak{p}$, $\mathfrak{q}$ *in the prime spectrum* $\mathrm{Spec}\,\mathrm{B}(G)$ *belong to the same connected component of* $\mathrm{Spec}\,\mathrm{B}(G)$ *if and only if* $\mathfrak{h}^{\mathrm{s}} = \mathfrak{k}^{\mathrm{s}}$*, where*

$$\mathfrak{h} := \mathrm{Min}\left(\{\mathfrak{h} \in \mathfrak{C}_G \mid \mathfrak{x}_{\mathfrak{h}} \notin \mathfrak{p}\}, \preccurlyeq\right) \text{ and } \mathfrak{k} := \mathrm{Min}\left(\{\mathfrak{k} \in \mathfrak{C}_G \mid \mathfrak{x}_{\mathfrak{k}} \notin \mathfrak{q}\}, \preccurlyeq\right)$$

(see Corollary 4.3).

(2) *The map*

$$\{\mathfrak{h} \in \mathfrak{C}_G \mid H = [H,H] \text{ for } H \in \mathfrak{h}\} \xrightarrow{\sim} \pi_0(\mathrm{Spec}\,\mathrm{B}(G)), \quad \mathfrak{h} \mapsto \mathrm{V}(\mathfrak{p}(\mathfrak{h}, 0)),$$

*is a bijection from the set of conjugacy classes of perfect* (see Footnote 9) *subgroups* $H \in \mathbf{Sub}(G)$ *onto the set* $\pi_0(\mathrm{Spec}\,\mathrm{B}(G))$ *of connected components of* $\mathrm{Spec}\,\mathrm{B}(G)$.

(3) *The number of minimal primes in the connected component of* $\mathfrak{p}(\mathfrak{h}, p)$ *is equal to the number of conjugacy classes* $\{\mathfrak{k} \in \mathfrak{C}_G \mid H^s \text{ and } K^s \text{ conjugates in } GK \in \mathfrak{k}\}$.

**Proof:** Note that for every subgroup $H \in \mathbf{Sub}(G)$ and every prime number $p \in \mathbb{P}$, we have $H^{\mathrm{s}} = (H^p)^{\mathrm{s}}$. With this the assertion (1) immediate from Proposition 4.9 (4), (5). The assertions (2) and (3) are immediate from (1). □

**4.11 Corollary** *Let $G$ be a finite group and let* $\mathrm{B}(G)$ *be the Burnside algebra of $G$. Then the prime spectrum* $\mathrm{Spec}\,\mathrm{B}(G)$ *is connected if and only if $G$ is a solvable group.*

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Received 7 January 2011; Revised 7 March 2011


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