ON A SERIES OF PRODUCTS OF THREE GEGENBAUER POLYNOMIALS*

BY S. K. LAKSHMANA RAO

(Department of Power Engineering, Civil and Hydraulic Section, Indian Institute of Science, Bangalore-3)

Received August 5, 1955

In a note in the *Proceedings of the American Mathematical Society*, John P. Vinti[1] has established the following theorem:

THEOREM.—*If* x, y, z *are real variables and* P_n *denotes the Legendre polynomial of order* n, *then*

$$\sum_{n=0}^{\infty} (n + \tfrac{1}{2}) P_n(x) P_n(y) P_n(z) = \begin{cases} \pi^{-1} g^{-\frac{1}{2}} & (g > 0) \\ 0 & (g < 0) \end{cases} \tag{1}$$

$$-1 < x, y, z < +1$$

where

$$g \equiv g(x, y, z) = 1 - x^2 - y^2 - z^2 + 2xyz \tag{2}$$

The object of the present note is to prove a similar relation involving the Gegenbauer (ultraspherical) polynomials.

Let x, y, z be real variables and C_n^{ν} the Gegenbauer polynomial of order n. We prove

$$\sum_{n=0}^{\infty} \left(\frac{n!}{\Gamma(n + 2\nu)}\right)^2 (n + \nu)\, C_n^{\nu}(x)\, C_n^{\nu}(y)\, C_n^{\nu}(z)$$

$$= \begin{cases} \dfrac{4^{1-2\nu}\,\pi}{(\Gamma(\nu))^4} \dfrac{g^{\nu-1}}{\{(1 - x^2)(1 - y^2)(1 - z^2)\}^{\nu-\frac{1}{2}}} & (g > 0) \\ 0 & (g < 0) \end{cases} \tag{3}$$

$$-1 < x, y, z < +1;\ \nu > 0$$

As in [1], if we denote by T_+ and T_- the regions as bounded above, *i.e.*, $(-1 < x, y, z < +1)$, wherein $g > 0$ and $g < 0$ respectively, the left-hand side of (3) converges uniformly with respect to x or y or z alone in any closed interval (parallel to the x or y or z axis) interior to T_+ or T_-.

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*After completing the work of this paper, I have learnt that Dr. Brafman (Wayne University Detroit, U.S.A.) has communicated a paper on the topic.

Introduce the function

$$f(x,y,z) = \begin{cases} \dfrac{4^{1-2\nu}\pi}{(\Gamma(\nu))^4} \dfrac{g^{\nu-1}}{\{(1-x^2)(1-y^2)(1-z^2)\}^{\nu-\frac{1}{2}}} & (g>0) \\ 0 & (g\leqslant 0) \end{cases} \tag{4}$$

$$-1\leqslant x, y, z\leqslant +1\ ;\ \nu>0$$

From the expansion

$$f(x,y,z) = \sum_{n=0}^{\infty} A_n f_n(y,z)\, C_n^{\nu}(x) \tag{5}$$

we have after a formal calculation

$$\int_{-1}^{+1} f(x,y,z)(1-x^2)^{\nu-\frac{1}{2}} C_n^{\nu}(x)\,dx = A_n f_n(y,z) \frac{2^{1-2\nu}\pi\,\Gamma(n+2\nu)}{n!\,(n+\nu)\,(\Gamma(\nu))^2}. \tag{6}$$

As shown in[1], $g>0$ if and only if

$$x_1 = yz - \sqrt{(1-y^2)(1-z^2)} < x < x_2 = yz + \sqrt{(1-y^2)(1-z^2)},$$

so that the integral on the left-hand side of (6) can be written as

$$\frac{4^{1-2\nu}\pi}{(\Gamma(\nu))^4} \int_{x_1}^{x_2} \frac{g^{\nu-1}}{\{(1-y^2)(1-z^2)\}^{\nu-\frac{1}{2}}} \quad C_n^{\nu}(x)\,dx.$$

By the substitution

$$x = yz + \sqrt{(1-y^2)(1-z^2)}\cos\phi,$$

the above reduces to

$$\frac{4^{1-2\nu}\pi}{(\Gamma(\nu))^4}\int_0^{\pi} C_n^{\nu}\{yz + \sqrt{(1-y^2)(1-z^2)}\cos\phi\}(\sin\phi)^{2\nu-1}\,d\phi.$$

Using the addition formula for Gegenbauer polynomials,[2] the expression is seen to be

$$\frac{2^{1-2\nu}\,n!\,\pi}{(\Gamma(\nu))^2\,\Gamma(n+2\nu)} C_n^{\nu}(y)\, C_n^{\nu}(z).$$

From (6) we have then

$$A_n f_n(y,z) = \left(\frac{n!}{\Gamma(n+2\nu)}\right)^2 (n+\nu)\, C_n^{\nu}(y)\, C_n^{\nu}(z). \tag{7}$$

Comparing (5) and (7), we have

$$f(x,y,z) = \sum_{n=0}^{\infty}\left(\frac{n!}{\Gamma(n+2\nu)}\right)^2 (n+\nu)\, C_n^{\nu}(x)\, C_n^{\nu}(y)\, C_n^{\nu}(z). \tag{8}$$

It remains to examine the validity of the expansion (5). As $f(x,y,z)$ is piecewise continuous in $-1\leqslant x, y, z\leqslant 1$, we observe[3] that if the integrals

$$I_1 = \int_{-1}^{+1}(1-x^2)^{\nu-\frac{1}{2}}\,|f(x,y,z)|\,dx,\quad I_2 = \int_{-1}^{+1}(1-x^2)^{\nu/2-1/2}\,|f(x,y,z)|\,dx$$

exist, the series expansion (5) is valid in the interior of $(-1, +1)$ and the convergence is uniform in every closed interval interior to $(-1, +1)$. Also the expansion (5) is valid at the end points $x = \pm 1$ if $\nu < 0$.

Now we show that both I_1 and I_2 exist if $\nu > 0$. For

$$I_1 = \frac{4^{1-2\nu}\pi}{(\overline{|\nu})^4} \{(1-y^2)(1-z^2)\}^{-(\nu-\frac{1}{2})} \int_{x_1}^{x_2} g^{\nu-1}\, dx,$$

and with the substitution

$$x = yz + \sqrt{(1-y^2)(1-z^2)} \cos \phi$$

$$I_1 = \frac{4^{1-2\nu}\pi}{(\overline{|\nu})^4} \int_0^{\pi} (\sin \phi)^{2\nu-1}\, d\phi.$$

The last integral exists if $\nu > 0$.

$$I_2 = \frac{4^{1-2\nu}\pi}{(\overline{|\nu})^4} \{(1-y^2)(1-z^2)\}^{-(\nu-\frac{1}{2})} \int_{x_1}^{x_2} (1-x^2)^{-\nu/2} g^{\nu-1}\, dx$$

exists if $x_1^2, x_2^2 \neq 1$. If x_1^2 or $x_2^2 = 1$, we have $y = -z$ or $y = z$ and in both these cases we can write

$$I_2 = \frac{4^{1-2\nu}(2\pi)}{(\overline{|\nu})^4} \cdot \frac{|a|^{3\nu-2}}{(1-y^2)^{2\nu-1}} \int_0^{\pi/2} (\cos\theta)^{\nu-1} (\sin\theta)^{2\nu-1} (2 - a^2\cos^2\theta)^{-\nu/2}\, d\theta$$

where

$$0 < a^2 = 2(1-y^2) \leqslant 2.$$

As

$$(2 - a^2\cos^2\theta)^{-\nu/2} \leqslant (2\sin^2\theta)^{-\nu/2} \text{ for } \nu > 0,$$

$$I_2 \leqslant \frac{4^{1-2\nu}\pi}{(\overline{|\nu})^4} \cdot \frac{a^{3\nu-2}}{(1-y^2)^{2\nu-1}} \cdot 2^{-\nu/2}\, B\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$$

which certainly exists for $\nu > 0$. If $y^2 = z^2 = 1$, I_2 reduces to 0.

At the end points $x = \pm 1$, the expansion (5) is not valid as that requires $\nu < 0$. For the same reason we conclude that (8) is not valid at the points $y = \pm 1$ and $z = \pm 1$.

My best thanks are due to Professor N. S. Govinda Rao for kind encouragement.

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