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Nature of the spectrum associated with a matrix differential operator

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Abstract

The asymptotic solution of a self-adjoint matrix differential system are obtained by means of an integral transformation, using Titchmarsh's complex variable method. This leads to finding the nature of the spectrum of the differential system.

Key words and phrases: Matrix differential operator, eigenvalue parameter, self-adjoint, integral equation, Conte and Sangren's lemma, spectrum, L^2-solution.

1. Introduction

We consider the differential system

$$(L - \lambda F)\, U = 0 \tag{1}$$

where,

$$L = \begin{pmatrix} \dfrac{d}{dx}\left(p_0(x)\dfrac{d}{dx}\right) + p_1(x) & r(x) \\ r(x) & \dfrac{d}{dx}\left(q_0(x)\dfrac{d}{dx}\right) + q_1(x) \end{pmatrix}$$

is a matrix differential operator, F, a 2×2 matrix and $U \equiv \{u, v\}$, λ, a scalar. The system (eqn 1) is equivalent to

$$p_0\frac{d^2u}{dx^2} + p_0'\frac{du}{dx} + p_1 u + rv - \lambda(F_{11}u + F_{12}v) = 0,$$

$$q_0 \frac{d^2v}{dx^2} + q_0' \frac{dv}{dx} + q_1 v + ru - \lambda(F_{21}u + F_{22}v) = 0 \tag{2}$$

$0 \leq x < \infty$, where λ is an eigenvalue parameter; p_0, p_0', q_0, q_0', $\in c[0, \infty)$; p_1, q_1, $r \to \infty$, as $x \to \infty$; p_1, q_1, $r \in c[0,b]$ for all $b > 0$; p_0, $q_0 > 0$ in $[0,\infty]$; F is PDC $[0, \infty]$, along with the boundary conditions at $x = 0$, *viz.*,

i) $p_0(0)\ [a_{j1}u(0) + a_{j2}u'(0)] + q_0(0)[a_{j3}v(0) + a_{j4}v'(0)] = 0,\ (j = 1,2).$ (3)

ii) $a_{jk}(j = 1,2;\ k = 1,2,3,4)$ are real-valued constants.

iii) the set $\{a_{1k};\ k = 1,2,3,4\}$ is linearly independent of the set $\{a_{2k};\ k = 1,2,3,4\}$.

iv) $q_0(0)\ (a_{14}a_{23} - a_{24}a_{13}) + p_0(0)(a_{12}a_{21} - a_{14}a_{22}) = 0$ (4)

The relation (4) ensures the self-adjointness of the system (2–3).

In the present paper, we study the nature of the spectrum of the self-adjoint differential system[2-3], the system being considered earlier by Bhagat[1] in solving some other eigenvalue problem, and obtain some generalisation of the results in Paladhi[2].

The results and notations of Bhagat[1,3], Chakraborty[5], Paladhi[2] and Titchmarsh[5] are followed.

2. Notations

We use the following notations:

$$z(x) = (\lambda - p_1(x))\ (\lambda - q_1(x))$$

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \equiv \{u(x),\ v(x)\} \equiv \{u, v\}$$

$$K(p_0p_0') = [(p_0z(x)^{-1/4})' + p_0z(x)^{1/2}(z(x)^{1/4})'][z(x)^{-3/4}p_0'p_0^{-1} + (z(x)^{1/4})'\, z(x)^{-1}]$$

$$(Fij) = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$$

$F = (Fij)$ is PDC $[0, \infty]$ means that F is positive definite and continuous over the interval $[0, \infty]$.

$(\alpha, \beta) = \alpha_1\bar{\beta}_1 + \alpha_2\bar{\beta}_2$ for any two vectors $\alpha = \{\alpha_1, \alpha_2\}$, $\beta = \{\beta_1, \beta_2\}$

$$T(x) = -\ ip_0^{-0}p_0'z(x)^{-3/4}$$

$$T_1(x) = \{1/4\ p_0z(x)^{-5/4}\ z'(x) + p_0'(x)(z(x)^{-1/4}\}'(z(x)^{-1/4}\ p_0^{-1})$$

$$\xi_1(t) = -z(t)^{1/2} + p_0^{-1}(p_1 - \lambda F_{11})\, z(t)^{-1/2} - T'(t) - iT_1(t)$$
$$+ i\, p_0^{-2}(p_0 z(t)^{1/4})'\, z(t)^{-1} p_0'$$

$$\eta_1(t) = \{(r - \lambda F_{12})\, q_0^{-1} z(t)^{-1/2}\}$$
$$l(t) = \{\xi_1(t), \eta_1(t)\}, \ \Omega(t) = \{\eta(t), \xi(t)\}$$
$$\zeta_1(t) = \{-z(t)^{1/2} + q_0^{-1}(q_1 - \lambda F_{22})\, z(t)^{-1/2} - S'(t) - iS_1(t)$$
$$+ iq_0^{-2}\,(q_0 z(t)^{1/4})\, z(t)^{-1} q_0'\}.$$

This may be noted that $S(t)$, $S_1(t)$ denote the same expressions for $T(t)$, $T_1(t)$, respectively, p_0 and p_0' being replaced by q_0, q_0', respectively.

3. Downgrading of the coefficients of the differential system (2)

By using the transformation

$$\xi(x) = i\int_0^x z(t)^{1/2}\, dt.$$

$$\begin{pmatrix}\eta(x)\\ \xi(x)\end{pmatrix} \equiv \{\eta,\xi\} = iz(x)^{1/4}\ \{p_0 u(x),\ q_0 v(x)\} \tag{5}$$

(compare Paladhi[2]), we obtain

$$\frac{d\eta}{d\xi} = [z(x)^{-1/4}\,(p_0 u' + p_0' u) + \{z(x)^{1/4}\}' z(x)^{-1/2} p_0 u(x)]$$

$$\frac{d^2\eta}{d^2\xi^2} = \frac{d}{d\xi}\left(\frac{d\eta}{d\xi}\right) = \frac{d}{dx}\left(\frac{d\eta}{d\xi}\right)\frac{dx}{d\xi}$$
$$= -\frac{1}{p_0}\,[z(x)^{-1}(\lambda F_{11} + p_0'' - p_1) + \{(z(x)^{-1/4})'(z(x)^{1/4} \times z(x)^{-1/2}$$
$$p_0' z(x)^{-3/4} + p_0 z(x)^{-3/4}\{z(x)^{-1/2}(z(x)^{1/4}) - (p_0 z(x)^{-1/4}$$
$$+ p_0 z(x)^{-1/2}(z(x)^{1/4})'\}\ \{z(x)^{-3/4}\frac{p_0'}{p_0} + (z(x)^{1/4})'\ z(x)^{-1}\}]\eta$$
$$-\frac{1}{q_0}\,[z(x)^{-1}(\lambda F_{12} - r)]\ \zeta - iz(x)^{-1/4}\ \{(z(x)^{-1/4} + (z(x)^{1/4})'$$

$$z(x)^{-1/2} + \frac{p_0'}{p_0} z(x)^{-1/4}\} \frac{d\eta}{d\xi}$$

with similar expression for $\frac{d^2\xi}{d\xi^2}$.

Thus, (2) transforms to

$$\frac{d^2\eta}{d\xi^2} = -\frac{1}{p_0}[z(x)^{-1}(\lambda F_{11} + p_0'' - p_1) + \{(z(x)^{-1/4})' + (z(x)^{1/4})' z(x)^{-1/2}\}$$

$$p_0' z(x)^{-3/4} + p_0 z(x)^{-3/4} + p_0 z(x)^{3/4} \{z(x)^{-1/2}(z(x)^{1/4})'\}'$$

$$-(p_0 p_0')]\eta - \frac{1}{q_0}[z(x)^{-1} (\lambda F_{12} - r)]\} - iz(x)^{-1/4}\{(z(x)^{-1/4})'$$

$$+ (z(x)^{1/4})' z(x)^{-1/2} + \frac{p_0'}{p_0} z(x)^{-1/4}\} \frac{d\eta}{d\xi} \tag{6}$$

and

$$\frac{d^2\zeta}{d\xi^2} = -\frac{1}{q_0}[z(x)^{-1}(\lambda F_{22} + q_0'' - q_1) + \{(z(x)^{-1/4})' + (z(x)^{1/4})' z(x)^{-1/2}\}$$

$$q_0' z(\mathrm{x})^{-3/4} + q_0 z(x)^{-3/4} \{z(x)^{-1/2} (z(x)^{1/4})'\}' - K(q_0 q_0')]\zeta - \frac{1}{p_0}$$

$$[z(\mathrm{x})^{-1}(\lambda F_{21} - r)]\eta - i\, z(x)^{-1/4} \{(z(x)^{-1/4})\ ' + (z(x)^{-1/4} + \frac{q_0'}{q_0}$$

$$z(x)^{-1/4} \frac{d\zeta}{d\xi}. \tag{7}$$

In (6), the coefficient of η

$$= -\frac{1}{p_0}[z(x)^{-1}(\lambda F_{11} + p_0'' - p_1 + \{(z(x)^{-1/4})' + (z)x)^{1/4})' \times z(x)^{-1/2}\}$$

$$p_0' z(x)^{-3/4} + p_0 z(x)^{-3/4} \{z(x)^{-1/2}(z(x)^{1/4})'\}' - K(p_0 p'_0)]$$

$\rightarrow$ a finite number under suitable conditions on p_0, p_0'', p_1, q_1, F_{11} and the coefficient of ξ and $\frac{d\eta}{d\xi}$ are $0(1)$ as $x \rightarrow \infty$.

Similarly, in (7), the coefficient of η, ξ, $\frac{d\zeta}{d\xi}$ is $0(1)$ as $x \rightarrow \infty$.

4. Derivation of an integral solution of the system (6,7)

Let $p(\mathrm{x}) = z(x)^{1/4} H(x)$, $H(\mathrm{x}) = \{H_1(x), H_2(x)\}$

where

$$H_1(x) = \frac{d}{dx}\left[z(x)^{-1/2} \frac{d\eta}{dx}\right] - i\, z(x)^{-1/4} \frac{d}{dx}\left(p_0 \frac{du}{dx}\right) \tag{8}$$

$$H_2(x) = \frac{d}{dx}\left[z(x)^{-1/2}\frac{d\zeta}{dx}\right] - i\ z\ (x)^{-1/4}\frac{d}{dx}\left(q_0\frac{dv}{dx}\right) \tag{9}$$

Then, $$p(x) = z(x)^{1/4}\begin{pmatrix} H_1(x) \\ H_2(x) \end{pmatrix}$$

where $$z(x)^{1/4}H_1(x) = z(x)^{1/4}\frac{d}{dx}\left[z(x)^{-1/2}\frac{d\eta}{dx}\right] - i\frac{d}{dx}\left(p_0\frac{du}{dx}\right)$$

$$= i\ z(x)^{1/4}\left[\left[\left\{\frac{p_0}{4}z(x)^{-5/4}\ z'(x) + p_0'\ z(x)^{-1/4}\right\}'\ u(x)\right.\right.$$

$$+ (p_0'\ z(x)^{-1/4})u'(x). \tag{10}$$

Similarly,

$$z(x)^{1/4}\ H_2(x) = iz(x)^{1/4}\left[\left[\left\{\frac{q_0}{4}z(x)^{-5/4}\ z'(x) + q_0'z(x)^{-1/4}\right\}'v(x) + (q_0'z(x)^{-1/4})v'(x).\right.\right. \tag{11}$$

We have, $\int_0^x \sin\ (\xi(x) - \xi(t))\ H_1(t)\ dt$

$$= \int_0^x\left[\frac{d}{dt}\left\{z(t)^{-1/2}\frac{d\eta}{dt}\right\} + i\ \{p_1u + rv - \lambda(F_{11}u + F_{12}v)\}z(t)^{-1/4}\right]$$

$$\times \sin\ (\xi(x) - \xi(t))\ dt \text{ by (8 and 2)}$$

$$= -\sin\ \xi(x)\ \lambda^{-1}\ \eta'(0) + i\eta(x) - i\eta(0)\ \cos\ \xi\ (x) - \int_0^x \sin(\xi(x)$$

$$- \xi(t))z(t)^{1/2}$$

$$\times \eta(t)\ dt + \int_0^x \sin\ (\xi\ (x) - \xi(t))\left\{\frac{p_1}{p_0}\eta(t) + \frac{r}{q_0}\xi(t) - \lambda\left(F_{11}\frac{\eta(t)}{p_0}\right.\right.$$

$$\left.\left. + F_{12}\frac{\zeta(t)}{q_0}\right)\right\}z(t)^{-1/2}dt. \tag{12}$$

Also by (10) we have

$$\int_0^x \sin\ (\xi(x) - \xi(t))\ H_1(t)\ dt.$$

$$= i \int_0^x \text{Sin } (\xi(x) - \xi(t)) \left[\left\{ \frac{p_0}{4} z(t)^{-5/4} z'(t) + p_0' z(t)^{-1/4} \right\}' u(t) \right.$$

$$\left. + \{p_0' z(t)^{-1/2}\} u'(t) \, dt. \right. \tag{13}$$

From (12–13), we obtain

$$\eta(x) = \eta(0) \cos \xi(x) - i\eta^{-1}\eta' (0) \sin \xi(x) + i \int_0^x \sin (\xi(x) - \xi(t))$$

$$\times [\{ - z(t)^{1/2} + p_0^{-1} (p_1 - \lambda F_{11}) z(t)^{-1/2}\}\eta(t) + (r - \lambda F_{12}) q_0^{-1}$$

$$z(t)^{-1/2}\zeta(t) + K(t, \lambda)] \, dt, \tag{14}$$

where, $$K(t, \lambda) = \left\{ \frac{p_0(t)}{4} z(t)^{-5/4} z'(t) + p_0'(t) z(t)^{-1/4} \right\}' u(t) + p_0' \, z(t)^{-1/2}\} u'(t). \tag{15}$$

We have from (5)

$$\eta(x) = i \, z(x)^{1/4} \, p_0 \, (x) \, u(x)$$

$$\eta'(x) = i[(z(x)^{1/4} \, p_0')' \, u(x) + z(x)^{1/4} \, p_0(x) u'(x)]$$

so that

$$u'(x) = i \, [-z(x)^{-1/4} \, p_0^{-1} \, \eta' \, (x) + (z(x)^{1/4} p_0)' \, z(x)^{-1/2} \, p_0^{-2} \, \eta(x)] \tag{16}$$

It then follows from (15–16) that

$$K(t, \lambda) = T(t) \, \eta'(t) - i \left\{ \frac{p_0}{4} z(t)^{-5/4} z'(t) + p_0' \, z(t)^{-1/4} \right\}'$$

$$(z(t)^{-1/4} \, p_0^{-1}) \, \eta(t) + i \, (p_0 \, z(t)^{1/4})' \, p_0' \, p_0^{-2} \, z(t)^{-1}) \eta(t)$$

$$= T(t) \, \eta'(t) - iT_1(t) \, \eta(t) + i \, p_0^{-2} \, (p_0 \, z(t)^{1/4})' \, z(t)^{-1} \, p_0' \eta(t) \tag{17}$$

Now, $\int_0^x \sin (\xi (x) - \xi(t)) \, T(t) \, \eta' \, (t) \, dt$

$$= - T(0) \, \eta(0) \sin \xi(x) - \int_0^x [\sin (\xi(x) - \xi(t)) \, T'(t)$$

$$- i \cos (\xi (x) - \xi(t)) \, z(t)^{1/2} \, T(t)] \, \eta(t) \, dt. \tag{18}$$

It follows then from (14, 15, 17 and 18) that

$$\eta(x) = \eta(0)\cos\xi(x) + (-i\lambda^{-1}\eta'(0) - i\,T(0)\eta(0)\sin\xi(x)$$
$$+ i\int_0^x [\sin(\xi(x) - \xi(t))[\{-z(t)^{1/2} +$$
$$+ p_0^{-1}(p_1 - \lambda F_{11})z(t)^{-1/2} - T'(t) - iT_1(t) + ip_0^{-2}(p_0 z(t)^{-1/4})'$$
$$z(t)^{-1}p_0'\}\eta(t) + (r - \lambda F_{12})\,q_0^{-1}\,z(t)^{-1/2}\}\,\zeta(t)\,dt$$
$$-\int_0^x \cos(\xi(x) - \xi(t))\,z(t)^{1/2}\,T(t)\,\eta(t)\,dt. \tag{19}$$

Proceeding in the same way, taking (9) and (11) into consideration, we obtain

$$\zeta'(x) = \zeta(0)\cos\zeta(x) + (-i\lambda^{-1}\zeta'(0) - iS(0)\zeta(0))\sin\xi(x)$$
$$+ i\int_0^x \sin(\xi(x) - \xi(t))\,[\{-z(t)^{1/2} + q_0^{-1}(q_1 - \lambda F_{22})z(t)^{-1/2}$$
$$- S'(t) - iS_1(t) + iq_0^{-2}(q_0 z(t)^{1/4})'z(t)^{-1}q_0'\}\,\zeta(t) + \{(r - \lambda F_{12})$$
$$\times\, p_0^{-1}\,z(t)\,z(t)^{-1/2}\}\eta(t)]\,dt$$
$$-\int_0^x \cos(\xi(x) - \zeta(t))\,z(t)^{1/2}\,S(t)\,\xi(t)\,dt \tag{20}$$

where $S(t)$ and $S_1(t)$ can be obtained from $T(t)$ and $T(t)$, respectively, replacing p_0 by q_0 in the expressions for $T(t)$, $T_1(t)$ (see § 2).

Equations (20) can be written in the form

$$\Omega(x) = \begin{pmatrix} \eta(0) \\ \xi(0) \end{pmatrix}\cos\xi(x) + i\begin{pmatrix} -\lambda^{-1}\eta'(0) - T(0)\,\eta(0) \\ -\lambda^{-1}\zeta'(0) - S(0)\,\zeta(0) \end{pmatrix}\sin\xi(x)$$
$$+ i\int_0^x \sin(\xi(x) - \xi(t))\begin{pmatrix} \xi_1 & \eta_1 \\ \eta_1 & \zeta_1 \end{pmatrix}\begin{pmatrix} \eta(t) \\ \zeta(t) \end{pmatrix}dt \tag{21}$$
$$- i\int_0^x \cos(\xi(x) - \xi(t))\,z(t)^{1/2}\begin{pmatrix} T(t)\;\eta(t) \\ S(t)\;\zeta(t) \end{pmatrix}dt. \tag{21}$$

5. Some propositions

Proposition 1: If the coefficients p_0, q_0, p_1, q_1, $r(x)$ in the differential system (2) satisfy the following conditions, *viz.*,

(i) $p_1(x)$, $q_1(x) \to \infty$ as $x \to \infty$, *i.e.*, $p_1(x)$, $q_1(x) > Q(x)$ whenever $Q(x) \geq \delta > 0$; $x \geqslant 0$; $F_{12}\, q_0^{-1}$, $r(x)q_0^{-1}$, $\in L[O, \infty)$; $p_0^{-1}\, \{p_1, F_{11}\}$, $\{q_1\ \ F_{22}\}\ \ q_0^{-1}$ have all its elements square integrable over $[0, \infty)$.

(ii) $p_1'(x)$, $q_1'(x) \geqslant 0$

(iii) $p'_1(x) = 0[p_1(x)]^c$, $q_1(x) = 0[q_1(x)]^c$, $0 \times c < 7/4$

(iv) $p_1''(x)$, $q_1''(x)$ maintain their signs

(v) $\dfrac{1}{Q(x)} \in L[0, \infty)$

(vi) $p_0'\, p_0^{-1}$, $q_0 q_0^{-1}$, $(p_0^{-1}p_0')'$, $(q_0^{-1}q_0)' \in L\ [0, \infty)$ and $p_0 p_0'$, $q_0 q_1'$, $p_0'' p_0^{-1}$, $q_0''\, q_0^{-1} = 0(1)$

(compare Titchmarsh[5], (Ch. V, p. 121) and Paladhi[2] p. 448), then the integrals

$$\int_0^\infty [\,|\,\xi(t)\,| + |\,\eta_1(t)\,|\,]\,|\;z\,(t)^{-1}\,|\;\; dt,\; \int_0^\infty [\,|\,\eta_1(t)\,| + |\;\;\zeta_1(t)\,|\,]\,|\,z(t)^{-1}\,|\,dt$$

and

$$\int_0^\infty |\,T(t)\,|\;\;|\,z(t)^{-1/2}\,|\,dt,\; \int_0^\infty |\,S(t)\,|\;\;|\,z(t)^{-1/2}\,|\,dt$$ are uniformly convergent

with respect to λ over $|\,\lambda - p_1(x)\,|$, $|\,\lambda - q_1(x) \geq \delta_1 > 0$ for $0 \leqslant x < \infty$.

Proof. We have,

$$\int_A^x [\,|\,\xi_1(t)\,| + |\,\eta_1(t)\,|\,]\,|\,z(t)^{-1}\,|\;\; dt$$

$$\leqslant \int_A^x |\,z(t)^{-1/2}\,|\,dt + \int_A^x |\,p_0^{-1}(p_1 - \lambda F_{11})\,|\;\;|\;\;z(t)^{-3/2}\,|\;\;dt + \int_A^x |\,T'(t)\,|\;\;|\,z(t)^{-1}\,|\,dt$$

$$+ \int_A^x |\,T_1(t)\,|\;\;|\,z(t)^{-1}\,|\;\;dt + \int_A^x |\,p_0^{-2}(p_0 z(t)^{1/4} p_0'\,|\;\;\;|\,z(t)^{-1}\,|\,dt$$

$$+ \int_A^x |\,(r - \lambda F_{12})\,q_0^{-1}\,|\;\;|\,z(t)^{-3/2}\,|\;\;dt = \mathrm{I}_{11} + I_{12} + I_{13} + I_{14} + I_{15} + I_{16}, \text{ say.}$$

Now, $I_{11} = \int_A^x | z(t)^{-1/2} | \; dt$

$$\leqslant \int_A^x Q(t)^{-1} \, dt, \text{ since } | z(t) | > Q^2(t) \text{ for } t > A$$

$<\infty$

$$\leqslant \int_A^x | p_0^{-1} (p_1 - \lambda F_{11}) | \; Q(t)^{-3} \, dt$$

$$\leqslant \left[\int_A^x | p_0^{-1} (p_1 - \lambda F_{11}) |^2 \, dt \right]^{1/2} \left[\int_A^x Q(t)^{-6} dt \right]^{1/2}, \text{ by Schwarz}$$

inequality.

$= 0(1)$ as $x \to \infty$, by conditions (i) and (v).

$$I_{13} = \int_A^x | T'(t) | \; | z(t)^{-1} \, dt$$

$$\leqslant \int_A^x | 3/4 \, p_0^{-1} \, p_0' \, \{(\lambda - p_1(t)) \, q_1'(t) + (\lambda - q_1(t)) p_1'(t)\} |$$

$$| z(t)^{-11/4} | \, dt + \int_A^x | (p_0^{-1} p_0')' | \; | z(t)^{-7/4} | \, dt$$

$= 0$ (1), provided $0 < c < 11/4$.

$$I_{14} = \int_A^x | T_1(t) | \; | z(t)^{-1} | \; dt$$

$$= 0 \int_A^x [| z(t) |^{-5/2} | z''(t) | + | p_0'(t) p_0^{-1} | \; | z(t)^{-5/2} | \; | z'(t) |$$

$$+ | z(t)^{-7/2} | \; | z'(t)^2 | + | p_0''(t) | \, z(t)^{-3/2} p_0^{-1} | \,] dt$$

$$= 0 \int_A^x [p_1^{-5/2} q_1^{-5/2} (p_1^c \, q_1^c + p_1 q_1'' + q_1 p_1'') + p_1^{-5/2} \, q_1^{-5/2} \, (p_1 q_1^c + q_1 p_1^c)$$

$$+ p_1^{-7/2}\, q_1^{-7/2}(p_1^2\, q_1^{2c} + p_1^{2c}\, q_1^2) + |\, p_0''(t)p_0^{-1}\,|\,(p_1^{-3/2}q_1^{-3/2})\; dt$$

$= 0\ (1)$, as $x \to \infty$ if $0 < c < 7/4$.

$$I_{15} = \int_A^x |\, p_0^{-2}\,(p_0\, z(t)^{1/4})'\, p_0'\,| \quad |\, z(t)^{-1}\,| \quad dt$$

$$= 0 \int_A^x p_0'^2 p_0^{-2}\, p_1^{-3/4}\, q_1^{-3/4} + \frac{p_0 p_0'}{4} p_1^{-7/4}\, q_1^{-7/4}\,(p_1 q_1^c + q_1 p_1^c)\; dt$$

$= 0(1)$, $0 < c < 7/4$, by conditions (v) and (vi).

$$I_{16} = \int_A^x |\ (r - F_{12})\, q_0^{-1}\,| \quad |\, z(t)^{-3/2}\,| \quad dt \text{ as } x \to \nu$$

$= 0(1)$, as $x \to \infty$, since rq_0^{-1}, $F_{12}q_0^{-1} \in [L[0,\infty)$ by condition (i).

Thus, the integral

$$\int_0^\infty |\, z(t)^{-1}\,| \ \ (\,|\, \xi_1(t)\,| \ + \ |\, \eta_1(t)\,|\,)\ dt < \infty$$

over $|\, \lambda - p_1(x)\,|$, $|\, \lambda - q_1(x)\,| \geqslant \delta_1 > 0$ for $0 \leqslant x < \infty$.

Similar results hold for the other integral

$$\int_0^\infty |\, z(t)^{-1}\,| \ \ (\,|\, \eta_1(t)\,| \ + \ |\ \zeta_1(t)\,|\,)\ dt.$$

Now, $$\int_0^\infty |\, T(t)\,| \quad |\, z(t)^{-1/2}\,|\, dt$$

$$= 0 \int_0^\infty |\, p_0^{-1} p_0'\,| \ \ p_1^{-5/4}\, q_1^{-5/4}\; dt = 0\left[\int_0^\infty Q(t)^{-5/2}\; dt\right] = 0(1)$$

as $x \to \infty$, by condition (vi).

Similarly, $\int_0^\infty |\, s(t)\,| \quad |\, z(t)^{-1/2}\,| \quad dt = 0(1)$ as $x \to \infty$, by condition (vi).

Thus, the proposition is established.

Proposition 2: If $Im\ \lambda > 0$, $0 < \arg \lambda < \pi$, then $\exp[i\ \xi(x)] \to \infty$ as $x \to \infty$ [see Paladhi[2], Lemma II].

6. Estimates for $\eta(x)$, $\zeta(x)$

Setting $\Omega^{\dagger}(x) = \{\eta^{\dagger}(x), \zeta^{\dagger}(x)\} = \exp(-i\xi(x))z(x)\Omega(x)$ (22)

in (19) and (20), we have

$$\eta(x) = 1/2\eta(0)e^{-i\xi(x)}[1+e^{-2i\xi(x)}]+i(-\lambda^{-1}\eta'(0)-T(0)\eta(0)) \times$$

$$\times \frac{1}{2i}e^{i\xi(x)}[1-e^{-2i\xi(x)}]+i\int_0^x (\xi_1(t)\eta(t)+\eta_1(t)\zeta(t)) \times$$

$$\times e^{i(\xi(x)-\xi(t))}\frac{1}{2i}[1+e^{-2i(\xi(x)-\xi(t))}]\,dt$$

$$-\frac{1}{2}\int_0^x e^{i(\xi(x)-\xi(t))}[1+e^{-2i(\xi(x)-\xi(t))}]\,z(t)^{1/2}T(t)\eta(t)\,dt$$

or, $$\eta^+(x) = \frac{1}{2}\eta(0)\,[1+e^{-2i\xi(x)}]\,z(x) + i(-\lambda^{-1}\eta(0) - T(0)\,\eta(0))\,z(x)$$

$$\times\frac{1}{2i}[1-e^{-2i\xi(x)}] + iz(x)\int_0^x (\xi_1(t)\eta^+(t) + \eta_1(t)\,\xi^+(t))$$

$$\times z(t)^{-1}\frac{1}{2i}[1-e^{-2i(\xi(x)-\xi(t))}]\,dt$$

$$-\frac{z(x)}{2}\int_0^x \eta^+(t)\,z(t)^{-1/2}[1+e^{-2i(\xi(x)-\xi(t))}]\,T(t)\,dt.$$

Therefore, $$\left|\frac{\eta^+(x)}{z(x)}\right| \leqslant |\eta(0)| + |-\lambda^{-1}\eta'(0)-T(0)\eta(0)| +$$

$$\int_0^x (|\xi_1(t)|\,|\eta^+(t)| + |\eta_1(t)\,\zeta^+(t)|)\,|z(t)^{-1}|\,dt + \int_0^x |T(t)z(t)^{-1/2}|\,|\eta^+(t)|\,dt \quad (23)$$

Similarly,

$$\left|\frac{\zeta^+(x)}{z(x)}\right| \leqslant |\zeta(0)| + |-\lambda^{-1}\zeta'(0)-S(0)\zeta(0)| + \int_0^x (|\eta_1(t)|\,|\eta^+(t)| +$$

$$+ |\zeta_1(t)|\,|\zeta^+(t)|)\,|z(t)^{-1}|\,dt + \int_0^x |S(t)z(t)^{-1/2}|\,|\zeta^+(t)|\,dt. \quad (24)$$

Now, (23) and (24) can be written in the form

$$\left|\frac{\eta^+(x)}{z(x)}\right| \leqslant A + \int_0^x \left[\left|\frac{\xi_1^+(t)}{z(t)}\right|\,|\eta^+(t)| + \left|\frac{\eta_1(t)}{z(t)}\right|\,|\zeta^+(t)|\right]dt$$

and $\left|\frac{\zeta^+(x)}{z(x)}\right| \leqslant A + \int_0^x \left[\left|\frac{\eta_1(t)}{z(t)}\right| \left|\eta^+(t)\right| + \left|\frac{\zeta_1^+(t)}{z(t)}\right| \left|\zeta^+(t)\right|\right] dt$ (25)

where $A = \max\,[1-\lambda^{-1}\eta'(0)-T(0)\,\eta(0)\,| + |\,\eta(0)\,|\,,\,|-\lambda^{-1}\zeta'(0)-$
$S(0)\zeta(0)\,| + |\,\zeta(0)\,|\,]$

and $|\,\xi_1^+(t)\,| = |\,\xi_1(t)\,| + |\,T(t)\,z(t)^{1/2}\,|$

$|\,\zeta_1^+(t)\,| = |\,\zeta_1(t)\,| + |\,S(t)\,z(t)^{1/2}\,|\,.$ (26)

We can take $|\,\xi_1^+(t)\,| = \max[\,|\,\xi_1^+(t)\,|\,,\,|\,\zeta_1^+(t)\,|\,] = M_1(t)$ say, (27)

for otherwise the same lines of argument will follow with

$|\,\zeta_1^+(t)\,| = \max[\,|\,\xi_1^+(t)\,|\,,\,|\,\zeta_1^+(t)\,|\,]$. It follows then from (25) that

$$\left|\frac{\eta^+(x)}{z(x)}\right|,\ \left|\frac{\zeta^+(x)}{z(x)}\right| \leqslant A + \int_0^x (\alpha,\ \beta)\ dt \tag{28}$$

where $(\alpha,\beta) = \max\,[(\{M_1(t),\,|\,\eta_1(t)\,|\,\},\,\beta),\,(\{\,|\,\eta_1(t)\,|\,,\,M_1(t)\},\,\beta)]$

and $\beta = \left\{\left|\frac{\eta^+(t)}{z(t)}\right|,\ \left|\frac{\zeta^+(t)}{z(t)}\right|\right\}$; if $\alpha = \{\alpha_1,\,\alpha_2\}$, then

$$\alpha_1 + \alpha_2 = M_1(t) + |\,\eta_1(t)\,| \tag{29}$$

Thus, $\left|\frac{\eta^+(x)}{z(x)}\right|,\ \left|\frac{\zeta^+(x)}{z(x)}\right| \leqslant A + \int_0^x \left[\alpha_1 \left|\frac{\eta^+(t)}{z(t)}\right| + \alpha_2 \left|\frac{\zeta^+(t)}{z(t)}\right|\right] dt$ (30)

$\leqslant A\,\exp\,[\int_0^x (\alpha_1+\alpha_2)\,dt$, by Conte and Sangren[7]

$\leqslant A\,\exp\,[\,|\,z(x)\,|\int_0^\infty (\alpha_1^+(t) + \alpha_2^+(t))\,dt]$, where $\alpha_j^+(t) = \alpha_j(t)\,|\,z(t)\,|^{-1}$.

It follows by definition of $\Omega^+(x)$ that

$|\,\eta(x)\exp(-i\xi(x))\,|\,,\,|\,\zeta(x)\exp(-i\xi(x))\,|$

or $\leqslant A\,\exp\int_0^\infty \left[\frac{M_1(t)}{|\,z(t)\,|} + \frac{|\,\eta_1(t)\,|}{|\,z(t)\,|}\right]|\,z(x)\,|\,dt$

$|\,\eta(x)\,|\,,\,|\,\zeta(x)\,| \leqslant A\,|\exp\,(i\xi(x))\,|$

$$\times \exp\left[| z(x) | \int_0^\infty \frac{M_1(t) + | \eta_1(t) |}{| z(t) |} dt \right]. \tag{31}$$

Now, $\int_0^\infty \frac{M_1(t) + | \eta_1(t) |}{| z(t) |} dt$

is either equal to $\int_0^\infty \frac{| \xi_1^+(t) | + | \eta_1(t) | dt}{| z(t) |}$

$$= \int_0^\infty \frac{| \xi_1(t) | + | \eta_1(t) |}{| z(t) |} dt + \int_0^\infty | T(t) \ z(t)^{-1/2} | dt$$

or equal to $\int_0^\infty \frac{| \zeta_1^+(t) | + | \eta_1(t) |}{| z(t) |} dt$

$$= \int_0^\infty \frac{| \zeta_1(t) | + | \eta_1(t) |}{| z(t) |} dt + \int_0^\infty | s(t) | \ | z(t)^{-1/2} | dt.$$

The two integrals above in the R.H.S. are convergent by Proposition 1. Therefore, we obtain from (31)

$$| \eta(x) | , | \zeta(x) | \leqslant A \exp(i\xi(x))) \exp [k | z(x) |]$$
$$= 0[| \exp(i\xi(x)) | \ | \exp(K p_1(x)q_1(x)) |]. \tag{32}$$

2.7 *Derivation of asymptotic expressions for* $\{\eta(x),\zeta(x)\} = \Omega(x)$

From (21), we have

$$\eta(x) = \exp(i\xi(x) [\tfrac{1}{2}(1+\exp(-2i\xi(x))) \eta(0) + \tfrac{1}{2}(1-\exp(-2i\xi(x))$$
$$(-\lambda^{-1}\eta^1(0) - T(0) - \eta(0) + \tfrac{1}{2} z(x) \exp(Kp_1(x)q_1(x))$$
$$\times \int_0^x z(x)^{-1} \exp(-Kp_1(x)q_1(x)) \exp(-i\xi(t))(1-\exp(-2i(\xi(x)-\xi(t)))$$
$$\times (\xi_1(t)\eta(t) + \eta_1(t)\zeta(t)) dt$$
$$-\tfrac{1}{2} \exp (Kp_1 \ q_1) \ z(x^{-1}) \int_0^x z(x^{-1}) \ \{1+\exp(-2i(\xi(x)-\xi(t)))\}$$
$$\times z(t)^{1/2} \ T(t) \ \eta(t) \exp(-K \ p_1(x)q_1(x) \exp(-i\xi(t)) \ dt]. \tag{33}$$

Now,

$$| z(x)^{-1}\exp(-Kp_1(x)q_1(x)) \exp(-i\xi(t)) \{1-\exp)-2i\xi(x)-\xi(t))\}$$

$$\times (\xi_1(t)\ \eta(t) + \eta_1(t)\ \zeta(t)) |$$

$$\leq z(t^{-1}) \exp(-Kp_1(t)q_1(t) \exp(-i\xi(t)) (| \xi_1(t) | \ | \eta(t) | +$$

$$+ | \eta_1(t) | \ | \zeta(t) |)$$

$$= | \exp (-Kp_1(t)q_1(t) | \ | \exp (-i\xi(t) |$$

$$\times\left(\frac{| \xi_1(t) |}{| z(t) |} | \eta(t) | + \frac{| \eta_1(t) |}{| z(t) |} | \zeta(t) | \right)$$

$$\leq K_1\left(\frac{| \xi_1(t) | + | \eta_1(t) |}{| z(t) |}\right), \text{ by (32) when } K_1 \text{ is a constant.} \tag{34}$$

Also, $| z(x)^{-1}\{1+\exp(-2i(\xi(x)-\xi(t))\} z(t)^{1/2} T(t)\eta(t)\exp(-Kp_1(x)q_1(x))$

$$\times \exp (-i\xi(t)) |$$

$$\leq | \{1+\exp(-2i(\xi(x)-\xi(t)))\}z(t)^{-1/2} | \ | T(t)\eta(t) | \ | \exp(-i\xi(t) |$$

$$\times | \exp(-Kp_1(t)q_1(t) |$$

$$\leq K_2 | T(t)z(t)^{1/2} | \text{ by (32), where } K_2 \text{ is a constant.} \tag{35}$$

But, $\int_0^\infty (| \xi_1(t) | + | \eta_1(t) |) \ | z(t)^{-1} | \, dt$, $\int_0^\infty | T(t)\ z(t)^{-1/2} | \, dt$ are convergent by Proposition 1.

Therefore, from (34) and (35), it follows that

$$L_1 = \lim_{x\to\infty} \int_0^x +z(x)^{-1} \exp(-Kp_1(x)q_1(x))\exp(-i\xi(t)) \times \{1+\exp(-2i(\xi(x)-\xi(t))\}$$

$$\times (\xi_1(t)\ \eta(t) + \eta_1(t)\zeta(t))\, dt, \text{ and}$$

$$L_2 = \lim_{x\to\infty} \int_0^x -z(x)^{-1} \exp(-Kp_1(x)q_1(x)) \exp(-i\xi(t))$$

$$\times\{1+\exp(-2i(\xi(x)-\xi(t)))\}z(t)^{1/2}T(t)\eta(t)\, dt$$

are finite numbers.

Hence, it follows from (33)

$$\eta(x) \sim \frac{L^*}{2} z(x) \exp(Kp_1(x)q_1(x) + i\xi(x)) \text{ where } L^* = L_1 + L_2. \tag{36}$$

Proceeding in a similar way, it follows that

$$\zeta(x) \sim S^*/2\ z(x)\ \exp(Kp_1(x)q_1(x)+i\xi(x)) \text{ where } S^* = S_1+S_2 \tag{37}$$

with $S_1 = \lim_{x\to\infty} \int_0^x z(x)^{-1} \exp(-Kp_1(x)q_1(x)-i\xi(t))\{1-\exp(-2i(\xi(x)-\xi(t))\}$

$\times (\eta_1\eta(t) + \zeta_1\zeta(t)\ dt$

and $S_2 = \lim_{x\to\infty} \int_0^x -z(x)^{-1}\exp(-Kp_1(x)q_1(x)-i\xi(t))\{1+\exp)-2i(\xi(x)-\xi(t))\}$

$\times z(t)^{1/2}\ S(t)\ \zeta(t)\ dt.$

We take

$$U_j(x) = \begin{pmatrix} U_{j1} \\ U_{j2} \end{pmatrix} = \{U_{j1},\ U_{j2}\},\ V_j(x) = \begin{pmatrix} V_{j1} \\ V_{j2} \end{pmatrix}$$

where $\{U_{j1},\ U_{j2}\} \equiv U_j(x) = iz(x)^{1/4}\{p_0x_j(x),\ q_0y_j(x)\}$

$$V_j(x) = iz(x)^{1/4}\{P_0u_j(x),\ q_0v_j(x)\}$$
$$= \{V_{j1},\ V_{j2}\},\ j = 1,2.$$

$\phi_k = \{u_k,\ v_k\}$, $k = 1,2$ are two solution vectors of (2) satisfying (2–4) and $\theta_k = \{x_k, y_k\}$ are also two solution vectors of (2) such that

$[\theta_1\theta_2] = 0$, $[\phi_j\theta_k] = \delta_{jk}(j,k = 1,2)$ and

$$[\phi_j\phi_k] = p_0 \begin{vmatrix} u_j & u_k \\ u_j' & u_k' \end{vmatrix} + q_0 \begin{vmatrix} v_j & v_k \\ v_j' & v_k' \end{vmatrix}$$

It follows from (36–37)

$$U_j(x,\lambda) \sim \frac{1}{2} z\ (x)\ \exp\ [K_1p_1(x)q_1(x)+i\xi(x)]\ L_j(\lambda)$$
$$V_j(x,\lambda) \sim \frac{1}{2} z(x)\ \exp\ [K_1p_1(x)q_1(x)+i\xi(x)]\ S_j\ (\lambda) \tag{38}$$

where $L_j(\lambda) \equiv \{L_{1j}(\lambda),\ L_{2j}\ (\lambda)\}$, $S_j(\lambda) \equiv \{S_{1j}(\lambda),\ S_{2j}(\lambda)\}$.

$$U_j(x,\lambda) \equiv \{U_{j1},\ V_{j2}\},\ V_j(x,\lambda) \equiv \{V_{j1},\ V_{j2}\};$$

and L_{ij} are independent of x; $U_j\ (x) = iz(x)^{1/4}\ \{p_0\ x_j(x),\ q_0y_j(x)\}$

and $V_j(x) = i\ z(x)^{1/4}\{p_0u_j(x),\ q_0v_j(x)\}$

8. Main theorem

Theorem. The spectrum of the differential system (2–4) is a countably infinite poi[nt] spectrum in $(-\infty,\infty)$ if all the conditions (i)–(vi) of Proposition 1 are satisfied by th[e] potentials p_0, q_0, p_1, q_1, $r(x)$.

Proof. We consider two solutions of (2–4), *viz.*,

$$\psi_j = \theta_j(x,\lambda) + \sum_{r=1}^{2} m_{jr}(\lambda)\ \phi_r(x,\lambda) \qquad (j = 1,2$$

$$= \begin{pmatrix} x_j + m_{j1}\, u_1 + m_{j2}\, u_2 \\ y_j + m_{j1}\, v_1 + m_{j2}\, v_2 \end{pmatrix}$$

$$= -iz(x)^{-1/4} \begin{pmatrix} p_0^{-1}\ U_{j1}(x) + m_{j1}p_0^{-1}\ V_{11} + m_{j2}p_0^{-1}\ V_{21} \\ q_0^{-1}\ U_{j2}(x) + m_{j1}\ q_0^{-1}\ V_{12} + m_{j2}q_0^{-1}V_{22} \end{pmatrix}$$

Therefore,

$$\psi_j \sim (-\frac{1}{2}i)\ z(x)^{3/4} \exp[K_1p_1(x)q_1(x)+i\xi(x)] \begin{pmatrix} p_0^{-1}[L_{1j} + m_{j1}S_{11} + m_{j2}S_{12}] \\ q_0^{-1}[L_{2j} + m_{j1}\ S_{21} + m_{j2}S_{22}] \end{pmatrix}$$

$$= +\frac{1}{2c}\ z(x)^{3/4} \exp[K_1p_1(x)q_1(x) + i\xi(x)]\ W_j(\lambda,\ p_0^{-1},\ q_0^{-1})$$

$$(j = 1,2). \qquad (39)$$

But for non-real values of λ, we have in the singular case $0 \leqslant x < \infty$ at least two linearly independent solutions of (2) say $W_r\ (x,\ \lambda)$ $(r = 1,2)$ such that

$$W_r(x,\lambda) \in L^2\ [0,\ \infty)\ \text{(Bhagat}^{1,3}\text{)}.$$

If ψ_1 be an L^2–solution of (2), we must have

$$p_0^{-1}(L_{11} + m_{11}\, s_{11} + m_{12}\, S_{12}) = 0$$

$$q_0^{-1}\ (L_{21} + m_{11}\ S_{21} + m_{12}\, S_{22}) = 0$$

that is,

$$\begin{aligned} L_{11} + m_{11}\ S_{11} + m_{12}\ S_{12} &= 0 \\ L_{21} + m_{11}\ S_{21} + m_{12}\ S_{22} &= 0 \end{aligned} \qquad (40)$$

since $1/2i\ \exp\ [i\xi(x) + K_1p_1(x)q_1(x)]\ z(x)^{3/4} \to \infty$ as $x \to \infty$.

Similarly, if ψ_2 be an L^2-solution of (2) we have

$$\begin{aligned} L_{12} + m_{21}\ S_{11} + m_{22}\ S_{12} &= 0 \\ L_{22} + m_{21}\ S_{21} + m_{22}\ S_{22} &= 0. \end{aligned} \qquad (41)$$

It follows from (40–41) that

$$m_{rs}(\lambda) = \frac{U_{rs}(\lambda)}{D(\lambda)} \text{ where } U_{rs}(\lambda),\ D(\lambda)$$

are functions of λ_1 expressible in terms of L_{rs}, $S_{rs}(\lambda)$. It follows by argument similar to Paladhi[2] pp. 457–458) that $U_{rs}(\lambda)$, $D(\lambda)$ are integral functions of λ. Therefore, $m_{rs}(\lambda)$ are meromorphic functions of λ over $(-\infty, \infty)$.

The spectrum of the system (2–4) is, therefore, a countably infinite point-spectrum in the interval $(-\infty, \infty)$.

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