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REVIEWS

Stochastic Methods for the Analysis of Uncertain Composites

B.N. Singh and Neeraj Grover*

Abstract | The involvement of various processes/parameters in the manufacturing and fabrication of laminated composites and the lack of control over these constituent processes cause the uncertainty in the system parameters. Therefore, the consideration of these random parameters in the analysis of composite structures is essential. The objective of the present study is to present state-of-the-art on the stochastic modeling of composite structures. The various techniques available for the uncertainty characterization and their propagation in the deterministic solver are discussed. The significance of various approaches is presented and their applicability to quantify various uncertainties is concluded. The essential requirements to accurately model the multilayered structures are discussed by presenting the classification of available theories. The incorporation of the uncertainty quantification techniques with the deterministic structural mechanics solvers leads to the stochastic techniques such as stochastic finite element analysis. The wide range of applicability and the accuracy of finite element analysis in deterministic environment are both—an added advantage and a priori requirement of an accurate stochastic solver. The finite element method in the stochastic framework is presented to determine the governing equations for the structural behavior of laminated composite and sandwich plates. The stochastic finite element method is then presented in the context of Monte Carlo simulation and perturbation technique. Further, the influence of various uncertain parameters such as material properties, loading conditions etc. on the structural behavior of laminated composite structures is discussed by presenting the observations made in the available literature on such analyses. It is hoped that the present work could facilitate the research community involved in stochastic or deterministic analysis of laminated composites.

1 Introduction

Over the past few decades, the use of new structural materials such as composite and sandwich construction for the design of structures in aerospace, automotive, civil and other applications has improved the performance and reliability of structural system due to their mechanical advantage of specific modulus and specific strength over monolithic materials, improved fatigue and impact resistance, and design flexibility. The panels form an essential part of many structures. Laminated composite panels are made of orthotropic layers that are bonded together while in sandwich construction, the inner layers are replaced by relatively low strength thicker layers called core.

Conventionally, a structural analysis is accomplished assuming deterministic behavior of the loads, material properties and other system

Department of Aerospace Engineering, Indian Institute of Technology, Kharagpur 721 302, West Bengal, India.
**bnsingh@aero.iitkgp.ernet.in*

parameters. However, the real structural systems employed in aerospace, naval, civil, and mechanical applications are characterized by inherent uncertainties in the definition of structural parameters. The major sources of such uncertainties in real structural problems can be due to randomness in material properties, loading conditions, geometric properties etc. As an inevitable consequence of the uncertainties in these system parameters, the response of structural system will always exhibit some degree of uncertainty. The conventional deterministic analysis based on an exact reliable model would not help in proper accounting of variation in the response and therefore, the analysis based on deterministic material properties may vary significantly from the real behaviour.[1] Incorporation of these uncertainties into the analysis and design enables the prediction of the performance variation in the presence of uncertainties and more importantly their sensitivity for targeted testing and quality control. In order to provide useful and accurate information about the safe and reliable design of the structures, it is essential to incorporate these uncertainties into account for modeling and analysis procedure. The steady development of powerful computational technologies in recent years has led to high-resolution numerical models of real-life engineering structural systems. It is also required to quantify uncertainties and robustness associated with a numerical model. As a result, the quantification of uncertainties plays a key role in establishing the credibility of a numerical model. Therefore, the development of an efficient mathematical model possessing the capability to quantify the uncertainties present in the structures is extremely essential in order to accurately assess the laminated composite and sandwich structures.

In addition to quantify the uncertainties, an accurate and reliable deterministic solver for structural behavior is mandatory.[2] The laminated composite and sandwich panels differ significantly from the monolithic material since these are multi-phase, heterogeneous materials. The complicating effects[3] of these layered structures must be properly taken care prior to the analysis. The approximations and assumptions are always associated with the mathematical model which governs the physical system. The developments in computational technologies have enabled the implementation of numerical methods to solve the governing mathematical model for large scale structural problems. A precise mathematical representation of the physical system and a compendious solution technique ensure the accuracy of the deterministic solver.

The available literature provides a wide range of methodologies to model the laminated composite and sandwich plates along with numerous solution techniques. The present study aims to highlight the significant aspects of stochastic modeling of uncertain structures. It explores the theories that have been proposed and developed for laminated composite and sandwich panels. The purpose of the survey paper herein is to provide a review of the current state-of the-art for various methods for the stochastic analysis of composite structures. The various techniques to model the uncertainties present in the system are discussed. The consequences of various uncertain parameters involved in the composite structures are discussed and their impact on the structural behavior of composite structures is studied. The present paper deals with the developments related to stochastic analysis of composites in the following sequence: uncertainty characterization in composites, modeling of the composites, deterministic structural analysis, stochastic finite element analysis, and the influence of uncertain parameters on response characteristics of composite structures. It is felt that the present work will be of interest to the research community already involved in the stochastic analysis of composite structures.

2 Uncertainty Characterization in Composites

The laminated composite and sandwich structures are characterized by inherent uncertainties and variability (randomness) in the structural parameters. The response characteristics of these structures are strongly influenced by these uncertainties. It is, therefore, desired to identify the causes of uncertainties, their classification and hence seek for appropriate approach to model them.

2.1 *Causes of uncertainties*

The manufacturing technique of the composites involves complex processes and due to lack of control over these constituent processes, considerable amount of uncertainties in the material properties and other parameters arise. In the performance of composite structures, uncertainty can be addressed by means of material, geometric and structural considerations.[4–7] Material and geometric uncertainty mainly arise due to lack of control over the manufacturing and fabrication techniques. The manufacturing of the composites is strongly influenced by the volume fractions of its constituents i.e. fiber and matrix, the amount of resins in the plies, bonding between fiber and matrix, curing technique, proper alignment of fibers, voids in the matrix etc.[8–10] The incomplete control over these factors causes the variability in the material properties e.g. young's modulus, shear modulus, Poisson's ratio, coefficient of thermal expansion etc. The uncertainties in these parameters propagate to a higher level and enhance the randomness in strength and stiffness thereby affecting the overall performance. In addition to these, the manufacturing processes such as thermal treatment, filament winding, braiding, etc. severely influence the randomness in material properties. These uncertainties should be quantified for the design and analysis of composite structures. Many researchers[11–14] have considered random material properties and analyzed their effect on the performance of composites. In the assembly process, geometric and structural uncertainties are experienced due to different joint types, joining and machining techniques. The interaction between tooling and the composite lay-up processes contributes towards the geometric uncertainty.[8,15] The variability in the loading conditions, boundary conditions, operating temperature conditions, and the environmental conditions further contribute towards the uncertainty which must be taken care for modeling and analysis of laminated composite and sandwich structures.

2.2 *Classification of uncertainty*

Uncertainties associated with geometric properties, material properties, boundary conditions, loading conditions etc widely exist in practical engineering problems. Moreover, uncertainty occurs during different phases of modeling and analysis such as conceptual modeling of the physical system, mathematical modeling of the physical system, discretization of the mathematical model, computer programming of the discrete model, numerical solution of the computer program and representation of the numerical solution.[16] Therefore, the quantification, propagation and management of the concerned uncertainty are essential for the product design and analysis.

There are two different types of uncertainties (see Fig. 1.). These are: (a) aleatory uncertainty (also known as objective uncertainty, stochastic uncertainty, irreducible uncertainty, inherent uncertainty) and (b) epistemic uncertainty (also known as subjective uncertainty, reducible uncertainty, modal form uncertainty).[16,17]

Aleatory uncertainties arise due to random nature of input parameters and describe the inherent variation of the physical system. The aleatory uncertainties are represented as randomly distributed quantities and their sources can usually be separated from the other sources of uncertainties in the system. The availability of sufficient experimental data or evidences enables the representation of aleatory uncertainties in terms of probability distribution. On the other hand, epistemic uncertainty arises due to incomplete information of the processes, lack of experimental data and ignorance of the characteristics of the system. There is one more class of uncertainty called 'numerical uncertainty' or error; however some researchers consider it different from uncertainty.[16] In the framework of composite materials, the uncertainties in the fiber and matrix characteristics, manufacturing variations etc. correspond to the aleatory uncertainties e.g. the spatial variation in material properties, loading conditions etc. The randomness associated with the experimental and modeling methods contributes to the epistemic uncertainty e.g. randomness due to choice of shear deformation.[18]

2.3 *Quantification of structural uncertainties*

The quantification of structural response uncertainties can be achieved by three basic methodologies. These are: Monte Carlo Simulation (MCS),[19,20] Perturbation Techniques (PT),[7,21–25] and expansion methods.[26,27]

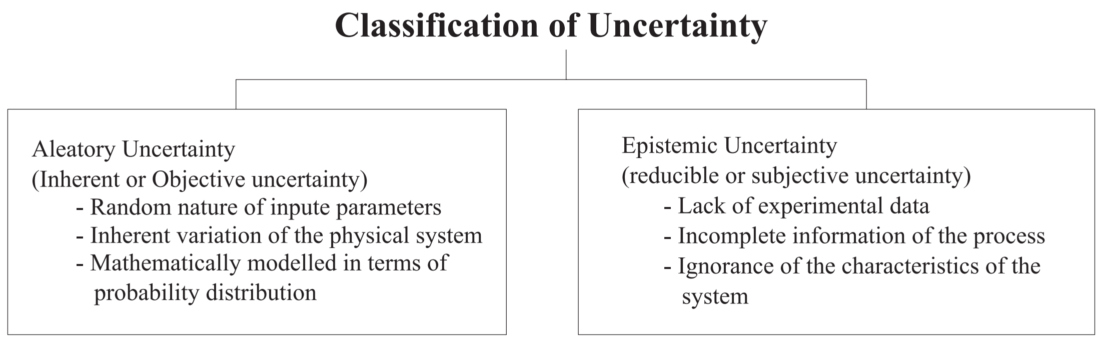

Figure 1: Classification of uncertainty.

2.3.1 Monte Carlo Simulation: In this process, the probability of certain outcome is approximated by multiple trial simulations using random variables. It is a numerical method to evaluate approximate solutions of physical and engineering problems by means of digital simulation and statistical analysis of random variables. In MCS, a set of random samples is generated to represent the variability present in the structures. These random samples are then substituted in the deterministic solver to obtain the response of the sample structures. It has the capability to provide an accurate solution for random variables. Various researchers have implemented MCS to investigate the random behavior of structures. However, as the number of degree of freedom and the number of random parameters starts increasing, the MCS requires extremely high amount of time and for large complicated problems, the computational efforts may make the method impractical. For example, it requires 12, 000 samples to evaluate the stochastic deflection statistics of soft core sandwich plate using MCS.[28] That is why this method is usually adopted for validation purpose.

2.3.2 Perturbation Techniques: The perturbation techniques are based on the formulation of analytical expression of random parameters around their mean values. This method utilizes Taylor's series expansion to formulate linear relationship between statistical characteristics of the response and the random structural parameters. Depending upon the order of derivative chosen, the perturbation techniques can be categorized as: First Order Perturbation Technique (FOPT), Second Order Perturbation Technique (SOPT) and Higher Order Perturbation Technique (HOPT). It should be noted that FOPT is significant for very small dispersion of input random variable (coefficient of variation less than 0.1). The advantage of the PT lies in the fact that higher-order moments and coefficients deciding the distributions of the structural response parameter can be obtained. Moreover, sensitivity analysis can be performed to examine the dominancy of the input random variables on the output response parameters. The perturbation techniques are very convenient with the variations in the system parameters. Many researchers[29–32] have implemented perturbation techniques to examine the structural responses in stochastic environment.

2.3.3 Expansion methods: In these methods, the responses of structural variability are expanded in terms of a suitable expression (polynomial or expansion) in the probability space.[25] The orthogonal polynomial expansion method (OPEM)[26,27] utilizes the orthogonal polynomials such as Hermite, Legendre, or Gegenbauer polynomials to represent the uncertainties.[25] Neumann expansion and polynomial chaos expansion[23] have also been considered for the structural analysis in stochastic environment. In polynomial chaos methods, the random output parameters are approximated by means of functions of random input parameters. These functions are orthogonal stochastic polynomial. Polynomial chaos method has been frequently used for the structural analysis because of its simplicity and ease of implementation with deterministic numerical methods such as finite element method.

The methods of uncertainty quantification can also be classified on the basis of Gaussian and non-Gaussian probability distribution.[33] The spectral representation method[34–35] and Karhunen-Loeve (K-L) expansion[36] are quite popular for the simulation of Gaussian stochastic process i.e. when the uncertain quantities associated with the structural system are Gaussian in nature. Moreover, it should be noted that above mentioned quantification techniques are generally applicable for aleatory uncertainties. Various quantification techniques such as convex models,[37] possibility theory,[38–39] Dempster-Shafer theory (evidence theory),[16,40] fuzzy sets,[41] probability box (p-box) method[42] etc. have been implemented to mathematically represent the epistemic uncertainties. In the convex models, all the possible values of uncertainties are bounded within a convex set without any assumption of inner distribution. Due to lack of sufficient samples, the distribution of random parameters is not always available and therefore this approach is significant in such instances.[37] The possibility theory utilizes the membership functions for the epistemic uncertainties to deal with the randomness involved in the system parameters. The intrinsic nature of the random variables is retained in the possibilistic analysis. Dempster et al.[43] proposed the evidence theory to deal with epistemic uncertainties which was further refined by Shafer.[44] In the framework of evidence theory, the belief and plausibility measures are implemented to quantify the random parameters. The possibility theory based on the theory of fuzzy sets can be an alternative method to consider the uncertainties. The fuzzy numbers are employed in the possibility theory by using interval representation.[41] In the probability box or the probability bound methods, the epistemic uncertainties are represented by a structure called p-box.[42] It is worth to mention that in the framework of composite structures, aleatory uncertainties due to variability in young's modulus, shear modulus, Poisson's ratio, loading conditions etc. are of significant interest and MCS and PT are popularly implemented to investigate the variability statistics of the stochastic structural system.[7,23,24,28–32,45]

3 Modeling of the Composites

In order to ensure the efficient and reliable usage of the layered structures, an authentic and compendious technique for the modeling and analysis of these structures is mandatory. The three-dimensional (3D) approaches presented by Pagano and Hatfield,[46] Srinivas and Rao,[47] Noor[48] and Demasi[49] yield the exact response; however, their range of applicability is limited to a particular class of configurations. In order to ensure the applicability to general plate configurations, the researchers have focused on the development of accurate two-dimensional (2D) approaches. The classification of the plate theories has been depicted in Fig. 2.

Since the thickness of the panel structures is less as compared to other two dimensions, it facilitates to reduce the 3D problem to a 2D problem.[3] However, various critical points must be taken care of in the modeling and analysis of these structures via 2D approaches. The significance of the shear deformation had been studied by various authors. Notably among them are due to Mindlin,[50] Yang et al.,[51] Whitney,[52] Whitney and Pagano,[53] and Reissner.[54,55] However, in these studies, First order Shear Deformation Theory (FSDT) was employed which leads to linear transverse shear strain and hence the corresponding stresses. Therefore, a shear correction factor is required in order to ensure the traction free top and bottom surfaces. However, the choice of shear correction factor depends upon the lamination sequence, loading conditions etc. which makes the FSDT less reliable for the analysis of layered structures.[56] There are various Higher order Shear Deformation Theories (HSDTs) which possess the non-linear shear deformation and in addition they satisfy the traction free boundary conditions on top and bottom surfaces of the plate. The development of HSDTs can be observed in terms of Polynomial Shear Deformation Theories (PSDTs) and Non-Polynomial Shear Deformation Theories (NPSDTs). The influence of shear deformation is expressed by means of Taylor's series coefficients in case of PSDTs. The significant contributions towards the development of PSDTs are due to Lo et al.,[57] Levinson,[58] Reddy,[59] Pandya and Kant,[60] Kant et al.,[61] Khdeir and Reddy,[62] Kant and Swaminathan,[63] Swaminathan and Patil,[64] Ferreira et al.,[65,66] and Talha and Singh.[67] On the contrary, a shear strain function (function of thickness co-ordinate) is employed to express the shear deformation in NPSDTs. The various shear strain functions such as trigonometric (Stein,[68] Touratier,[69] Soldatos,[70] Mantari et al.,[71] Grover et al.[72]), exponential (Karama et al.,[73] Aydogdu,[74] Mantari et al.[75]), hyperbolic (Meiche et al.[76]) and inverse hyperbolic (Grover et al.[77,78]) have been implemented for the realistic consideration of shear deformation. The FSDT and HSDT are Equivalent Single Layer (ESL) theories since the layered structures are modeled as a single plate. In addition to shear deformation and traction free conditions, a few researchers have also focused on the Interlaminar Continuity (IC) of the layered structures. These theories are classified as Layerwise (LW) and Zigzag (ZZ) theories.[3] Some of the significant contributions towards LW and ZZ theories are due to Murakami,[79] Di Sciuva,[80] Lee and Liu,[81] Cho and Parmerter,[82] Carrera,[83] Ferreira,[84] Roque et al.,[85] Kukarni and Kapuria,[86] Pandit et al.,[87,88] Rodrigues et al.,[89] Ferreira et al.,[90] Neves et al.,[91] Mantari et al.,[92] Thai et al.[93] and Sahoo and Singh.[94] The LW and ZZ theories are accurate for the analysis of laminated composite and sandwich plates, however, the involvement of variables and hence the computational efforts are quite large as compared to ESL theories. The significant literature reviews on the modeling of plate structures have been presented by Reissner,[95] Noor and Burton,[96] Kapania and Reciti,[97,98] Reddy,[99] Mallikarjuna and Kant,[100] Liu and Li,[101] Kant and Swaminathan,[102] Carrera,[3,103] Zhang and Yang,[104] Carrera and Brischetto,[105] Khandan et al.,[106] and Jha et al.[107]

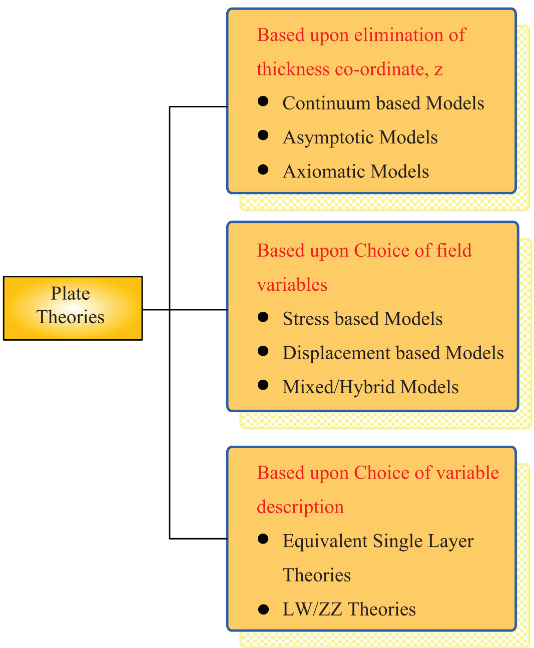

Figure 2: Classification of plate theories.

Along with the development of plate theories, there have been significant developments towards the solution methodologies. Analytical solutions are most accurate and different analytical techniques have been proposed for structural responses of laminated composite plates; however, they are applicable for a particular class of plate configuration and hence less appropriate for practical applications. The development of computational technologies in the present industry makes it quite possible to implement numerical/approximate methods for the practical applications. In the numerical investigations, Ritz methods,[108] finite strip method,[109,110] Discrete Singular Convolution (DSC) method,[111] Finite Element Method (FEM),[112,113] and mesh-less method[65,66,84,85] have been widely used for the structural responses. Among these, FEM is most popular and versatile method for investigating the structural behavior of arbitrary shaped components.

4 Deterministic Structural Analyses

In view of the discussions made in section 3, FEM is the favorable choice as the deterministic solver for the structural analysis for laminated composite and sandwich structures. Here, in this section, a brief overview of the finite element approach for the analysis of laminated composite and sandwich plates in the deterministic environment is presented. The general procedure of finite element implementation for laminated composite and sandwich plates is also outlined in Fig. 3. One may refer to the works related to FEM[114,115] for the detailed description. In the framework of FEM, the domain is decretized into small elements and the various energies such as strain energy, kinetic energy, work due to external load etc are calculated first at the elemental level in terms of elemental matrices such as element stiffness, element mass, element geometric stiffness etc. These elemental matrices are then assembled together by performing adequate assembly procedure so as to retain the contribution of stiffness coefficients to the corresponding degree of freedom.

The governing equation of motion for the deterministic behavior is obtained by implementing the Lagrange equations. The Lagrange equation for a conservative system is as:

$$\frac{d}{dt}\left(\frac{\partial U_l}{\partial\{\dot{q}_i\}}\right)+\frac{\partial U_l}{\partial\{q_i\}}+\frac{\partial U_{nl}}{\partial\{q_i\}}+\frac{\partial W}{\partial\{q_i\}}=0 \tag{1}$$

here, T is the kinetic energy, U_l is the strain energy due to linear strains contributing towards stiffness matirix, U_{nl} is the strain energy due to non-linear strains contributing towards geometric stiffness, W is the work done due to external load and q is the vector containing nodal field variables. The length of vector q depends upon the choice of plate theory and the element which is used for discretization. In addition to these energies, Eq. (1) may contain the energy contribution due to additional quantities such as penalty parameter, elastic foundation etc. depending upon the analysis to be performed. The expression for strain energies, kinetic energy and work done are as:

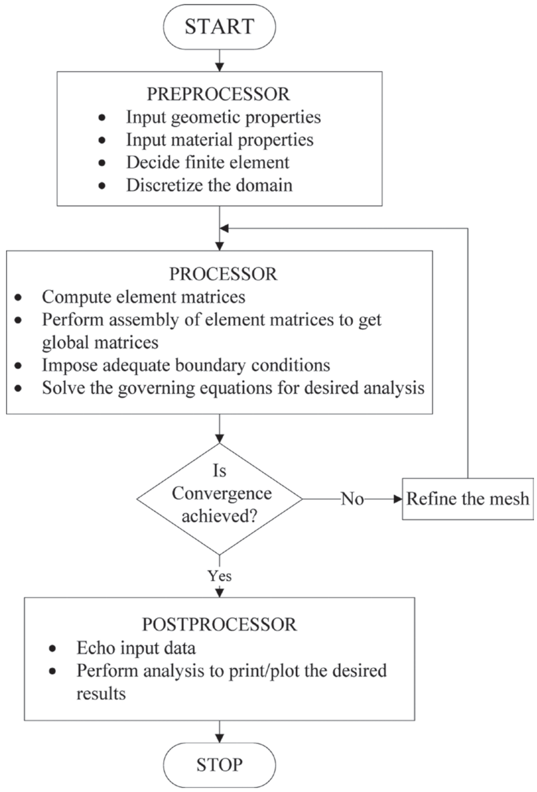

Figure 3: Flow chart of the finite element analysis.

$$U_l=\sum_{j=1}^{nel}U_l^{(j)}=\sum_{j=1}^{nel}\frac{1}{2}\{q\}_j^T[K]_j\{q\}_j =\frac{1}{2}\{q\}^T[K]\{q\} \tag{2}$$

$$U_{nl}=\sum_{j=1}^{nel}U_{nl}^{(j)}=\sum_{j=1}^{nel}\frac{1}{2}\{q\}_j^T[K_G]_j\{q\}_j =\frac{1}{2}\{q\}^T[K_G]\{q\} \tag{3}$$

$$T=\sum_{j=1}^{nel}\frac{1}{2}\{\dot{q}\}_j^T[M]_j\{\dot{q}\}_j=\frac{1}{2}\{\dot{q}\}^T[M]\{\dot{q}\} \tag{4}$$

$$W = \sum_{j=1}^{nel} W_j^{(e)} = \sum_{j=1}^{nel} \frac{1}{2} \int_{A_j^{(e)}} p(x,y) w \cdot dA_j^{(e)}$$
$$= \sum_{j=1}^{nel} \frac{1}{2} p \left\{ f^{(e)} \right\}_j \{q\}_j = \frac{p}{2} \{F\}\{q\} \quad (5)$$

Here, $[K]_j$, $[K_G]_j$, $[M]_j$ and $\{f^{(e)}\}_j$ are the elastic stiffness, geometric stiffness, mass matrices and nodal force vector respectively for the jth element and $[K]$, $[K_G]$, $[M]$ and $[F]$ are the global stiffness matrices, global mass matrix and global load vector obtained by the assembly of elemental matrices or the vectors over the number of elements (nel).

Substitution of the energy expressions in the Lagrange equation given by Eq. (1) yields the governing algebraic equation which is as follows:

$$[M]\{\ddot{q}\} + [K - \lambda K_G]\{q\} = \{F\} \quad (6)$$

The above set of equations can be solved to get the desired response e.g. static, buckling and free vibration response in deterministic environment.

5 Stochastic Finite Element Analyses

The deterministic analysis based on Eq. (6) is based on the assumption that the system properties are represented by their constant or mean values over the entire domain of the structure. However, the uncertainties involved with various parameters, as discussed in section 2, necessitates the consideration of randomness in the system parameters such as material properties, loading conditions etc. The analysis of a structural system with uncertain parameters requires a stochastic approach and it is well versed in stochastic analysis that if the structural parameters become uncertain, the matrices and vectors involved in the Eq. (6) will be random in nature. For example, the stiffness matrix for jth element will take the following form:

$$[K]_j = [K_0]_j + [\Delta K]_j \quad (7)$$

where $[K_0]_j$ is the deterministic part of the stiffness matrix dependent on the mean values of material properties and $[\Delta K]_j$ is the fluctuating part of the stochastic finite element stiffness matrix.[33] The global stiffness matrix of the stochastic system is then obtained by assembly of stochastic finite element stiffness matrices.

$$[K] = \sum_{j=1}^{nel} [K]_j = [K_0] + [\Delta K] \quad (8)$$

The global mass matrix for the stochastic dynamic system is also obtained following the similar procedures and is given as:

$$[M] = \sum_{j=1}^{nel} [M]_j = [M_0] + [\Delta M] \quad (9)$$

where $[M_0]_j$ is the deterministic part of the mass matrix dependent on the mean values and $[\Delta M]_j$ is the fluctuating part of the stochastic finite element mass matrix. The global mass matrix of the stochastic system is then obtained by assembly of stochastic finite element mass matrices. Following the similar procedures, the geometric stiffness matrix and load vector can be represented in the stochastic domain as follows:

$$[K_G] = \sum_{j=1}^{nel} [K_G]_j = [K_{G0}] + [\Delta K_G] \quad (10)$$

$$\{F\} = \sum_{j=1}^{nel} \{f\}_j = \{F_0\} + \{\Delta F\} \quad (11)$$

The governing system of equations is then obtained by substituting Eq. (7)–(11) in Eq. (6). Thus, in the context of stochastic analysis, the equations for SFEA are as:

$$[M_0 + \Delta M]\{\ddot{q}\} + \left[[K_0 + \Delta K] - \lambda [K_{G0} + \Delta K_G] \right]\{q\}$$
$$= \{F_0\} + \{\Delta F\} \quad (12)$$

It should be noted that the above equation should be modified depending upon the nature of the variability involved with the system parameters. In order to solve the stochastic equations three main approaches are used in the framework of SFEA.[33] These are: (a) MCS (b) Perturbation techniques (c) Spectral Stochastic Finite Element Analysis (SSFEA).

5.1 *Stochastic Finite Element Analysis with Monte Carlo Simulation*

It is the simplest method to treat the uncertainties in the framework of SFEA. In this method, a large number of random samples of the uncertain parameters are generated according to the known probability distribution. The governing equation (Eq. (12)) is solved for each sample to get the response of individual samples. The statistical estimation (mean and variance) of the output random variables is then obtained considering the response of all random samples. For example, if there are n samples for the static analysis as represented in Eq. (12), the mean and variance of output random variable i.e. displacement are obtained as:

$$E(q)=\frac{1}{n}\sum_{k=1}^{n}q(k) \tag{13}$$

$$\sigma^2=\frac{1}{n-1}\sum_{k=1}^{n}q^2(k)-n\cdot E^2(q) \tag{14}$$

It is clear that number of samples strongly influence the statistical response characteristics of the output random variables. It requires quite large number of samples to accurately estimate the stochastic response of a structural system associated with large scale uncertainties. However, this method is quite useful and frequently used for the validation purposes.

5.2 *Stochastic Finite Element Analysis with perturbation techniques*

In the context of perturbation techniques, R random field variables are assumed such that

$$d(x,y)=\left\{d_1(x,y)\ d_2(x,y)\ d_3(x,y)\ \cdots\ d_R(x,y)\right\} \tag{15}$$

where, $d_r(x,y)$ represent the random field variables e.g. Young's modulus, mass density, Poisson's ratio etc. Using the finite element implementations, the random field variables (d) may be expressed in terms of random nodal field variables (d_ρ) and shape functions ($Y_{r\rho}$). For a structure with random parameters, the individual terms of Eq. (6) will be random in nature and thus these terms can be represented in terms of random variables as:

$$\begin{aligned}[K]&=[K](d_\rho)\\ [K_G]&=[K_G](d_\rho)\\ [M]&=[M](d_\rho)\\ \{q\}&=\{q\}(d_\rho)\\ \{F\}&=\{F\}(d_\rho)\end{aligned} \tag{16}$$

These random variables are expanded about their spatial expectations with the help of Taylor's expansion with a given small parameter ξ. The involvement of higher order terms increases the computationally complexity and it is believed that the first order terms are sufficient for the accurate response and the improvement in accuracy by considering second order and higher order terms is very small as compared to the computational efforts.[116] The random variables are expanded in terms of Taylor's expansion by omitting the third and higher order terms and consequently the matrices and vectors in the Eq. (6) will be random in nature up to second order. Thus, in Second Order Perturbation Technique (SOPT), the terms of Eq. (6) are expressed as:[23]

$$\begin{aligned}[K]&=[K_0]+\xi\left[K'^{\rho}\right]\Delta d_\rho+\frac{1}{2}\xi^2\left[K'^{\rho\sigma}\right]\Delta d_\rho\Delta d_\sigma\\ [K_G]&=[K_{G0}]+\xi\left[K_G'^{\rho}\right]\Delta d_\rho+\frac{1}{2}\xi^2\left[K_G'^{\rho\sigma}\right]\Delta d_\rho\Delta d_\sigma\\ [M]&=[M_0]+\xi\left[M'^{\rho}\right]\Delta d_\rho+\frac{1}{2}\xi^2\left[M'^{\rho\sigma}\right]\Delta d_\rho\Delta d_\sigma\\ \{F\}&=\{F_0\}+\xi\left\{F'^{\rho}\right\}\Delta d_\rho+\frac{1}{2}\xi^2\left\{F'^{\rho\sigma}\right\}\Delta d_\rho\Delta d_\sigma\\ \{q\}&=\{q_0\}+\xi\left\{q'^{\rho}\right\}\Delta d_\rho+\frac{1}{2}\xi^2\left\{q'^{\rho\sigma}\right\}\Delta d_\rho\Delta d_\sigma\end{aligned} \tag{17}$$

where, $\xi\Delta d_\rho=\delta d_\rho=\xi[d_\rho-d_\rho^0]$ is the first order variation of d_ρ about d_ρ^0. The symbol $(.)_0$ represents the value of functions taken at d_ρ^0. While $(.)'^{\rho}$ stands for the first partial derivative with respect to nodal random field variables d_ρ evaluated at their expectations, d_ρ^0. Substituting Eq. (17) in the governing equations and equating terms of like powers of small parameter ξ, the zeroth order, first order and second order equations for the stochastic structural problems are obtained as follows:[23]

- Zeroth order:

$$[M_0]\{\ddot{q}_0\}+[K_0-\lambda K_G]\{q_0\}=\{F_0\} \tag{18}$$

- First order:

$$[M_0]\left\{\ddot{q}'^{\rho}\right\}+\left[M'^{\rho}\right]\{\ddot{q}_0\}+[K_0-\lambda K_G]\left\{q'^{\rho}\right\}+\left[K'^{\rho}-\lambda K_G'^{\rho}\right]\{q_0\}=\left\{F'^{\rho}\right\} \tag{19}$$

- Second order:

$$[M_0]\left\{\ddot{q}'^{\rho\sigma}\right\}+2\left[M'^{\rho}\right]\left\{\ddot{q}'^{\sigma}\right\}+\left[M'^{\rho\sigma}\right]\{\ddot{q}_0\}+[K_0-\lambda K_G]\left\{q'^{\rho\sigma}\right\}+2\left[K'^{\rho}-\lambda K_G'^{\rho}\right]\left\{q'^{\sigma}\right\}+\left[K'^{\rho\sigma}-\lambda K_G'^{\rho\sigma}\right]\{q_0\}=\left\{F'^{\rho\sigma}\right\} \tag{20}$$

The zeroth order equation given by Eq. (18) is a deterministic equation which can be solved by any conventional solution technique and gives the mean response of the structure. On the other hand, first order and second order equations represents the random counterpart and solution to these provides the statistics of the structural response.

5.3 *Other methods*

In addition of the above, Spectral Stochastic Finite Element Method (SSFEM) has gained popularity for the structural analysis in random environment. Ghanem and Spanos[36] introduced SSFEM by extending the deterministic finite elements for the solution of boundary value problems with random material properties. He considered the random spatial variation of Young's modulus and described its randomness by Gaussian stochastic field using K-L expansion. The computational aspects of SSFEM are discussed by Stefanou.[33]

Although there have been various methods proposed for the stochastic analysis of structures, the availability of commercial software which is capable of performing the stochastic analysis is very less. Some examples of SFEA software are COSSAN, NESSUS, and FERUM. The probability design toolbox in ANSYS enhances its capability to uncertainty modeling of the random parameters in structural systems.[33]

6 Influence of Uncertain Parameters on the Response Characteristics of Laminated Composite and Sandwich Plates

6.1 *Free vibration behavior in stochastic environment*

Singh et al.[117] studied the influence of random material properties on natural frequencies of simply supported laminated composite plate. First order perturbation technique was implemented to obtain the random behavior. A higher order shear deformation theory including the effect of rotary inertia was employed to model the laminated composite plates and a closed form solution in the deterministic and stochastic environment was obtained. Further, Singh et al.[118] implemented a C^0 finite element to investigate the influence of boundary conditions. It was observed that the dispersions in the square of the natural frequency (w^2) show a linear variation with Standard Deviation (SD) of the material properties.[117] Moreover, the influence of SD of natural frequencies shows different sensitivity to different material properties. The sensitivity changes with the laminate construction, mode of vibration, a/h ratios and the material.[117] The increase in thickness of the plate results in reduction in the frequency scatters.[117] The results were also compared with the classical plate theory and it was concluded that the Classical Laminate Theory (CLT) over predicts the scatter in the fundamental frequency. The use of classical laminate theory may not be appropriate for study of scatter in the fundamental frequency even for the thin plates.[117] The influence of uncertainties in various material properties on the scatter of fundamental frequency was examined and the investigation showed that the scatter in the fundamental frequency increases with the individual variations of E_{11}, E_{22}, G_{12} and ν_{12} as the side to thickness ratio increases, while the scatter decreases with G_{13} and G_{23}.[118] The numerical experiments of the effects of uncertain parameters along with the change in boundary constraints indicate that a panel with all its edges clamped ($a/h = 5$) is most sensitive for the changes in fundamental frequency with simultaneous changes in the material properties, while a simply supported panel ($a/h = 5$) is least sensitive.[118] The dominance of material properties on the scatter in the fundamental frequency changes from E_{11} to G_{13} as a/h ratio changes from 10 to 5 and the composite panels are seen to be least affected with changes in the Poisson's ratio.[118]

Singh et al.[119,120] examined the free vibration behavior of spherical and cylindrical panels considering random material properties. Tripathi et al.[32] investigated the sensitivities of randomness in material properties on the free vibration frequency of laminated composite conical shells. The composite material properties were modeled as random variables while the panels are modeled using higher order shear deformation. First order perturbation technique was implemented to investigate the statistics of panels in terms of mean and variance of random frequency parameters and MCS results were obtained for the validation purpose. It was observed that the SD of the square of the natural frequency of spherical panel changes linearly with SD of the material properties.[119] Moreover, the fundamental frequency is most affected by simultaneous changes in SD of the material properties as compared to subsequent four natural frequencies for simply supported laminates.[119] The dominance of various material properties along with different boundary constraints was examined and it was observed that the effect of E_{11} is most dominant on dispersion in the natural frequencies and effect of ν_{12} is least dominant for a spherical shell with simply supported conditions. Furthermore, the fundamental frequency is most sensitive to the changes in E_{11}

as compared to all other frequencies.[119] The study on the effect of boundary conditions indicates that the panels with SFSS (one edge free while the other edges are simply supported) and SFSC (one pair of opposite edges is simply supported while the other pair has a clamped edge and a free edge) conditions are most sensitive while a simply supported spherical panel is least sensitive to simultaneous changes in the material properties.[119] It is worth to note that the effect of dispersion in longitudinal elastic modulus, E_{11} on the scatter in the fundamental frequency is most important for all support conditions considered while the effect of v_{12} is least important.[119] With regard to the analysis of cylindrical panels, it was observed that a clamped cylindrical panel ($a/h = 10$) is most sensitive, while a simply supported cylindrical panel ($a/h = 5$) is least sensitive against simultaneous change of material properties.[120] The influence of span-thickness ratio (a/h) on the dispersion in the natural frequency was investigated and it was observed that the sensitivity of the dispersion in natural frequency increases as a/h ratio increases with variation of SD of in-plane elastic and shear modulii and Poisson's ratio. The trend is reversed for out-of-plane shear modulii.[120] Moreover, the dominant effect of G_{13} on the scatter in fundamental frequency of cylindrical panels changes to E_{11} when a/h ratio changes from 5 to 10. However, the study on the simply supported cylindrical panels indicates that E_{11} is the most dominant property for both the thickness ratios.[120] Also, the panels showed very low sensitivity to changes in Poisson's ratio.[120]

The free vibration behaviors of sandwich plates with random material properties were investigated by Pandit et al.[121,122] The sandwich plate was modeled using higher order zig-zag theory taking into account the effect of core compressibility. It was observed that the sensitivity of the frequencies and mode shape changes with the variations in the core and face sheet properties and the thickness ratios of the plate.[121] Moreover, the variability in face-sheet properties has more pronounced effects on the frequencies as compared to that of the core.[121] Also, the thick plates were observed to be more sensitive to the material randomness than thin plates.[122]

The free vibration behavior of laminated composite plates considering uncertain system parameters was performed by Lal et al.[123] The FOPT in conjunction with FEM was employed to obtain the response statistics and the results were also obtained using MCS. Further, the work was extended to study the non-linear vibration of laminated composite plate resting on elastic foundation.[124,125] Lal and Singh[126] also examined the free vibration behavior in thermal environment. The smart composite plates associated with material uncertainties were examined for the free vibration response by Singh et al.[127,128] Dash and Singh[129] examined the geometrical non-linear free vibration behavior of composite plates embedded with piezoelectric layers having random material properties.

6.2 *Buckling behavior in stochastic environment*

The influence of uncertain parameters on the buckling behavior of laminated composite plates was studied by Singh et al.[130,131] A mean centered first order perturbation technique was implemented to examine second order statistics of the buckling load. The effectiveness of the theories in predicting the buckling load dispersions was examined.[130] The second-order statistics of the non-dimensional initial buckling loads were obtained for two stacking sequences of cross-ply graphite-epoxy laminates with all edges simply supported. It was observed that CLT over predicts the dispersions for thick plates as compared to FSDT and HSDT, although it does not always over predict dispersions for thin plates.[130] For mean analysis, FSDT under predicts the mean buckling load as compared to HSDT for cross-ply anti-symmetric laminates and over predicts for the symmetric laminates. However, for buckling load dispersion analysis, FSDT does not show a consistent prediction pattern compared to HSDT for cross-ply anti-symmetric and symmetric laminates. The relative magnitude depends on laminate construction, aspect ratio, and span-thickness ratio.[130] The dispersion in buckling loads always decreases as the aspect ratio increases for thick symmetric and anti-symmetric laminates considered for the study against individual increase in SD of longitudinal elastic modulus, transverse elastic modulus, in-plane shear modulus, and Poisson's ratio, and it showed a reverse trend with the out-of-plane shear modulus.[130] The study of variability in different material properties indicated that the longitudinal elastic modulus E_{11} is the dominant material property in the case of symmetric square laminates and anti-symmetric laminates of $b/a = 1$ and 2, whereas the out-of-plane shear modulus G_{13} is dominant for the plates with aspect ratio as 3.[130]

Singh et al.[132,133] examined the stability behavior of composite cylindrical panels with random material properties. They considered the effect of shear deformation and implemented first order perturbation technique to examine the second order statistics of buckling load. The influence of random material properties was studied and it was observed that the panel buckling loads for the anti-symmetric cross-ply laminates show larger scatter with change in thickness ratio as compared to the symmetric laminates.[132] Moreover, the dominant material property is found to be E_{11}.[132] It was concluded that the sensitivity of buckling load dispersion due to variation in material properties is dependent on the thickness ratio and boundary conditions of the laminate.[133] The analysis considering the influence of different boundary conditions indicates that the effects of E_{11}, E_{22}, G_{12} and ν_{12} on the scatter in non-dimensional buckling load increases as b/h ratio increases, while it decreases with changes in G_{13} and G_{23} for all the support conditions. However, for CFCF the effect of G_{12} and ν_{12} on dispersion also decreases as b/h ratio increases.[133] It was also observed that the dispersion in the transverse shear modulus G_{13} affects the buckling response to a large extent. However, for panel with CCCF and SSSS edges conditions and $b/h = 5$ the E_{11} is dominant and the buckling load is least affected with changes in ν_{12}.[133]

Lal et al.[134] studied the buckling behavior of laminated composite plates resting on elastic foundation. Pandit et al.[135] studied the effects of random material properties on the buckling behavior of sandwich plates by modeling the plate using higher order zig zag theory and taking into account the effects of core compressibility. It was observed that the scattering of the buckling-load parameters shows linear variation with standard deviation of the material properties. In general, the variations in longitudinal elastic modulus E_{11} of the face sheet have the most significant influence on the scattering of the buckling load of the sandwich plates, whereas it is the least sensitive to the changes in properties of the core material. Influence of geometric and material uncertainty on the buckling behavior in thermal environment was examined by Lal et al.[136] and Verma and Singh.[137] Singh and Verma[139] investigated buckling load of composite plates in hygrothermal environment. A C^0 finite element based on higher order shear deformation theory has been used for deriving the standard eigenvalue problem. The random material and geometric properties are modeled as basic random variables. A Taylor series based mean-centered first order perturbation technique ws used to find mean and standard derivation of buckling loads of composite plate in thermal and hygrothermal environment. It was observed that the characteristics of the thermal buckling load of plates are significantly influenced by the support conditions, the plate thickness ratios, the aspect ratios and the temperature changes. The mean and the dispersion of the thermal buckling load of the plates are higher when the plates are subjected to temperature dependent thermo-elastic properties as compared to temperature independent case.[136] The analysis considering the effect of boundary conditions reveal that the clamped plates buckle at slightly higher temperature compared to other supports. The buckling is more dominant in plates of temperature dependent material properties as compared to temperature independent case.[136] Moreover, it was noted that the thick plates are less affected by random input variables and others input variables compared to thin plates.[136] The consideration of the effects of environmental factors on the buckling load of laminated plates with uncertain geometric and material properties reveal that the random change in thickness has more impact on hygrothermal buckling load scattering compared to individual random changes in material property.[138] Moreover, the hygrothermal buckling load is most affected with random changes in E_{22}, while it is least affected with random changes G_{12}.[138] On the whole, the sensitivity of hygrothermal buckling load dispersion due to variation in geometric and material properties is dependent on thickness ratio and boundary conditions of the laminate.[138]

6.3 *Static analysis*

Lal et al.[139] investigated the influence of uncertain material properties on the static response of laminated composite plates resting on elastic foundation. Singh et al.[140] considered the geometrical non-linearity to examine the static response of composite plate on nonlinear elastic foundation in stochastic conditions. Lal et al.[141] investigated nonlinear bending response of laminated composite plate with material randomness under lateral pressure and thermal loading. Further they extended the work to study the bending response of composite spherical shell panel with material random properties and subjected to hygro-thermo-mechanical loading.[142] Effects of random system properties on the bending response of composite plates subjected to thermo-mechanical loads were examined by Lal and Singh.[143] Pandit et al.[28] examined the deflection statistics of soft core sandwich plates with uncertain material properties implementing stochastic perturbation based finite element method.

7 Conclusion

The present work focuses on the state-of-the-art of the evaluation of structural responses of composite structures in the stochastic environment.

The composite materials exhibit a considerable amount of scatter in their material properties due to large number of parameters associated with their manufacturing and fabrication process. The causes and consequences of the uncertainties present in the system are discussed. The different approaches such as MCS, perturbation techniques etc. are discussed to quantify the parametric randomness. The incorporation of these approaches with the deterministic finite element analysis gives rise to stochastic finite element analysis. The influence of uncertain material properties on the structural responses (static, free vibration and buckling) of composite structures is investigated. The variability in the elastic Young's modulus in the fiber direction is found to be most dominant for almost all the cases while the variability in the Poisson's ratio is found to be least dominant. It is felt that the present work will be of interest to the research community already involved in the stochastic analysis of composite structures. It is to be noted that, in this paper, no distinction is made on the behavior of materials and structural kinematics. However, it is observed that some researchers have also focused on the geometrical non linear structural kinematics and nonlinear material behavior. Therefore, the stochastic analysis of the composite structures considering geometrical and material nonlinearity will be the subject matter for the future work.

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Dr. B.N. Singh is a full Professor in the Department of Aerospace Engineering at the Indian Institute of Technology Kharagpur, India. Prof. Singh has more than 20 years of teaching and/or research experience in the institute of importance in India and abroad as well. He is working in the area of aerospace composite structures and its uncertainty quantification and has developed the several stochastic and deterministic mathematical models and its applications in aerospace structural components made of smart composites. His current research interests also include plates and shells, functionally graded materials, analytical and finite element modeling, shear deformation theories, aeroelasticity, multiscale modeling and nonlinear adaptive FEM. He has published 87 research papers in the Journals of repute such as AIAA J, ASME J, ASCE J, JSV, CM, CS, IJMS, etc and more than 73 research papers in refereed Conference proceedings. He has organised and edited two international conference proceedings (ICTACEM 2007 & 2010). He has guided 06 Ph.D., 39 M. Tech and several B. Tech students so far.

Neeraj Grover is a research scholar at the Department of Aerospace Engineering, Indian Institute of Technology, Kharagpur, India. His research interests include laminated composite and sandwich plates, analytical and finite element modeling, shear deformation theories, structural and aeroelastic behavior, and uncertainty quantification.