

STRESS SYSTEM IN A ROTATING AELOTROPIC SHAFT IN FINITE STRAIN AND ITS NUMERICAL STUDY*

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ABSTRACT

Stress system in a rotating aelotropic shaft has been obtained within the theory of finite strain. A numerical study of the above stress system has been made.

1. INTRODUCTION

1.1. The theory of finite strain in elastical problems has been developed on the hypothesis that the second order terms in the components of strain may not be neglected [1, 4].

Like the body-stress equations these components have been referred to the actual position of a point P of the material in the strained condition, and not to the position of a point considered before strain. Many applications of this theory have been already worked out [2].

Seth [3] has developed the stress of a rotating isotropic shaft in finite strain. The object of the present paper is to find out the stress of a rotating aelotropic shaft in finite strain. A numerical example has been considered.

1.2. We treat the problem as one of the plane strain, with an allowance for uniform longitudinal extension α . Since the shaft is strained symmetrically we can take the components of displacements

$$u = x(1 - \beta), \quad v = y(1 - \beta), \quad \omega = \alpha z \quad (1)$$

where β is a function of $r = (x^2 + y^2)^{1/2}$.

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Using cylindrical polar co-ordinates the components of displacements are given by

$$\begin{aligned} U_r &= u \cos \theta + v \sin \theta \\ U_\theta &= -u \sin \theta + v \cos \theta \\ U_z &= az. \end{aligned} \quad (2)$$

The stress-strain relation (for finite strain) in cartesian co-ordinates are given by

$$\begin{aligned} \widehat{xx} &= C_{11}\epsilon_{xx} + C_{12}\epsilon_{yy} + C_{13}\epsilon_{zz} \\ \widehat{yy} &= C_{12}\epsilon_{xx} + C_{11}\epsilon_{yy} + C_{13}\epsilon_{zz} \\ \widehat{zz} &= C_{13}\epsilon_{xx} + C_{13}\epsilon_{yy} + C_{33}\epsilon_{zz} \\ \widehat{xy} &= C_{66}\epsilon_{xy} \\ \widehat{yz} &= \widehat{zx} = 0. \end{aligned} \quad (3)$$

In cylindrical polar co-ordinates the stress corresponding to given displacements (2) are given by

$$\begin{aligned} \widehat{rr} &= \frac{1}{2} \cdot C_{11} (1 - \beta^2) - \frac{r^2}{2} \cdot C_{11} \left(\beta'^2 + \frac{2\beta\beta'}{r} \right) + \frac{1}{2} C_{12} (1 - \beta^2) \\ &\quad + \frac{1}{2} C_{13} (2a - a^2) \\ \widehat{\theta\theta} &= \frac{1}{2} \cdot C_{11} (1 - \beta^2) + \frac{1}{2} C_{12} (1 - \beta^2) + \frac{1}{2} \cdot C_{13} (2a - a^2) \\ &\quad - \frac{1}{2} \cdot C_{12} r^2 \cdot \left(\beta'^2 + \frac{2\beta\beta'}{r} \right) \\ \widehat{zz} &= \frac{1}{2} \cdot C_{13} (1 - \beta^2) - \frac{1}{2} \cdot C_{13} r^2 \cdot \left(\beta'^2 + \frac{2\beta\beta'}{r} \right) \\ &\quad + \frac{1}{2} \cdot C_{33} (2a - a^2) \\ \widehat{r\theta} &= \widehat{rz} = \widehat{\theta z} = 0. \end{aligned} \quad (4)$$

The only stress-equation of equilibrium which is not identically satisfied is given by

$$\frac{\partial \widehat{rr}}{\partial r} + \frac{\partial \widehat{rz}}{\partial z} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} + \rho r \omega^2 = 0 \quad (5)$$

which, on substituting from (4), reduces to

$$\frac{\partial}{\partial r} \left[(C_{11} + C_{12}) \beta^2 + r^2 \cdot C_{11} \cdot \left(\beta'^2 + \frac{2\beta\beta'}{r} \right) - r^2 \rho \omega^2 - (C_{12} - C_{11}) \int (r\beta'^2 + 2\beta\beta') dr \right] = 0$$

where ω is the angular velocity of the shaft. The differential equation satisfied by β is therefore

$$(C_{11} + C_{12}) \beta^2 + r^2 \cdot C_{11} \left(\beta'^2 + \frac{2\beta\beta'}{r} \right) - r^2 \rho \omega^2 - (C_{12} - C_{11}) \int (r\beta'^2 + 2\beta\beta') dr = K_1 \quad (6)$$

K_1 being a constant.

In the present paper, for mathematical simplicity we propose to discuss only the particular solution obtained by putting $K_1 = 0$.

We get

$$\beta = Ar = \left(\frac{2\rho\omega^2}{8C_{11} - C_{12}} \right)^{1/2} r \quad (7)$$

The radial displacement is therefore given by

$$U = r \left[1 - \left(\frac{2\rho\omega^2}{8C_{11} - C_{12}} \right)^{1/2} r \right] \quad (8)$$

The boundary condition over the curved surface is $\widehat{rr} = 0$ over $r = a$, a being the radius of the shaft in the strained condition. Using (4), we get

$$A^2 a^2 (4C_{11} + C_{12}) = K - C_{13} (1 - a)^2 \quad (9)$$

$$K = C_{11} + C_{12} + C_{13}$$

The boundary condition over the plane ends is $\widehat{zz} = 0$ over $x = \pm l$ $2l$ being the length of the shaft. This cannot be exactly satisfied. But we can make the resultant longitudinal stress vanish over the plane end

$$\text{This requires } \int_0^a r \widehat{zz} dr = 0$$

which gives

$$2a^2 A^2 C_{13} + C_{13} = (1 - a)^2 C_{33} - C_{33} \quad (10)$$

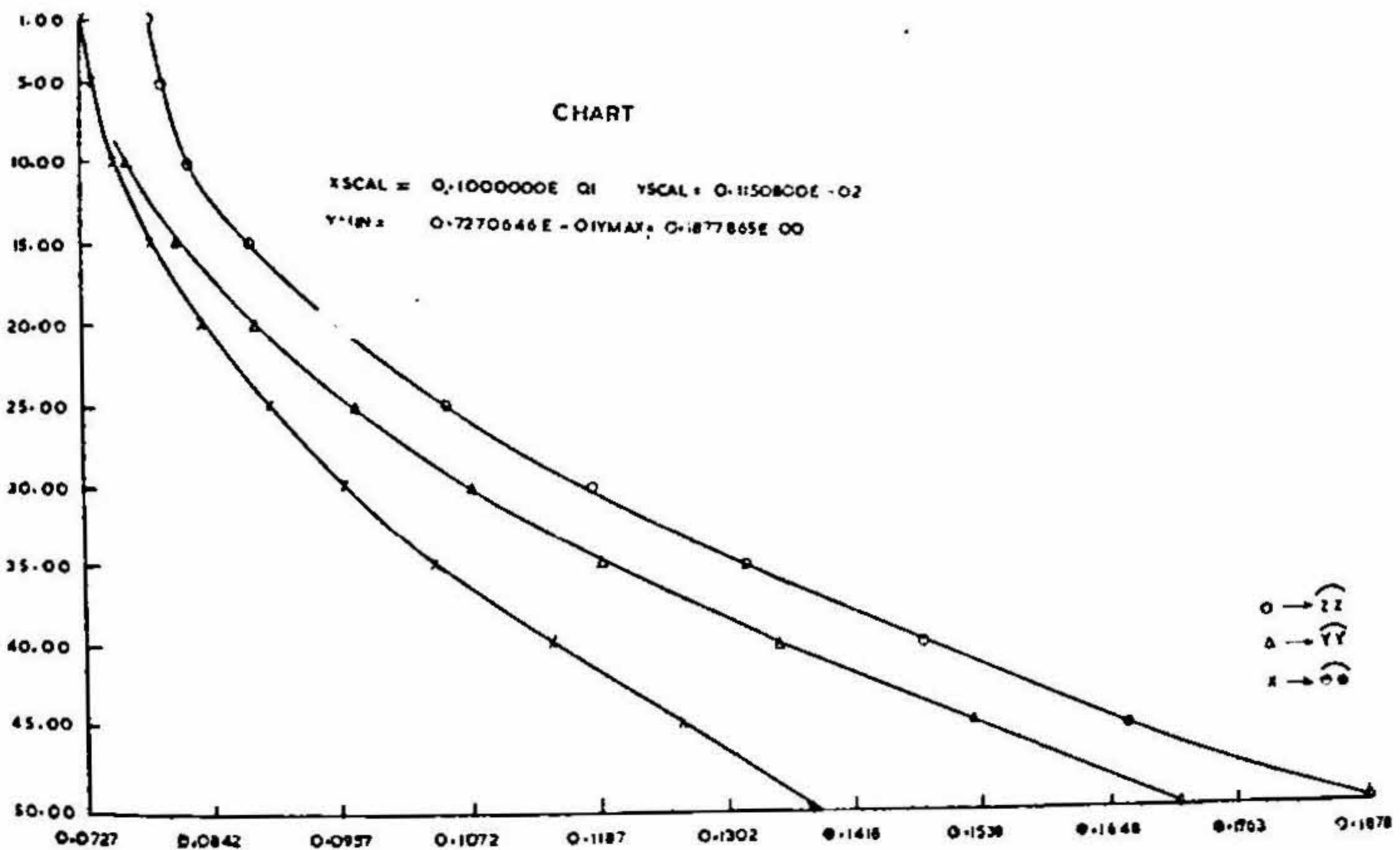
So, the stress system is given by

$$\begin{aligned} \widehat{rr} &= \frac{1}{2} C_{11} (1 - \beta^2) - \frac{1}{2} r^2 \cdot C_{11} \left(\beta'^2 + \frac{2\beta\beta'}{r} \right) \\ &\quad + \frac{1}{2} C_{12} (1 - \beta^2) + \frac{1}{2} C_{13} (2a - a^2) \\ \widehat{\theta\theta} &= \frac{1}{2} C_{11} (1 - \beta^2) + \frac{1}{2} C_{12} (1 - \beta^2) + \frac{1}{2} C_{13} (2a - a^2) \\ &\quad - \frac{1}{2} C_{12} \cdot r^2 \left(\beta'^2 + \frac{2\beta\beta'}{r} \right) \\ \widehat{zz} &= \frac{1}{2} C_{13} (1 - \beta^2) - \frac{C_{13}}{2} r^2 \left(\beta'^2 + \frac{2\beta\beta'}{r} \right) + C_{33} (2a - a^2) \\ \widehat{r\theta} &= \widehat{rz} = \widehat{\theta z} = 0. \end{aligned} \tag{11}$$

A and a are obtained from (9) and (10).

1.3. Numerical Study of the Stress:

Though the elastic constants are C_{ij} ($i = 1, 6; j = 1, 6$), here the stress components are dependent only on four elastic constants, *viz.*, C_{11} , C_{12} , C_{13} , C_{33} . Here we consider an aelotropic material, having the elastic constants.



$$C_{11} = \cdot 2, C_{12} = \cdot 1, C_{13} = \cdot 25, C_{33} = \cdot 15.$$

Let radius of a quartz shaft be 60 units.

Now we compute the stress \widehat{rr} , $\widehat{\theta\theta}$, \widehat{zz} of the shaft at different radius vector (from 1 to 50). The magnitude of the stress components are shown in a multiplot. The plotting has been made by IBM 360/44 computer.

The stress and the radii vector may be expressed in any of the system, *viz.*, either C.G.S. or F.P.S.

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