INVESTIGATION OF STRESSES AROUND A HOLE IN THIN ROTATING DISKS OF HYPERBOLIC AND PARABOLIC PROFILES

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Introduction

Stresses in rotating disks have been investigated by several workers in the past few years. Chree[1] was one of the earliest investigators and his paper on stress in a rotating ellipsoid has become a classic. Stresses in thin rotating disks of hyperbolic profiles have been mentioned by Stodola[2], Biezeno and Grammel[3]. Stresses in a rotating disk of exponential variation of thickness have been investigated recently by Lee[4].

Symbols Used:

r, θ = polar co-ordinates.

ω = angular velocity of the disk.

ρ = density of the material of the disk.

σ_r = radial stress.

σ_θ = tangential stress.

$\tau_{r\theta}$ = shearing stress in the r, θ plane.

R = body force.

h = thickness of the disk.

F = stress function.

n = a parameter used to define the profile of the disk.

n_s = a singularity of n.

f = the variable part of σ_θ at the inner boundary.

ν = Poisson's ratio of the material of the disk

Summary

Hyperbolic and parabolic profiles used in defining a rotating disk and having a circular hole at the centre are obtained by the same law of thickness distribution, *i.e.*

$$h = Cr^n$$

only by giving negative and positive values to n (Fig. 1 & 2). For the simplification of the complete process, the scale is taken such that the radius of the hole is unity. It is investigated as to how the stress concentration at the hole varies with the shape of the disk. Indeterminacy sets in at

$$n_s = -\frac{8}{3+\nu}$$

Leaving this point, the curve of f vs. n is continuous from $-\infty$ to $+\infty$. The curve corresponding to parabolic profiles shows nearly a linear variation. In the negative n region the curve goes asymptotically to $f = o$ with decreasing n, showing that the stress concentration at the hole is reduced with n.

A circular rotating disk being symmetrical about its axis of rotation, polar co-ordinates are suitable in such investigations.

In polar co-ordinates the equations of equilibrium[5] are

$$\left.\begin{aligned} \frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + R &= 0 \\ \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} &= 0 \end{aligned}\right\} \quad (1)$$

If the thickness of the disk is small in comparison to its radial dimensions, we can neglect the variation of the tangential and radial stresses over the thickness of the disk. Also the stress distribution being symmetrical about the axis of rotation, $\tau_{r\theta}$ is zero and the stresses are not dependent on θ but on r alone.

In case of a disk of a variable thickness

$$R = \rho \omega^2 r^2 h$$

and the first of the equations (1) becomes

$$\frac{\partial}{\partial r} (h r \sigma_r) - h \sigma_\theta + \rho \omega^2 r^2 h = 0 \tag{2}$$

The second of equations (1) vanishes altogether. We now define the stress function F such that

$$\sigma_r = \frac{F}{hr}$$

$$\sigma_\theta = \frac{1}{h} \left(\frac{\partial F}{\partial r} \right) + \rho \omega^2 r^2$$

which evidently satisfy equation (2). From these and the expression for the strain components in polar co-ordinates, we get the compatibility equation:

$$r^2 \frac{d^2 F}{dr^2} + r \frac{dF}{dr} - F + (3 + \nu) \rho \omega^2 h r^3$$

$$- \frac{r}{h} \frac{dh}{dr} \left(r \frac{dF}{dr} - \nu F \right) = 0 \tag{3}$$

Since, here r is the only variable $\frac{\partial F}{\partial r} = \frac{dF}{dr}$ and $\frac{\partial^2 F}{\partial r^2} = \frac{d^2 F}{dr^2}$. This differential equation is easily solvable in case of the substitution

$$h = C r^n$$

C being an arbitrary constant and n any real number.

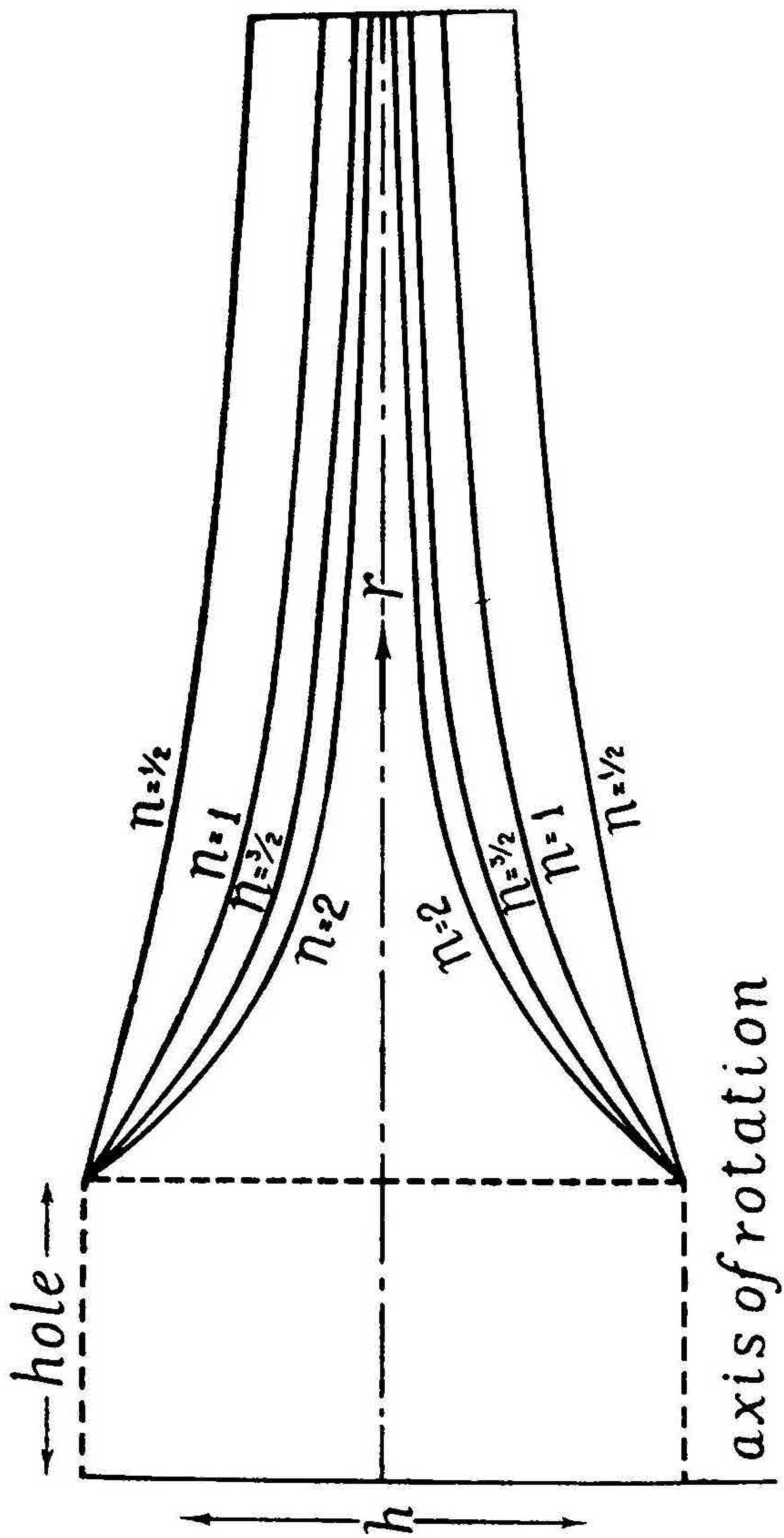

FIG. 1. Hyperbolic profiles of Rotating Disks corresponding to $h = \frac{2}{r^n}$

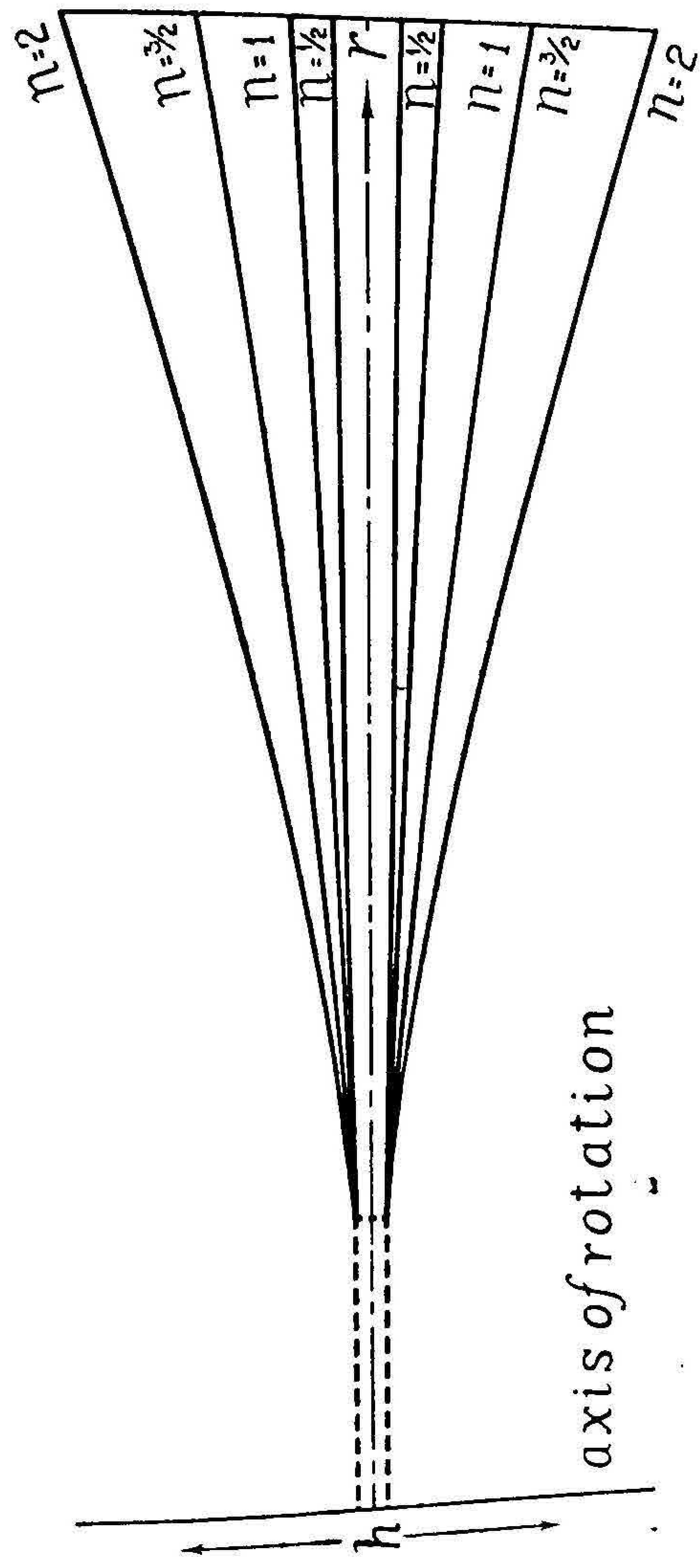

FIG. 2. Parabolic profiles of rotating disks corresponding to $h=r^n/10$

If we keep n positive we get parabolae of different degrees all passing through $h = r = 0$ and tangential to $h = 0$, at the origin. Again if n is negative the curves are hyperbolae of different degrees (Figs. 1 and 2). In the solution of the differential equation (3) the complementary function is

$$C_1 r^{\psi_1} + C_2 r^{\psi_2} \tag{4}$$

where C_1, C_2 are arbitrary constants and ψ_1, ψ_2 are the roots of the equation

$$y^2 - ny + \nu n - 1 = 0 \tag{5}$$

and the particular integral is

$$= - \frac{(3 + \nu)\, \rho\omega^2 C \cdot r^{n+3}}{(3n + \nu n + 8)} = m r^{n+3} \text{ say}$$

Therefore the general solution of equation (3) is

$$F = m r^{n+3} + C_1 r^{\psi_1} + C_2 r^{\psi_2} \tag{6}$$

Now let the disk have a circular concentric hole of radius a_1 at the centre of the disk the radius of the disk being b_1.
From equation (2),

$$\sigma_\theta = \frac{1}{h} \left\{ \frac{dF}{dr} + h\rho\omega^2 r^2 \right\}$$

Substituting the value of F and h we have

$$\sigma_\theta = \frac{1}{Cr^n} \left[- \frac{(n+3)(3+\nu)\rho\omega^2 C}{(\nu n + 3n + 8)} r^{n+2} + C_1 \psi_1 r^{\psi_1 - 1} \right.$$

$$\left. + C_2 \psi_2 r^{\psi_2 - 1} \right] + \rho\omega^2 r^2 \tag{7}$$

We know that stress concentration exists at the boundary of the hole and in the case of a rotating disk, σ_r being zero at the boundaries, only σ_θ exists. Since the maximum tangential stress occurs at the inner boundary of the disk, *i.e.*, at $r = a_1$, let us investigate the stress concentration at the hole where the disk is liable to give way.

From equation (5)

$$\psi_1 = \frac{n + \sqrt{n^2 - 4n\nu + 4}}{2} = \frac{n+p}{2} \text{ say}$$

and $\psi_2 = \dfrac{n - \sqrt{n^2 - 4n\nu + 4}}{2} = \dfrac{n-p}{2}$

From the boundary conditions we have,

$$(\sigma_r)_{r=a_1} = (\sigma_r)_{r=b_1} = 0$$

Therefore $ma_1^{n+3} + C_1 a_1^{\psi_1} + C_2 a_1^{\psi_2} = 0$

and $mb_1^{n+3} + C_1 b_1^{\psi_1} + C_2 b_1^{\psi_2} = 0$

Solving these simultaneous equations we have

$$C_1 = \frac{ma_1^{p/2} b_1^{p/2} (a_1^{-p/2} b_1^{\frac{n}{2}+3} - b_1^{-p/2} a_1^{\frac{n}{2}+3})}{(a_1^p - b_1^p)}$$

and a similar expression for C_2.

Therefore

$$(\sigma_\theta)_{r=a_1} = -\frac{(3+\nu)\rho\omega^2 C}{(\nu n + 3n + 8)} \cdot \frac{1}{Ca_1^n}\Big[(n+3)a_1^{n+2}$$

$$+ \frac{a_1^{p/2} b_1^{p/2} (a_1^{-p/2} b_1^{\frac{n}{2}+3} - b_1^{-p/2} a_1^{\frac{n}{2}+3})}{a_1^p - b_1^p} \cdot \frac{n+p}{2} \cdot a_1^{\frac{n+p}{2}-1}$$

$$+ \frac{a_1^{p/2} b_1^{p/2} (a_1^{n/2+3} b_1^{p/2} - b_1^{n/2+3} a_1^{p/2})}{a_1^p - b_1^p} \cdot \frac{n-p}{2} \, a_1^{\frac{n-p}{2}-1}\Big] + \rho\omega^2 a^2 \qquad \ldots \quad (8)$$

This expression for σ_θ can be simplified by taking our unit of length to be the radius of the inner circle *i.e.* setting

$$b=\frac{b_1}{a_1} \text{ and } 1=a=\frac{a_1}{a_1}$$

$$\therefore (\sigma_\theta)_{r=a=1} = -\frac{(3+\nu)\rho\omega^2}{(\nu n+3n+8)}\left[n+3+b^{p/2}\left(\frac{b^{n/2+3}-b^{-p/2}}{1-b^p}\right)\frac{n+p}{2}\right.$$

$$\left.+\,b^{\frac{p}{2}}\left(\frac{b^{p/2}-b^{\frac{n}{2}+3}}{1-b^p}\right)\frac{n-p}{2}\right]+\rho\omega^2$$

$$= -\frac{(3+\nu)\rho w^2}{(\nu n+3n+8)}\left[n+3+\frac{b^{\frac{n+p}{2}+3}-1}{1-b^p}\cdot\frac{n+p}{2}\right.$$

$$\left.+\left(\frac{b^p-b^{\frac{n+p}{2}+3}}{1-b^p}\right)\frac{n-p}{2}\right]+\rho\omega^2 \tag{9}$$

Since we want to find the mode of variation of $(\sigma_\theta)_{r=a=1}$ for different shapes of the disk which are here obtained by the variation of the parameter n in equation $h=Cr^n$, constants multiplying the variable terms in equation (9) can for the present, be neglected. Also the constant $\rho\omega^2$, which is a separate term can similarly be neglected. So for studying the variation of σ_θ let us consider the equation

$$f = -\frac{1}{(\nu n+3n+8)}\left[n+3+\frac{b^{\frac{n+p}{2}+3}-1}{1-b^p}\cdot\frac{n+p}{2}\right.$$

$$\left.+\left(\frac{b^p-b^{\frac{n+p}{2}+3}}{1-b^p}\right)\frac{n-p}{2}\right] \tag{10}$$

So f and σ_θ are connected by the equation

$$\sigma_\theta = (3+\nu)\,\rho\,\omega^2 f + \rho\,\omega^2$$

$$\text{or } f = \frac{1}{(3+\nu)}\left(\frac{\sigma_\theta}{\rho\omega^2} - 1\right) \tag{11}$$

For convenience of calculation let us take ν to be a constant.

Now if we differentiate (9) with respect to n and then equate to zero to find the value of n corresponding to the maxima and minima of σ_θ, we find that the roots come to be $+\infty$ and $-\infty$. This shows that the curve of f vs. n will not have maxima or minima in the finite region.

Equation (10) can further be simplified and rewritten as

$$f = -\frac{1}{(\nu n + 3n + 8)}\left[\frac{n}{2} + 3 + \frac{1}{2}\left(\frac{p}{1-b^p}\right)\left(2b^{\frac{n+p}{2}+3} - b^p - 1\right)\right] \tag{12}$$

Now to find the behaviour of f with the variation of n, let us give some specific value to b. Let $b = 5$.

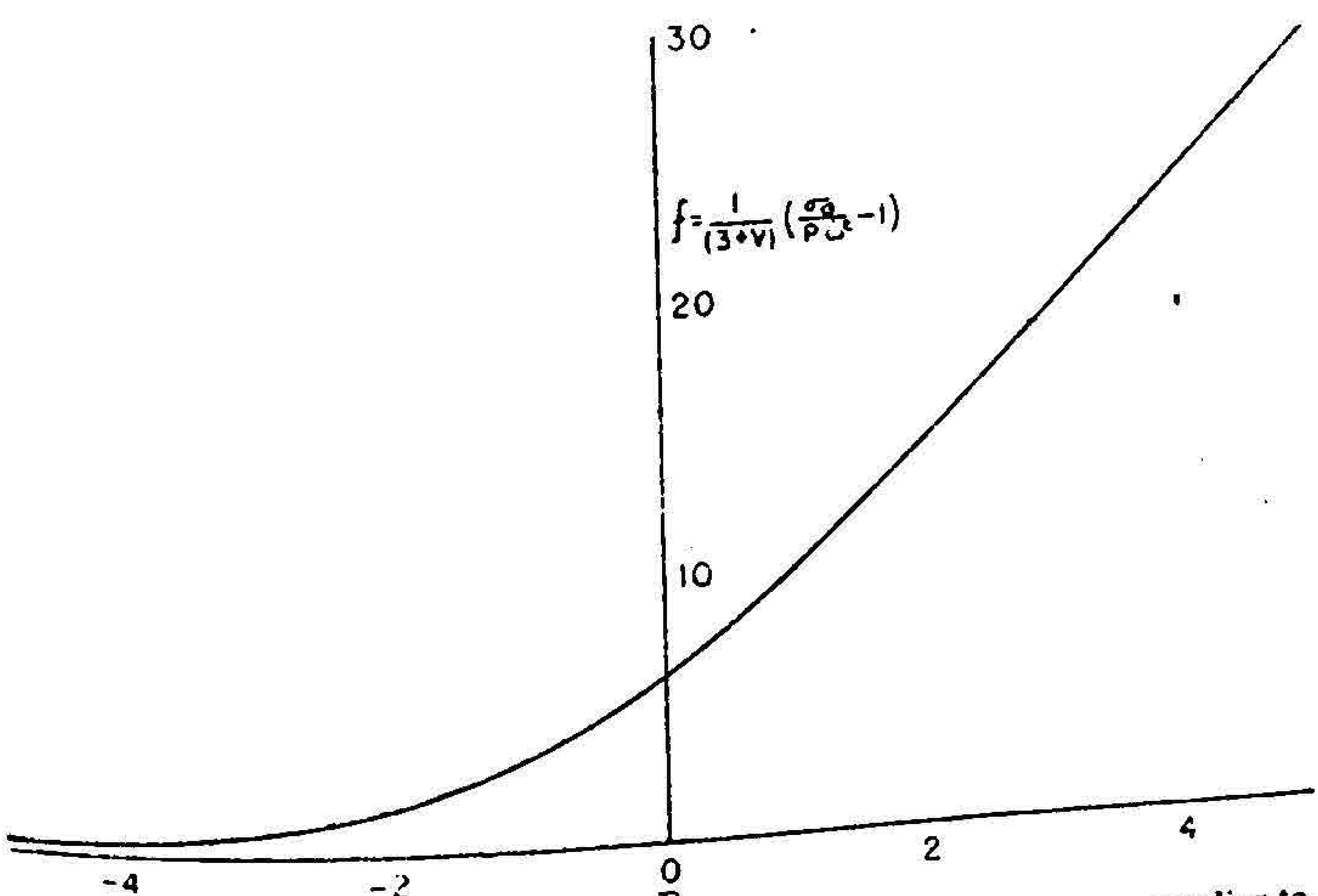

FIG. 3. A typical mode of variation of the stress at the hole of disks corresponding to $h = Cr^n$, for different values of n

It was felt that, if the expression in parenthesis in equation (12) remained finite and negative throughout the variation of n in the finite region, f would become infinite at two points, *i.e.*, at

$$n = +\infty$$

and
$$n_s = -\frac{8}{3+\nu}$$

Hence a minimum of f should have appeared between these two values of n. On investigating it is found that the expression in parenthesis in equation (12) also goes on decreasing as the denominator tends to zero. But exactly at n_s the numerator as well as the denominator are both exactly zero and so f at n_s may be said to be indeterminate. Again as we proceed below n_s, the points are obtained in continuation to the previous curve. Hence we decide to call the value of n_s only as a singularity and smoothen the graph over this particular value also. (Fig. 3 for steel).

Conclusion.—As we find, that the disks of a parabolic profile and a hole at the centre, have a high stress concentration at the inner boundary they are not the advisable shapes. The disks with a hyperbolic profile do have a small stress concentration. The higher the degree (here n) of the hyperbola, the smaller is the stress concentration. But this value cannot be increased indefinitely due to some practical reasons, *e.g.*, a thin metallic sheet at the periphery is not advisable, since it cannot take high loads either when the disk is rotating or when it is stationary. So the final shape of the disk should be an outcome of a compromise between the practical requirements and the degree of the hyperbolic profile.

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References

1. Chree, C. ... Proceedings of the Royal Society of London, England, Vol. 58, 1895.
2. Stodola, A. ... Steam and Gas Turbines, 1927, Vol. I.
3. Biezeno, C. B. and Grammel, R.... Technische Dynamik, 1939, Julius Springer, Berlin.
4. Lee, T. C. ... Journal of Applied Mechanics, Sept. 1952, Vol. 19, No. 3.
5. Timoshenko, S. ... Theory of Elasticity, 1934, McGraw Hill Book Co. Inc.

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