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On some dual integral equations involving Legendre and associated Legendre functions

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Abstract

Dual integral equations involving Legendre and associated Legendre functions as kernels are considered in this paper. Except for one pair, these dual integral equations are first reduced to solving some appropriate ordinary differential equations. Invoking the inversion formulae for Abel integral equations, closed-form solutions to these dual integral equations are obtained in most cases and in other cases they are reduced to some appropriate Fredholm integral equation of the second kind. For the exceptional dual integral equations pair which involves the associated Legendre function as kernel, a direct method of the use of Abel integral equation is applied to obtain the closed-form solution. As an example of application of these dual integral equations a problem arising from mathematical physics is considered.

Key words: Dual integral equations, Legendre and associated Legendre function kernel, Fredholm integral equation of the second kind.

1. Introduction

Dual integral equations involving Legendre functions as kernels are encountered in the study of certain mixed boundary value problems in mathematical physics. Babloian[1] first considered some dual integral equations involving $P_{-1/2+i\tau}(\cosh\alpha)$ as kernel. He solved these equations by using the Mehler–Fok inversion formulae and applied these to solve a torsion problem involving a spherical segment in the theory of elasticity. Certain dual integral equations involving the associated Legendre function $P^m_{-1/2+i\tau}(\cosh\alpha)$, where $m = 0, 1, 2, \ldots$, were considered by Rukhovets and Ufliand[2], who reduced these equations to the solution of a Fredholm integral eqaution of the second kind. They also applied these to solve the problem of an elastic half-space twisted by a hollow cylindrical die. Later, Pathak[3] considered some dual integral equations involving $P^\mu_{-1/2+i\tau}(\cosh\alpha)$ as kernel, where μ is not an integer. He exploited the results of some integrals involving $P^\mu_{-1/2+i\tau}(\cosh\alpha)$ to handle these dual integral equations and obtained the closed-form solution in some cases and in other cases expressed the solutions in terms of one unknown function which satisfies a Fredholm integral equation of the second kind. For $\mu = 0$, the corresponding integral equations were mostly considered by Babloian[1]. Recently, Mandal[4] also considered certain dual integral equations involving $P^\mu_{-1/2+i\tau}(\cosh\alpha)$, where $\mathrm{Re}\,\mu < \frac{1}{2}$ and obtained a closed-form solution.

In this paper we consider some new classes of dual integral equations involving both Legendre function $P_{-1/2+i\tau}(\cosh\alpha)$ and the associated Legendre function $P^{\mu}_{-1/2+i\tau}(\cosh\alpha)$ $(-\frac{1}{2} < \operatorname{Re}\mu \leq 0)$ as kernels. These dual integral equations are solved by a method based on reducing them to solving ordinary differential equations, followed by an appropriate inversion formula for Abel integral equations. A somewhat similar idea has recently been used by Srivastava and Srivastava[5] for studying some dual integral equation involving Bessel functions as kernel. We obtain the closed-form solution in some cases and in other cases the dual integral equations are reduced to solving Fredholm integral equations of the second kind. As an application to physical problems, a boundary value problem in elasticity involving a half-space under torsion due to an attached rigid annular die is considered.

2. Dual integral equations involving the Legendre function as kernel

In this section, we consider three pairs of dual integral equations with the Legendre function as kernel.

(A) The first is

$$\left.\begin{aligned} &\int_0^{\infty} \tau^2 A(\tau) P_{-1/2+i\tau}(\cosh\alpha)\,d\tau = f(\alpha), \quad 0 \leq \alpha \leq a, \\ &\int_0^{\infty} \tau \tanh \pi\tau\, A(\tau) P_{-1/2+i\tau}(\cosh\alpha)\,d\tau = 0, \quad a \leq \alpha < \infty, \end{aligned}\right\} \tag{1}$$

where $f(\alpha)$ is a prescribed function and $A(\tau)$ is an unknown.

To solve these, we assume that

$$\int_0^{\infty} \tau \tanh \pi\tau\, A(\tau) P_{-1/2+i\tau}(\cosh\alpha)\,d\tau = \phi(\alpha), \quad 0 \leq \alpha \leq a, \tag{2}$$

where $\phi(\alpha)$ is an unknown for $0 \leq \alpha \leq a$. It follows from the second equation of (1) and (2), by using Mehler–Fok transform formulae (see Sneddon[6]), that

$$A(\tau) = \int_0^a \phi(\alpha)\, P_{-1/2+i\tau}(\cosh\alpha) \sinh\alpha\, d\alpha. \tag{3}$$

Substituting this expression for $A(\tau)$ in the first equation of (1), we find

$$\int_0^{\infty} \tau^2 P_{-1/2+i\tau}(\cosh\alpha) \left[\int_0^a \phi(\beta)\, P_{-1/2+i\tau}(\cosh\beta) \sinh\beta\, d\beta\right] d\tau = f(\alpha), \quad 0 \leq \alpha \leq a. \tag{4}$$

Equation (4) is now equivalent to the differential equation

$$\frac{1}{\sinh\alpha} \frac{d}{d\alpha}\left(\sinh\alpha \frac{du}{d\alpha}\right) + \frac{1}{4} u = -f(\alpha), \tag{5}$$

where

$$u(\alpha) = \int_0^{\infty} P_{-1/2+i\tau}(\cosh\alpha) \left[\int_0^a \phi(\beta) P_{-1/2+i\tau}(\cosh\beta) \sinh\beta\, d\beta\right] d\tau, \quad 0 \leq \alpha \leq a. \tag{6}$$

The solution of (5), by the method of variation of parameter, is

$$u(\alpha) = C_1 P_{-1/2}(\cosh\alpha) + C_2 Q_{-1/2}(\cosh\alpha) + P_{-1/2}(\cosh\alpha)\int_0^\alpha f(t)\, Q_{-1/2}(\cosh t)\sinh t\, dt$$

$$-Q_{-1/2}(\cosh\alpha)\int_0^\alpha f(t) P_{-1/2}(\cosh t)\sinh t\, dt, \tag{7}$$

where the constants C_1 and C_2 are arbitrary. They are to be determined from physical considerations of the problems. Of these, the constant C_2 must be zero in order that $u(\alpha)$ is finite for $\alpha = 0$. The constant C_1 will be determined later.

Interchanging the order of integration and then using the result (which can be established by using the integral representation of the Legendre function)

$$\int_0^\infty P_{-1/2+i\tau}(\cosh\alpha)\, P_{-1/2+i\tau}(\cosh\beta)\, d\tau = \frac{1}{\pi}\int_0^{\min(\alpha,\beta)} \frac{dt}{\{(\cosh\alpha - \cosh t)(\cosh\beta - \cosh t)\}^{1/2}} \tag{8}$$

and again interchanging the order of integration, (6) reduces to the Abel's integral equation

$$\int_0^\alpha \frac{dt}{(\cosh\alpha - \cosh t)^{1/2}} \int_t^a \frac{\phi(\beta)\sinh\beta\, d\beta}{(\cosh\beta - \cosh t)^{1/2}} = \pi u(\alpha), \quad 0 \le \alpha \le a,$$

so that

$$\int_x^a \frac{\phi(\beta)\sinh\beta\, d\beta}{(\cosh\beta - \cosh x)^{1/2}} = \psi(x),$$

where

$$\psi(x) = \sinh x \int_0^x \frac{u'(t)}{(\cosh x - \cosh t)^{1/2}}\, dt. \tag{9}$$

Another use of Abel's inversion formula[6] gives

$$\sinh x\, \phi(x) = -\frac{1}{\pi}\frac{d}{dx}\int_x^a \frac{\psi(t)\sinh t}{(\cosh t - \cosh x)^{1/2}}\, dt. \tag{10}$$

This gives the complete solution of dual integral equations (1) provided the constant C_1 is determined. This is found from the fact that $\psi(\alpha) = 0$, which arises from the physical requirement involving the continuity of $\phi(\alpha)$ at $\alpha = a$. Hence, the constant C_1 is determined by the equation

$$\int_0^a \frac{u'(t)}{(\cosh a - \cosh t)^{1/2}}\, dt = 0. \tag{11}$$

(B) The second pair is

$$\left.\begin{aligned} &\int_0^\infty (\tau^2 - \mu^2) A(\tau) P_{-1/2+i\tau}(\cosh\alpha)\, d\tau = f(\alpha), \quad 0 \le \alpha \le a, \\ &\int_0^\infty \tau \tanh \pi\tau\, A(\tau) P_{-1/2+i\tau}(\cosh\alpha)\, d\tau = 0, \quad a \le \alpha < \infty, \end{aligned}\right\} \tag{12}$$

where μ is real. The pair (12) is a generalisation of the pair (1) in the sense that for $\mu = 0$ these are reduced to the pair (1).

To solve the pair (12), we assume

$$\int_0^\infty \tau \tanh \pi\tau \, A(\tau) P_{-1/2+i\tau}(\cosh\alpha)\, d\tau = \phi(\alpha), \quad 0 \le \alpha \le a, \tag{13}$$

where $\phi(\alpha)$ is unknown for $0 \le \alpha \le a$. Thus,

$$A(\tau) = \int_0^a \phi(\beta) P_{-1/2+i\tau}(\cosh\beta) \sinh\beta \, d\beta. \tag{14}$$

Substituting (14) into the first equation of (12), we find

$$\int_0^\infty (\tau^2 - \mu^2) P_{-1/2+i\tau}(\cosh\alpha) \left[\int_0^a \phi(\beta) P_{-1/2+i\tau}(\cosh\beta) \sinh\beta \, d\beta \right] d\tau = f(\alpha), \ 0 \le \alpha \le a. \tag{15}$$

This is equivalent to the differential equation

$$\frac{1}{\sinh\alpha} \frac{d}{d\alpha}\left(\sinh\alpha \frac{du}{d\alpha} \right) + \left(\frac{1}{4} + \mu^2 \right) u = -f(\alpha), \quad 0 \le \alpha \le a, \tag{16}$$

where now

$$u(\alpha) = \int_0^\infty P_{-1/2+i\tau}(\cosh\alpha) \left[\int_0^a \phi(\beta) P_{-1/2+i\tau}(\cosh\beta) \sinh\beta \, d\beta \right] d\tau. \tag{17}$$

The solution of (16) is

$$\begin{aligned} u(\alpha) = {} & C_1 P_{-1/2+i\mu}(\cosh\alpha) + C_2 Q_{-1/2+i\mu}(\cosh\alpha) \\ & + P_{-1/2+i\mu}(\cosh\alpha) \int_0^\alpha f(t) Q_{-1/2+i\mu}(\cosh t) \sinh t \, dt \\ & - Q_{-1/2+i\mu}(\cosh\alpha) \int_0^\alpha f(t) P_{-1/2+i\mu}(\cosh t) \sinh t \, dt. \end{aligned} \tag{18}$$

Since $u(\alpha)$ must be finite at $\alpha = 0$, the constant C_2 must be equal to zero. Now, proceeding as before, the solution of the pair (12) can be obtained.

(C) Next we consider the pair

$$\left.\begin{aligned} & \int_0^\infty \coth \pi\tau \, A(\tau) P_{-1/2+i\tau}(\cosh\alpha)\, d\tau = f(\alpha), \quad 0 \le \alpha \le a, \\ & \int_0^\infty [\lambda\tau + \mu(\tau^2 + c^2) \coth \pi\tau] \, A(\tau) \, P_{-1/2+i\tau}(\cosh\alpha)\, d\tau = 0, \ a \le \alpha < \infty, \end{aligned}\right\} \tag{19}$$

where λ, μ are real constants and $c > \frac{1}{2}$.

When $\mu = 0$, equation (19) reduces to the pair considered by Babloian[1] (writing $B(\tau) = \coth \pi\tau A(\tau)$).

When $\lambda = 0$ but $\mu \neq 0$, (19) reduces to the pair

$$\left.\begin{aligned} &\int_0^\infty \coth \pi\tau\, A(\tau) P_{-1/2+i\tau}(\cosh\alpha)\,d\tau = f(\alpha), \quad 0 \le \alpha \le a, \\ &\int_0^\infty (\tau^2 + c^2) \coth \pi\tau\, A(\tau) P_{-1/2+i\tau}(\cosh\alpha)\,d\tau = 0, \quad a \le \alpha < \infty, \end{aligned}\right\} \tag{20}$$

whose solution can be obtained in a similar manner as in case (A), and the solution in this case is given by (after utilizing the result (2) on p. 170 of Erdelyi *et al*[7] involving Legendre functions in the simplification)

$$\begin{aligned} A(\tau) = \tau \tanh^2 \pi\tau \int_0^a f(\alpha) P_{-1/2+i\tau}(\cosh\alpha) \sinh\alpha\, d\alpha \\ + \frac{\tau \tanh^2 \pi\tau . \sinh a . f(a)}{(\tau^2 + c^2) Q_{c-1/2}(\cosh a)} \Big[Q_{c-1/2}(\cosh a)\, P^1_{-1/2+i\tau}(\cosh a) \\ - P_{-1/2+i\tau}(\cosh a) Q^1_{c-1/2}(\cosh a) \Big]. \end{aligned} \tag{21}$$

When both λ and μ are not zero, we can write the second equation of (19) as

$$\int_0^\infty (\tau^2 + c^2) \left[1 + \frac{\lambda\tau}{\mu(\tau^2 + c^2)} \tanh \pi\tau \right] \coth \pi\tau\, A(\tau) P_{-1/2+i\tau}(\cosh\alpha)\,d\tau = 0, \quad a \le \alpha < \infty. \tag{22}$$

This is equivalent to the differential equation

$$\frac{1}{\sinh\alpha} \frac{d}{d\alpha} \left(\sinh\alpha \frac{du}{d\alpha} \right) + \left(\frac{1}{4} - c^2 \right) u = 0, \quad a \le \alpha < \infty, \tag{23}$$

where now

$$u(\alpha) = \int_0^\infty \left[1 + \frac{\lambda\tau}{\mu(\tau^2 + c^2)} \tanh \pi\tau \right] \coth \pi\tau\, A(\tau) P_{-1/2+i\tau}(\cosh\alpha)\,d\tau. \tag{24}$$

The solution of (23) is

$$u(\alpha) = C_1 P_{c-1/2}(\cosh\alpha) + C_2 Q_{c-1/2}(\cosh\alpha), \quad a \le \alpha < \infty.$$

Since $u(\alpha)$ must remain finite as α tends to infinity and $c > \frac{1}{2}$, we must have $C_1 = 0$. Thus, (24) gives

$$\begin{aligned} &\int_0^\infty A(\tau) \coth \pi\tau\, P_{-1/2+i\tau}(\cosh\alpha)\,d\tau \\ &= C_2 Q_{c-1/2}(\cosh\alpha) - \frac{\lambda}{\mu} \int_0^\infty \frac{\gamma A(\gamma)}{(\gamma^2 + c^2)} P_{-1/2+i\gamma}(\cosh\alpha)\,d\gamma, \quad a \le \alpha < \infty. \end{aligned} \tag{25}$$

The first equation of (19) and (25) produces $A(\tau)$ by Mehler–Fok inversion. After some simplification, we find

$$\frac{A(\tau)}{\tau} = C_2 \frac{\tanh^2 \pi\tau}{(\tau^2+c^2)} \sinh a \Big[Q_{c-1/2}(\cosh\alpha) P^1_{-1/2+i\tau}(\cosh a) - P_{-1/2+i\tau}(\cosh a) Q^1_{c-1/2}(\cosh a) \Big]$$

$$+ \tanh^2 \pi\tau \int_0^a f(\alpha) P_{-1/2+i\tau}(\cosh\alpha) \sinh\alpha \, d\alpha - \frac{\lambda \tanh^2 \pi\tau}{\mu} \Bigg[\frac{A(\tau)}{(\tau^2+c^2)\tanh \pi\tau}$$

$$- \int_0^\infty \frac{\gamma A(\gamma) \sinh a}{(\gamma^2+c^2)(\tau^2-\gamma^2)} \Big\{ P_{-1/2+i\tau}(\cosh a) P^1_{-1/2+i\gamma}(\cosh a)$$

$$- P_{-1/2+i\gamma}(\cosh a) P^1_{-1/2+i\tau}(\cosh a) \Big\} d\gamma \Bigg]. \tag{26}$$

The continuity requirement at $\alpha = a$ gives

$$C_2 = \frac{f(a)}{Q_{c-1/2}(\cosh a)} + \frac{\lambda}{\mu Q_{c-1/2}(\cosh a)} \int_0^\infty \frac{\gamma A(\gamma)}{(\gamma^2+c^2)} P_{-1/2+i\gamma}(\cosh a) \, d\gamma. \tag{27}$$

Using (27) in (26), we obtain the Fredholm integral equation (FIE) of the second kind for $A(\tau)$ as

$$\frac{A(\tau)}{\tau} - \lambda \int_0^\infty \frac{A(\gamma)}{\gamma} K(\gamma, \tau) \, d\gamma = L(\tau), \tag{28}$$

where

$$K(\gamma, \tau) = \frac{\gamma^2 \sinh a \cdot \tanh^2 \pi\tau}{(\gamma^2+c^2)\{\lambda\tau \tanh \pi\tau + \mu(\tau^2+c^2)\}}$$

$$\times \Bigg[\frac{P_{-1/2+i\gamma}(\cosh a)}{Q_{c-1/2}(\cosh a)} \Big\{ P^1_{-1/2+i\tau}(\cosh a) \, Q_{c-1/2}(\cosh a) - P_{-1/2+i\tau}(\cosh a)$$

$$\times Q^1_{c-1/2}(\cosh a) \Big\} + (\tau^2+c^2)(\tau^2-\gamma^2)^{-1} \Big\{ P^1_{-1/2+i\gamma}(\cosh a) P_{-1/2+i\tau}(\cosh a)$$

$$- P_{-1/2+i\gamma}(\cosh a) P^1_{-1/2+i\tau}(\cosh a) \Big\} \Bigg]$$

and

$$L(\tau) = \frac{\mu \tanh^2 \pi\tau}{\lambda\tau \tanh \pi\tau + \mu(\tau^2+c^2)} \Bigg[\frac{\sinh a}{Q_{c-1/2}(\cosh a)} \Big\{ P^1_{-1/2+i\tau}(\cosh a) Q_{c-1/2}(\cosh a)$$

$$- P_{-1/2+i\tau}(\cosh a) Q^1_{c-1/2}(\cosh a) \Big\} f(a) + (\tau^2+c^2) \int_0^a f(\alpha) P_{-1/2+i\tau}(\cosh\alpha) \sinh\alpha \, d\alpha \Bigg].$$

3. Dual integral equation involving associated Legendre function as kernel

In this section, we consider four pairs of dual integral equations with associated Legendre function as kernel.

(A) The first pair is

$$\left.\begin{aligned}&\int_0^\infty A(\tau)P^{\mu}_{-1/2+i\tau}(\cosh\alpha)\,d\tau = f(\alpha), \quad 0\le\alpha\le a,\\ &\int_0^\infty \tau\,\Gamma(\tfrac{1}{2}-\mu+i\tau)\Gamma(\tfrac{1}{2}-\mu-i\tau)\sinh\pi\tau\,A(\tau)P^{\mu}_{-1/2+i\tau}(\cosh\alpha)\,d\tau = 0, \quad a\le\alpha<\infty,\end{aligned}\right\} \tag{29}$$

where $-\frac{1}{2} < \operatorname{Re}\mu \le 0$.

To solve (29), we assume the right-hand side of the second equation to be equal to the unknown function $\phi(\alpha)$ for $0 \le \alpha \le a$, so that by applying generalized Mehler–Fok inversion formulae (cf. (2.14) and (2.15) of Pathak[3]) we find

$$\pi A(\tau) = \int_0^a \phi(\beta)P^{\mu}_{-1/2+i\tau}(\cosh\beta)\sinh\beta\,d\beta. \tag{30}$$

Using (30), then interchanging the order of integration and using the result (which can be established by using the integral representation of associated Legendre function)

$$\begin{aligned}&\int_0^\infty P^{\mu}_{-1/2+i\tau}(\cosh\alpha)P^{\mu}_{-1/2+i\tau}(\cosh\beta)\,d\tau\\ &\quad=\left[\Gamma\left(\frac{1}{2}-\mu\right)\right]^{-2}\left[\sinh\alpha\sinh\beta\right]^{\mu}\\ &\quad\times\int_0^{\min(\alpha,\beta)}\frac{dt}{\{(\cosh\alpha-\cosh t)(\cosh\beta-\cosh t)\}^{1/2+\mu}}\left(\operatorname{Re}\mu<\frac{1}{2}\right),\end{aligned} \tag{31}$$

in the first equation of (29), we obtain the Abel's integral equation

$$\begin{aligned}&\int_0^\alpha\frac{dt}{(\cosh\alpha-\cosh t)^{1/2+\mu}}\int_t^a\frac{\phi(\beta)\sinh^{1+\mu}\beta}{(\cosh\beta-\cosh t)^{1/2+\mu}}\,d\beta\\ &\quad=\frac{\pi[\Gamma(\frac{1}{2}-\mu)]^2 f(\alpha)}{\sinh^{\mu}\alpha};\quad 0\le\alpha\le a,\quad -\frac{1}{2}<\operatorname{Re}\mu\le 0,\end{aligned} \tag{32}$$

so that

$$\int_x^a\frac{\phi(\beta)\sinh^{1+\mu}\beta}{(\cosh\beta-\cosh x)^{1/2+\mu}}d\beta=\cos\mu\pi\frac{d}{dx}\int_0^x\frac{[\Gamma(\frac{1}{2}-\mu)]^2 f(t)\sinh^{1-\mu}t}{(\cosh x-\cosh t)^{1/2-\mu}}dt=\Psi(x),\ 0\le x\le a. \tag{33}$$

Another Abel's inversion gives

$$\phi(x)\sinh^{1+\mu}x=-\frac{\cos\mu\pi}{\pi}\int_x^a\frac{\Psi(t)\sinh t}{(\cosh t-\cosh x)^{1/2-\mu}}\,\mathrm{d}t. \tag{34}$$

Hence, the solution of the dual integral equations (29) is obtained, after some elementary calculations and using the result (2.12) of Pathak[3] as

$$A(\tau)=\frac{\sqrt{2}}{\pi\sqrt{\pi}}\Gamma(\tfrac{1}{2}-\mu)\cos\mu\pi\int_0^a\cos x\tau\left[\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\frac{\sinh^{1-\mu}\alpha\,f(\alpha)}{(\cosh x-\cosh\alpha)^{1/2-\mu}}\mathrm{d}\alpha\right]\mathrm{d}x,\quad -\frac{1}{2}<\mathrm{Re}\,\mu\le 0. \tag{35}$$

This result was earlier obtained by Pathak[3]. For $\mu=0$, this also reduces to the known result obtained by Babloian[1].

(B) Next we consider the pair

$$\left.\begin{aligned}&\int_0^\infty\tau^2A(\tau)P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau=f(\alpha),\quad 0\le\alpha\le a,\\&\int_0^\infty\tau\,\Gamma(\tfrac{1}{2}-\mu+\mathrm{i}\tau)\Gamma(\tfrac{1}{2}-\mu-\mathrm{i}\tau)\sinh\pi\tau\,A(\tau)P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau=0,\quad a\le\alpha<\infty,\end{aligned}\right\} \tag{36}$$

where $-\frac{1}{2}<\mathrm{Re}\,\mu\le 0$.

As before, we assume the right-hand side of the second equation (36) to be the unknown function $\phi(\alpha)$ for $0\le\alpha\le a$. Then by the generalized Mehler–Fok inversion formulae (cf. (2.14) and (2.15) of Pathak[3]), we obtain

$$\pi A(\tau)=\int_0^a\phi(\alpha)P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\sinh\alpha\,\mathrm{d}\alpha. \tag{37}$$

Substituting $A(\tau)$ in the first equation of (36), we have

$$\int_0^\infty\tau^2P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\left[\int_0^a\phi(\beta)P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh\beta)\sinh\beta\,\mathrm{d}\beta\right]\mathrm{d}\tau=\pi f(\alpha),\quad 0\le\alpha\le a. \tag{38}$$

This is equivalent to the differential equation

$$\frac{1}{\sinh\alpha}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\sinh\alpha\frac{\mathrm{d}u}{\mathrm{d}\alpha}\right)+\left(\frac{1}{4}-\frac{\mu^2}{\sinh^2\alpha}\right)u=-\pi f(\alpha), \tag{39}$$

where now

$$u(\alpha)=\int_0^\infty P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\left[\int_0^a\phi(\beta)P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh\beta)\sinh\beta\,\mathrm{d}\beta\right]\mathrm{d}\tau,\quad 0\le\alpha\le a. \tag{40}$$

The solution of (39) is

$$u(\alpha)=C_1P^{\mu}_{-1/2}(\cosh\alpha)+C_2Q^{\mu}_{-1/2}(\cosh\alpha)+2^{-2\mu}\mathrm{e}^{\mathrm{i}\mu\pi}\frac{\Gamma(\frac{3}{4}-\frac{1}{2}\mu)\,\Gamma(\frac{1}{4}-\frac{1}{2}\mu)}{\Gamma(\frac{3}{4}+\frac{1}{2}\mu)\,\Gamma(\frac{1}{4}+\frac{1}{2}\mu)}$$

$$\times\left[P^{\mu}_{-1/2}(\cosh\alpha)\int_0^{\alpha}\sinh t\, Q^{\mu}_{-1/2}(\cosh t)f(t)\,dt - Q^{\mu}_{-1/2}(\cosh\alpha)\right.$$

$$\left.\times\int_0^{\alpha}\sinh t\, P^{\mu}_{-1/2}(\cosh t)f(t)\,dt\right],\quad 0\le\alpha\le a, \tag{41}$$

where C_1 and C_2 are arbitrary constants to be determined from physical considerations of the problems. Since $u(\alpha)$ must be bounded at $\alpha=0$ and $-\frac{1}{2}<\operatorname{Re}\mu\le 0$, C_2 must be zero. The constant C_1 will be determined later.

As before, (40) gives rise to

$$\int_0^{\alpha}\frac{dt}{(\cosh\alpha-\cosh t)^{1/2+\mu}}\int_t^{a}\frac{\phi(\beta)\sinh^{1+\mu}\beta}{(\cosh\beta-\cosh t)^{1/2+\mu}}\,d\beta=\frac{[\Gamma(\frac{1}{2}-\mu)]^2u(\alpha)}{\sinh^{\mu}\alpha}, \tag{42}$$

from which we find another Abel integral equation

$$\int_x^{a}\frac{\phi(\beta)\sinh^{1+\mu}\beta}{(\cosh\beta-\cosh x)^{1/2+\mu}}\,d\beta=\frac{\Gamma(\frac{1}{2}-\mu)}{\Gamma(\frac{1}{2}+\mu)}\frac{d}{dx}\int_0^{x}\frac{u(t)\sinh^{1-\mu}t}{(\cosh x-\cosh t)^{1/2-\mu}}\,dt$$

$$=\Psi(x),\ \text{say, for}\ -\tfrac{1}{2}<\operatorname{Re}\mu\le 0,\quad 0\le x\le a,$$

so that

$$\sinh^{1+\mu}x\,\phi(x)=-\frac{\cos\mu\pi}{\pi}\frac{d}{dx}\int_x^{a}\frac{\Psi(t)\sinh t}{(\cosh t-\cosh x)^{1/2-\mu}}\,dt. \tag{43}$$

This gives the complete solution of dual integral equations (36) provided the constant C_1 is determined. This constant C_1 is obtained from the fact that $\Psi(a)=0$ if $\phi(x)$ is to be continuous at $x=a$. Therefore, the constant C_1 is obtained from the equation

$$\int_0^{a}\frac{\frac{d}{dt}\{u(t)\sinh^{-\mu}t\}}{(\cosh a-\cosh t)^{1/2-\mu}}\,dt=0. \tag{44}$$

For $\mu=0$, the dual integral equations together with the solution are the same as in Section 2(A).

(C) Next, we consider the pair

$$\left.\begin{aligned}&\int_0^{\infty}(\tau^2-\nu^2)A(\tau)P^{\mu}_{-1/2+i\tau}(\cosh\alpha)\,d\tau=f(\alpha),\quad 0\le\alpha\le a,\\&\int_0^{\infty}\tau\,\Gamma(\tfrac{1}{2}-\mu+i\tau)\,\Gamma(\tfrac{1}{2}-\mu-i\tau)\sinh\pi\tau\,A(\tau)P^{\mu}_{-1/2+i\tau}(\cosh\alpha)\,d\tau=0,\quad a\le\alpha<\infty,\end{aligned}\right\} \tag{45}$$

where $-\frac{1}{2}<\operatorname{Re}\mu\le 0$ and ν is real.

As before, we assume the right-hand side of the second equation of (45) to be the unknown function $\phi(\alpha)$ for $0\le\alpha\le a$. By using the generalized Mehler–Fok inversion formulae cited above, we find

$$\pi A(\tau) = \int_0^a \phi(\beta) P^{\mu}_{-1/2+i\tau}(\cosh\beta) \sinh\beta \, d\beta.$$

Substituting this into the first equation of (45) we find

$$\int_0^{\infty} (\tau^2 - \nu^2) P^{\mu}_{-1/2+i\tau}(\cosh\alpha) \left[\int_0^a \sinh\beta \, P^{\mu}_{-1/2+i\tau}(\cosh\beta) \, \phi(\beta) \, d\beta \right] d\tau = \pi f(\alpha), \quad 0 \le \alpha \le a. \tag{46}$$

This is equivalent to the differential equation

$$\frac{1}{\sinh\alpha} \frac{d}{d\alpha} \left(\sinh\alpha \frac{du}{d\alpha} \right) + \left(\frac{1}{4} + \nu^2 - \frac{\mu^2}{\sinh^2\alpha} \right) u = -\pi f(\alpha) \tag{47}$$

where now

$$u(\alpha) = \int_0^{\infty} P^{\mu}_{-1/2+i\tau}(\cosh\alpha) \left[\int_0^a \phi(\beta) P^{\mu}_{-1/2+i\tau}(\cosh\beta) \sinh\beta \, d\beta \right] d\tau, \quad 0 \le \alpha \le a. \tag{48}$$

The solution of (47) is

$$u(\alpha) = C_1 P^{\mu}_{-1/2+i\nu}(\cosh\alpha) + C_2 Q^{\mu}_{-1/2+i\nu}(\cosh\alpha) + 2^{-2\mu} e^{-i\mu\pi} \frac{\Gamma(\frac{3}{4} + \frac{i\nu}{2} - \frac{\mu}{2}) \, \Gamma(\frac{1}{4} + \frac{i\nu}{2} - \frac{\mu}{2})}{\Gamma(\frac{3}{4} + \frac{i\nu}{2} + \frac{\mu}{2}) \, \Gamma(\frac{1}{4} + \frac{i\nu}{2} + \frac{\mu}{2})}$$

$$\times \left[P^{\mu}_{-1/2+i\nu}(\cosh\alpha) \int_0^{\alpha} \sinh t \, Q^{\mu}_{-1/2+i\nu}(\cosh t) f(t) \, dt \right.$$

$$\left. - Q^{\mu}_{-1/2+i\nu}(\cosh\alpha) \int_0^{\alpha} \sinh t \, P^{\mu}_{-1/2+i\nu}(\cosh t) f(t) \, dt \right]. \tag{49}$$

Since $u(\alpha)$ must be finite at $\alpha = 0$ and $-\frac{1}{2} < \operatorname{Re} \mu \le 0$, the constant $C_2 = 0$. Now proceeding as in Section 3(B) we obtain $\phi(x)$ $(0 \le x \le a)$ and the constant C_1 by the equations (43) and (44), respectively, with $u(\alpha)$ given by (49). This gives the complete solution of the dual integral equations (45).

(D) Finally, we consider the pair

$$\left.\begin{aligned} &\int_0^{\infty} \left[\Gamma(\tfrac{1}{2} - \mu + i\tau) \, \Gamma(\tfrac{1}{2} - \mu - i\tau) \right]^{-1} \operatorname{cosech} \pi\tau \, A(\tau) P^{\mu}_{-1/2+i\tau}(\cosh\tau) \, d\tau = f(\alpha), \quad 0 \le \alpha \le a \\ &\int_0^{\infty} \left[\lambda_1 \tau + \mu_1 (\tau^2 + c^2) \left\{ \Gamma(\tfrac{1}{2} - \mu + i\tau) \Gamma(\tfrac{1}{2} - \mu - i\tau) \right\}^{-1} \operatorname{cosech} \pi\tau \right] \\ &A(\tau) P^{\mu}_{-1/2+i\tau}(\cosh\alpha) \, d\tau = 0, \quad a \le \alpha < \infty, \end{aligned}\right\} \tag{50}$$

where $-\frac{1}{2} < \operatorname{Re} \mu \le 0$, λ_1, μ_1 are real and $c > \frac{1}{2}$.

For $\mu_1 = 0$, the equations (50) are reduced to the pair (29) by redefining $A(\tau)$ appropriately.

When $\lambda_1 = 0$ but $\mu_1 \neq 0$, the solution to the dual integral equations can be obtained as before and is given by (after utilizing the result (1) on p. 169 of Erdelyi *et al*[7] involving associated Legendre functions in the simplification)

$$\pi A(\tau) = \tau \sinh^2 \pi\tau \left[\Gamma(\tfrac{1}{2}-\mu+\mathrm{i}\tau)\,\Gamma(\tfrac{1}{2}-\mu-\mathrm{i}\tau)\right]^2$$
$$\times \left[\frac{f(a)}{(\tau^2+c^2)}\left\{(\mathrm{i}\tau - c)\cosh a\, P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh a)\, Q^{\mu}_{c-1/2}(\cosh a)\right.\right.$$
$$+\left(\mu + c - \tfrac{1}{2}\right) P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh a)\, Q^{\mu}_{c-3/2}(\cosh a)$$
$$\left.-\left(\mu + \mathrm{i}\tau - \tfrac{1}{2}\right) P^{\mu}_{-3/2+\mathrm{i}\tau}(\cosh a)\, Q^{\mu}_{c-1/2}(\cosh a)\right\}$$
$$\left.+\int_0^a f(\alpha)\, P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\sinh\alpha\,\mathrm{d}\alpha\right] \tag{51}$$

When both λ_1 and μ_1 are not zero, we can write the second equation of (50) as

$$\int_0^{\infty} (\tau^2+c^2)\left[\Gamma(\tfrac{1}{2}-\mu+\mathrm{i}\tau)\,\Gamma(\tfrac{1}{2}-\mu-\mathrm{i}\tau)\right]^{-1} \cdot \operatorname{cosech}\pi\tau$$
$$\times\left[1+\frac{\lambda_1\tau}{\mu_1(\tau^2+c^2)}\sinh\pi\tau\,\Gamma(\tfrac{1}{2}-\mu+\mathrm{i}\tau)\,\Gamma(\tfrac{1}{2}-\mu-\mathrm{i}\tau)\right]$$
$$\times A(\tau)\, P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau = 0, \quad a \leq \alpha < \infty. \tag{52}$$

As before, equation (52) is equivalent to the differential equation

$$\frac{1}{\sinh\alpha}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\sinh\alpha\frac{\mathrm{d}u}{\mathrm{d}\alpha}\right) + \left(\frac{1}{4} - c^2 - \frac{\mu^2}{\sinh^2\alpha}\right)u = 0, \tag{53}$$

where now

$$u(\alpha) = \int_0^{\infty}\left[1+\frac{\lambda_1\tau}{\mu_1(\tau^2+c^2)}\sinh\pi\tau\,\Gamma(\tfrac{1}{2}-\mu+\mathrm{i}\tau)\,\Gamma(\tfrac{1}{2}-\mu-\mathrm{i}\tau)\right]$$
$$\times\left[\Gamma(\tfrac{1}{2}-\mu+\mathrm{i}\tau)\,\Gamma(\tfrac{1}{2}-\mu-\mathrm{i}\tau)\right]^{-1} \operatorname{cosech}\pi\tau\, A(\tau) P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau, \quad a \leq \alpha < \infty. \tag{54}$$

The solution of (53) is given by

$$u(\alpha) = C_1 P^{\mu}_{c-1/2}(\cosh\alpha) + C_2 Q^{\mu}_{c-1/2}(\cosh\alpha).$$

Since $u(\alpha)$ must be finite as α tends to infinity and $c > \frac{1}{2}$, we must have $C_1 = 0$. Hence, by (54), we have

$$\int_0^{\infty}\left[\Gamma(\tfrac{1}{2}-\mu+\mathrm{i}\tau)\,\Gamma(\tfrac{1}{2}-\mu-\mathrm{i}\tau)\right]^{-1} \operatorname{cosech}\pi\tau\, A(\tau) P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau$$

$$= C_2 Q^{\mu}_{c-1/2}(\cosh\alpha) - \frac{\lambda_1}{\mu_1}\int_0^{\infty} \frac{\gamma A(\gamma)}{(\gamma^2+c^2)} P^{\mu}_{-1/2+i\gamma}(\cosh\alpha)\,d\gamma, \quad a \le \alpha < \infty. \tag{55}$$

Equation (55) and the first equation of (50) gives, after using the generalized Mehler–Fok inversion formulae cited above,

$$\pi\left[\Gamma(\tfrac{1}{2}-\mu+i\tau)\,\Gamma(\tfrac{1}{2}-\mu-i\tau)\right]^{-2} \operatorname{cosech}^2 \pi\tau \cdot \frac{A(\tau)}{\tau}$$

$$= C_2 \int_a^{\infty} P^{\mu}_{-1/2+i\tau}(\cosh\alpha) Q^{\mu}_{c-1/2}(\cosh\alpha)\,\sinh\alpha\,d\alpha + \int_0^a f(\alpha)\, P^{\mu}_{-1/2+i\tau}(\cosh\alpha)\sinh\alpha\,d\alpha$$

$$-\frac{\lambda_1}{\mu_1}\int_a^{\infty} P^{\mu}_{-1/2+i\tau}(\cosh\alpha)\sinh\alpha \left\{\int_0^{\infty} \frac{\gamma A(\gamma)}{(\gamma^2+c^2)} P^{\mu}_{-1/2+i\gamma}(\cosh\alpha)\,d\gamma\right\} d\alpha. \tag{56}$$

The continuity requirement at $\alpha = a$ gives

$$C_2 = \frac{f(a)}{Q^{\mu}_{c-1/2}(\cosh a)} + \frac{\lambda_1}{\mu_1 Q^{\mu}_{c-1/2}(\cosh a)} \int_0^{\infty} \frac{\gamma A(\gamma)}{(\gamma^2+c^2)} P^{\mu}_{-1/2+i\gamma}(\cosh a)\,d\gamma.$$

Substituting the value of C_2 in (56) and after some simplification, we find the FIE of the second kind for $A(\tau)$ as

$$\frac{A(\tau)}{\tau} - \lambda\int_0^{\infty} \frac{A(\gamma)}{\gamma} K(\gamma,\tau)\,d\gamma = B(\tau), \tag{57}$$

where

$$K(\gamma,\tau) = \frac{\gamma^2(\gamma^2+c^2)^{-1}\left[\Gamma(\tfrac{1}{2}-\mu+i\tau)\,\Gamma(\tfrac{1}{2}-\mu-i\tau)\right]^2 \sinh^2\pi\tau}{\pi\left\{\lambda_1\tau\,\Gamma(\tfrac{1}{2}-\mu+i\tau)\,\Gamma(\tfrac{1}{2}-\mu-i\tau)\sinh\pi\tau + \mu_1(\tau^2+c^2)\right\}}$$

$$\times\left[\frac{P^{\mu}_{-1/2+i\gamma}(\cosh a)}{Q^{\mu}_{c-1/2}(\cosh a)}\left\{(i\tau-c)\cdot\cosh a\cdot P^{\mu}_{-1/2+i\tau}(\cosh a)\,Q^{\mu}_{c-1/2}(\cosh a)\right.\right.$$

$$+\left(\mu+c-\tfrac{1}{2}\right)P^{\mu}_{-1/2+i\tau}(\cosh a)\,Q^{\mu}_{c-1/2}(\cosh a)$$

$$\left.-\left(\mu+i\tau-\tfrac{1}{2}\right)P^{\mu}_{-3/2+i\tau}(\cosh a)\,Q^{\mu}_{c-1/2}(\cosh a)\right\}$$

$$+(\tau^2+c^2)(\gamma^2-\tau^2)^{-1}\left\{i(\tau-\gamma)\cosh a\,P^{\mu}_{-1/2+i\tau}(\cosh a)\right.$$

$$\times P^{\mu}_{-1/2+i\gamma}(\cosh a)+\left(\mu+i\gamma-\tfrac{1}{2}\right)P^{\mu}_{-1/2+i\tau}(\cosh a)\,P^{\mu}_{-3/2+i\gamma}(\cosh a)$$

$$\left.\left.-\left(\mu+i\tau-\tfrac{1}{2}\right)P^{\mu}_{-3/2+i\tau}(\cosh a)\,P^{\mu}_{-1/2+i\gamma}(\cosh a)\right\}\right]$$

and

$$B(\tau)=\frac{\mu_1\left[\Gamma(\frac{1}{2}-\mu+\mathrm{i}\tau)\,\Gamma(\frac{1}{2}-\mu-\mathrm{i}\tau)\right]^2\sinh^2\pi\tau}{\pi\left\{\lambda_1\tau\,\Gamma(\frac{1}{2}-\mu+\mathrm{i}\tau)\,\Gamma(\frac{1}{2}-\mu-\mathrm{i}\tau)\sinh\pi\tau+\mu_1(\tau^2+c^2)\right\}}$$

$$\times\left[\int_0^a(\tau^2+c^2)f(\alpha)P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\sinh\alpha\,\mathrm{d}\alpha+\frac{f(a)}{Q^{\mu}_{c-1/2}(\cosh a)}\right.$$

$$\times\left\{(\mathrm{i}\tau-c)\cosh a\;P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh a)\,Q^{\mu}_{c-1/2}(\cosh a)\right.$$

$$+\left(\mu+c-\tfrac{1}{2}\right)P^{\mu}_{-1/2+\mathrm{i}\tau}(\cosh a)\,Q^{\mu}_{c-3/2}(\cosh a)$$

$$\left.\left.-\left(\mu+\mathrm{i}\tau-\tfrac{1}{2}\right)P^{\mu}_{-3/2+\mathrm{i}\tau}(\cosh a)\,Q^{\mu}_{c-1/2}(\cosh a)\right\}\right].$$

It is easy to verify that by putting $\mu=0$ in the dual integral equations of Section 3, some results obtained in Section 2 for the solutions of dual integral equations involving Legendre function as kernel are recovered.

4. An example

As application of the dual integral equations, we consider a mixed boundary value problem involving a half-space due to torsion of an attached rigid annular die. Using toroidal coordinates (α, β, θ) where

$$r=(x^2+y^2)^{1/2}=\frac{c\sinh\alpha}{\cosh\alpha+\cos\beta},\quad z=\frac{c\sin\beta}{\cosh\alpha+\cos\beta}\quad(c>0),$$

$0\le\alpha<\infty$, $0\le\beta\le\pi$, the half-space is $z\ge 0$, and the die is represented by $z=0$, $\alpha_0<\alpha<\infty$. The state of stress and strain does not depend on the angular coordinate θ and is determined by the non-zero component of displacements, $u_\theta=u(r,z)$. $u(r,z)$ satisfies

$$\frac{\partial^2u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2u}{\partial z^2}=0,\quad z\ge 0$$

with the boundary conditions

$$\left.\frac{\partial u}{\partial\beta}\right|_{\beta=0}=0,\quad 0\le\alpha\le\alpha_0,\quad u|_{\beta=0}=kr,\quad \alpha_0<\alpha<\infty,\quad u|_{\beta=\pi}=0$$

where k is the angle of rotation of the die.

We seek a solution of the above mixed boundary value problem in the form

$$u=kc(\cosh\alpha+\cos\beta)^{1/2}\int_0^{\infty}A(\tau)\frac{\sinh(\pi-\beta)\tau}{\cosh\pi\tau}P^1_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau,$$

then the boundary conditions produce the dual integral equations

$$\int_0^\infty \tau\, A(\tau) P^1_{-1/2+i\tau}(\cosh\alpha)\, d\tau = 0, \quad 0 \le \alpha < \alpha_0$$

$$\int_0^\infty \tanh \pi\tau\, A(\tau) P^1_{-1/2+i\tau}(\cosh\alpha)\, d\tau = \frac{\tanh \alpha/2}{(\cosh\alpha + 1)^{1/2}}, \quad \alpha_0 < \alpha < \infty,$$

for the unknown function $A(\tau)$. Using the formula $P^1_{-1/2+i\tau}(\cosh\alpha) = d/d\alpha\, P_{-1/2+i\tau}(\cosh\alpha)$ and taking $\tau\beta(\tau) = A(\tau)$, we obtain the dual integral equations discussed in Section 2(A).

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