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Effect of capillarity on fourth-order nonlinear evolution equations for two Stokes wave trains in deep water

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Abstract

Fourth-order nonlinear evolution equations, which are good starting point for the study of nonlinear water waves, are derived for a deep-water surface capillary gravity wave packet in the presence of a second wave packet. Stability analysis is made for a uniform wave train in the presence of a second uniform wave train. Graphs are plotted for maximum growth rate of instability and for wave number at marginal stability against wave steepness. Significant deviations are noticed from the results obtained from the third-order evolution equations which consist of two coupled nonlinear Schrodinger equations.

Key words: Evolution equation, capillary gravity wave, stability.

1. Introduction

A reasonable approach of studying stability of finite amplitude surface gravity waves in deep water is through the application of lowest order nonlinear evolution equation which is the nonlinear Schrödinger equation. This analysis is suitable for small wave steepness and for long wavelength perturbations. But, for wave steepness greater than 0.15, predictions from nonlinear Schrödinger equation do not agree with the exact results of Longuet–Higgins[1,2]. Dysthe[3] has shown that stability analysis made from fourth-order nonlinear evolution equation which is one order higher than nonlinear Schrödinger equation, gives results consistent with the exact results of Longuet–Higgins[1,2] and with the experimental results of Benjamin and Feir[4] for wave steepness up to 0.25. The fourth-order effects give a surprising improvement compared to ordinary nonlinear Schrödinger effects in many respects, and some of these points have been elaborated by Janssen[5]. The dominant new effect that comes in the fourth order is the influence of wave-induced mean flow and this produces a significant deviation in the stability character. So it can be concluded that a fourth-order evolution equation is a good starting point for studying nonlinear effects of surface gravity waves in deep water. Fourth-order nonlinear evolution equations for deep-water surface gravity waves in different contexts and stability analysis made from them were derived by several authors[5–8].

These analyses are for a single wave only. Stability analysis of a surface gravity wave in deep water in the presence of a second wave has been made by Roskes[9] based on the lowest order nonlinear evolution equation which consists of two coupled nonlinear Schrödinger equations. In his investigation, modulational perturbation is restricted to a direction along which group velocity projections of the two waves overlap and it is argued that the modulation will grow at a faster rate along this direction when the angle between two propagation directions of two waves lies between 0 and 70.5°.

The stability analysis of two Stokes wave trains in deep water starting from fourth-order nonlinear evolution equations has been made in a recent paper by Dhar and Das[10]. An extension of this paper to include capillarity is made in this paper. It is found that there is instability when θ lies in the intervals $0° \leq \theta < 74.53°$ and $80.21° < \theta \leq 180°$ for waves of wavelength 0.2 cm and in the intervals $0° \leq \theta < 31.62°$ and $149° < \theta \leq 180°$ for waves of wavelength 5 cm, where θ is the angle between the propagation directions of the two waves. Graphs are plotted for maximum growth rate of instability and for wave number at marginal stability against wave steepness for some different values of θ and for two values of k_0 ($k_0 = 31.32$ cm^{-1}, 1.25 cm^{-1}).

2. Basic equation

We take the free surface of the water in the undisturbed state as the $z = 0$ plane. We consider that the two waves move in the x–y plane with wave numbers $\vec{k}_1$ and $\vec{k}_2$, respectively. We take the x axis in a direction along which group velocity projections of the two waves overlap and consider the modulations along this line only. Let $z = \zeta(x, y, t)$ be the equation of the free surface at any time t in the perturbed state. We introduce the dimensionless quantities φ^*, ζ^*, (x^*, y^*, z^*), t^*, s^* which are, respectively, the perturbed velocity potential in water, elevation of the free surface, space coordinates, time and surface tension. These dimensionless quantities are related to the corresponding dimensional quantities by the following

$$\varphi^* = (k_o^3/g)^{1/2}\varphi,\ \zeta^* = k_o\zeta,\ t^* = (gk_o)^{1/2}\, t, \tag{1}$$

$$(x^*, y^*, z^*) = (k_o x, k_o y, k_o z),\ s^* = Tk_o^2/g,$$

where k_o is some characteristic wave number. In future, all these quantities will be written in their dimensionless form but with their stars dropped.

The perturbed velocity potential φ, from which perturbed velocity $\vec{u}$ of water can be obtained from the relation $\vec{u} = \vec{\nabla}\varphi$, satisfies the following Laplace's equation

$$\nabla^2\varphi = 0 \text{ in } -\infty < z < \zeta. \tag{2}$$

The kinematic boundary condition to be satisfied at the free surface is

$$\frac{\partial\varphi}{\partial z} - \frac{\partial\zeta}{\partial t} = \frac{\partial\varphi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\zeta}{\partial y}, \text{ when } z = \zeta. \tag{3}$$

The condition of continuity of pressure at the free surface gives

$$\frac{\partial\varphi}{\partial t} + \zeta = -\frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] + s\left[1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right]^{-3/2}$$

$$\cdot\left[\left(\frac{\partial\zeta}{\partial x}\right)^2\frac{\partial^2\zeta}{\partial y^2} + \left(\frac{\partial\zeta}{\partial y}\right)^2\frac{\partial^2\zeta}{\partial x^2} - 2\frac{\partial\zeta}{\partial x}\frac{\partial\zeta}{\partial y}\frac{\partial^2\zeta}{\partial x\partial y} + \frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right], \tag{4}$$

when $z = \zeta$.

Further, φ should satisfy the following condition at infinity

$$\varphi \to 0 \text{ as } z \to -\infty. \tag{5}$$

The linear dispersion relations for the first and second waves with wave number $\vec{k}_i$ and frequency ω_i, $(i = 1, 2)$, where $i = 1$ and $i = 2$ correspond, respectively, to the first and second wave, are given by

$$\omega_i^2 - (k_{ix}^2 + k_{iy}^2)^{1/2} - s(k_{ix}^2 + k_{iy}^2)^{3/2} = 0, \ (i = 1,2). \tag{6}$$

A case of interest occurs if we make the simplifying assumption that the wave numbers of the two waves are the same, which implies that $|\vec{k}_1| = |\vec{k}_2| = k$ (say). Let this common wave number be equal to k_o, the characteristic wave number. So we have $k = |\vec{k}_1| = |\vec{k}_2| = 1$ and consequently the linear dispersion relation (6) using $k = 1$ becomes

$$\lambda(\omega) \equiv \omega^2 - 1 - s = 0. \tag{7}$$

3. Evolution equation

By a standard procedure (Dhar and Das[10]) we find that $\zeta_1 = \epsilon\zeta_{101} + \epsilon^2\zeta_{102}$, $\zeta_2 = \epsilon\zeta_{011} + \epsilon^2\zeta_{012}$, where ζ_1 and ζ_2 are the complex amplitudes of the first and second wave, respectively, for the elevation of the free surface from the undisturbed state, satisfy the following fourth-order nonlinear evolution equations.

$$i\frac{\partial\zeta_1}{\partial\tau} + \gamma_{11}\frac{\partial^2\zeta_1}{\partial\xi^2} + i\gamma_{12}\frac{\partial^3\zeta_1}{\partial\xi^3} = \zeta_1(\beta_{11}|\zeta_1|^2 + \beta_{12}|\zeta_2|^2)$$

$$+i\alpha_{11}\zeta_1\zeta_1^*\frac{\partial\zeta_1}{\partial\xi} + i\alpha_{12}\zeta_1^2\frac{\partial\zeta_1^*}{\partial\xi} + i\lambda_{11}\zeta_2\zeta_2^*\frac{\partial\zeta_1}{\partial\xi} + i\lambda_{12}\zeta_1\zeta_2^*\frac{\partial\zeta_2}{\partial\xi}$$

$$+ i\lambda_{13}\zeta_1\zeta_2\frac{\partial\zeta_2^*}{\partial\xi} + \mu\zeta_1 H\frac{\partial}{\partial\xi}(\zeta_1\zeta_1^*) + \mu\zeta_1 H\frac{\partial}{\partial\xi}(\zeta_2\zeta_2^*) \tag{8}$$

and

$$i\frac{\partial\zeta_2}{\partial\tau} + \gamma_{11}\frac{\partial^2\zeta_2}{\partial\xi^2} + i\gamma_{12}\frac{\partial^3\zeta_2}{\partial\xi^3} = \zeta_2(\beta_{12}\,|\,\zeta_1\,|^2 + \beta_{11}\,|\,\zeta_2\,|^2)$$

$$+ i\alpha_{11}\zeta_2\zeta_2^*\frac{\partial\zeta_2}{\partial\xi} + i\alpha_{12}\zeta_2^2\frac{\partial\zeta_2^*}{\partial\xi} + i\lambda_{11}\zeta_1\zeta_1^*\frac{\partial\zeta_2}{\partial\xi} + i\lambda_{12}\zeta_2\zeta_1^*\frac{\partial\zeta_1}{\partial\xi}$$

$$+ i\lambda_{13}\zeta_2\zeta_1\frac{\partial\zeta_1^*}{\partial\xi} + \mu\zeta_2 H\frac{\partial}{\partial\xi}(\zeta_2\zeta_2^*) + \mu\zeta_2 H\frac{\partial}{\partial\xi}(\zeta_1\zeta_1^*) \tag{9}$$

where the coefficients γ_{11}, γ_{12}, β_{11}, β_{12}, α_{11}, α_{12}, λ_{11}, λ_{12}, λ_{13}, μ are given in Appendix A, and where

$$\xi = \epsilon(x - V.t),\ \tau = \epsilon^2 t \tag{10}$$

where V is the component of group velocity of any of the two waves along the x axis and is given by $V = \cos(\theta/2)\ d\omega/dk$, the derivative $d\omega/dk$ is to be evaluated from (6), and H is the Hilbert's transform given by

$$H(\Psi) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\Psi(\xi')d\xi'}{\xi' - \xi}. \tag{11}$$

For $\theta = 0$, and in the absence of the second wave the two coupled evolution equations given by (8) and (9) reduce to a single equation which becomes the same as equation (2.20) of Hogan[7] for one-dimensional case.

4. Stability analysis

The evolution equations (8), (9) admit Stokes wave solutions

$$\zeta_1 = \frac{\alpha_1}{2}\exp(i\Delta\omega_1\tau),\ \zeta_2 = \frac{\alpha_2}{2}\exp(i\Delta\omega_2\tau) \tag{12}$$

where α_1, α_2 are real constants and $\Delta\omega_1$, $\Delta\omega_2$ are the nonlinear frequency shifts of the two waves. As the two waves have the same wave number equal to 1, *i.e.*, $|\vec{k}_1| = |\vec{k}_2| = 1$, the change in the phase speeds of the two waves Δc_1 and Δc_2 are given by

$$\Delta c_1 = \frac{\Delta\omega_1}{|\vec{k}_1|} = \Delta\omega_1 = -\frac{1}{4}(\beta_{11}\alpha_1^2 + \beta_{12}\alpha_2^2)$$

$$\Delta c_2 = \frac{\Delta\omega^2}{|\vec{k}_2|} = \Delta\omega_2 = -\frac{1}{4}(\beta_{11}\alpha_2^2 + \beta_{12}\alpha_1^2). \tag{13}$$

The change in the phase speeds of each wave train is therefore made up of two parts. The first correction to c_1 is given by $-\frac{1}{4}\beta_{11}\alpha_1^2$ which is the well-known Stokes correction. This term is due to the nonlinearity of the wave train itself and is present even if the other wave train is absent. The second correction is given by $-\frac{1}{4}\beta_{12}\alpha_2^2$ and is entirely due to the presence of the other wave train. It is of the same order as the usual Stokes correction.

If in the two expressions given by (13) for the change in phase speeds of the two waves we set $s = 0$, we recover the expressions derived by Longuet-Higgins and Phillips[11] for the change in the phase speeds of the two waves in the absence of surface tension. Only we are to set $\sigma_1 = \omega$, $\sigma_2 = \omega$, $\beta = 180°-\theta/2$ in their expressions and also to make corrections in their expressions as noted by Willebrand[12].

In the particular case, $\sigma_1 = \sigma_2 = \omega$, $k_1 = k_2 = k$ and $\beta = 180° - \theta/2$, the expression for the change in the phase speed given by equation (5.25) of Hogan *et al*[13] reduce to an expression for Δc_2 given by our equation (13) for the change in the phase speed of the second wave in the presence of the first wave.

To study modulational instability of this uniform wave train, we introduce the following perturbations in the uniform solution.

$$\zeta_1 = \frac{\alpha_1}{2}(1 + \zeta_1') \exp i(\Delta\omega_1\tau + \theta_1') \tag{14}$$

$$\zeta_2 = \frac{\alpha_2}{2}(1 + \zeta_2') \exp i(\Delta\omega_2\tau + \theta_2') \tag{15}$$

where ζ_1', ζ_2', θ_1', θ_2' are small perturbations in amplitudes and phases, respectively, and are real. Substituting (14) and (15) in (8) and (9), linearizing and then assuming space time dependence of ζ_i', θ_i', $(i = 1,2)$ to be of the form $\exp i(\lambda\xi - \Omega'\tau)$, we arrive at the following nonlinear dispersion relation, the details of derivation of which are given in Appendix B.

$$\bar{P}_1 = -\left[\alpha^2(\alpha_{11} + \lambda_{11} + \lambda_{12})\frac{\lambda}{4}\right] \pm \left[\bar{P}_2\left\{\dot{\bar{P}}_2 + \frac{\alpha^2}{2}(\beta_{11} + \beta_{12} - 2\mu\lambda)\right\}\right]^{1/2} \tag{16}$$

where

$$\bar{P}_1 = \Omega - V\lambda + \gamma_{12}\lambda^3 \tag{17}$$

$$\bar{P}_2 = \gamma_{11}\lambda^2$$

and $\quad \Omega = \Omega' + V\lambda$.

From (16) it follows that for instability we must have

$$\gamma_{11}\lambda^2\left[\gamma_{11}\lambda^2 + \frac{\alpha^2}{2}(\beta_{11} + \beta_{12} - 2\mu\lambda)\right] < 0 \tag{18}$$

and if this condition is met, then the maximum growth rate γ_M is given by

$$\gamma_M = \frac{-(\beta_{11}+\beta_{12})\alpha^2}{4}\left[1+\frac{\mu\alpha}{\sqrt{-(\beta_{11}+\beta_{12})\gamma_{11}}}\right]$$

when $\gamma_{11} > 0,\ \beta_{11}+\beta_{12} < 0,$

$$= \frac{(\beta_{11}+\beta_{12})\alpha^2}{4}\left[1-\frac{\mu\alpha}{\sqrt{-(\beta_{11}+\beta_{12})\gamma_{11}}}\right]$$

when $\gamma_{11} < 0,\ \beta_{11}+\beta_{12} > 0.$ (19)

At marginal stability we have

$$\bar{P}_2\left[\bar{P}_2+\frac{\alpha^2}{2}(\beta_{11}+\beta_{12}-2\mu\lambda)\right] = 0$$

and this gives the following expression for λ at marginal stability.

$$\lambda = \sqrt{\frac{-(\beta_{11}+\beta_{12})}{2\gamma_{11}}}\,\alpha\left[1+\frac{\mu\alpha}{\sqrt{-2(\beta_{11}+\beta_{12})\gamma_{11}}}\right]$$

when $\gamma_{11} > 0,\ \beta_{11}+\beta_{12} < 0,$

$$= \sqrt{\frac{-(\beta_{11}+\beta_{12})}{2\gamma_{11}}}\,\alpha\left[1-\frac{\mu\alpha}{\sqrt{-2(\beta_{11}+\beta_{12})\gamma_{11}}}\right]$$

when $\gamma_{11} < 0,\ \beta_{11}+\beta_{12} > 0,$ (20)

where α is the common amplitude of the two waves.

From instability condition (18) it is found that there is instability when θ lies in the intervals $0° \le \theta < 74.53°$ and $80.21° < \theta \le 180°$ for waves of wavelength 0.2 cm and in the intervals $0° \le \theta < 31.62°$ and $149° < \theta \le 180°$ for waves of wavelength 5 cm.

In Figs 1(a) and 2(a) the maximum growth rate of instability γ_M and in Figs 1(b) and 2(b) the wave number λ at marginal stability have been plotted against wave steepness α for some different values of θ for the case of air water surface and for $k_o = 31.32\ \text{cm}^{-1}$ and $k_o = 1.25\ \text{cm}^{-1}$. For waves of wavelength 5 cm and for $0° \le \theta < 31.62°$ it is observed from the graph that the fourth-order effect produces a decrease in the growth rate giving a stabilizing influence and also a shrinkage of the instability region in λ–α plane. Effects reverse to these are observed for waves of wavelength 0.2 cm. There is no fourth-order contribution for $\theta = 180°$, since, when $\theta = 180°$, the Hilbert transform terms, which only contribute at the fourth order, vanish identically.

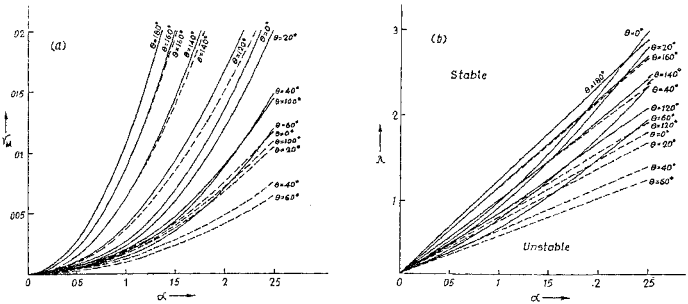

FIG. 1. k_o = 31.32 cm^{-1}. (a) Maximum growth rate γ_M as a function of dimensionless wave steepness α. (b) Plot of perturbed wave number λ at marginal stability against wave steepness α. ---- third-order result, — fourth-order result.

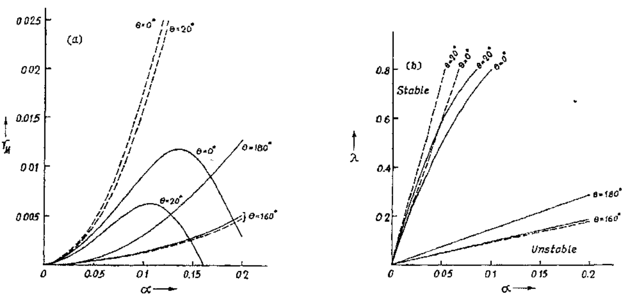

FIG. 2. k_o = 1.25 cm^{-1}. (a) Maximum growth rate γ_M as a function of dimensionless wave steepness α. (b) Plot of perturbed wave number λ at marginal stability against wave steepness α. ----third-order result, —fourth-order result.

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Appendix A

$$\gamma_{11} = \frac{1}{4}(1+s)^{-1}\left[3s(1+\cos^2\frac{\theta}{2}) + \sin^2\frac{\theta}{2}\right] - \frac{1}{8}(1+s)^{-2}(1+3s)^2\cos^2\frac{\theta}{2}$$

$$\gamma_{12} = \frac{1}{2}(1+s)^{-1}\gamma_{11}(1+3s)\cos\frac{\theta}{2} + \frac{1}{4}(1+s)^{-1}\left[\cos\frac{\theta}{2}\sin^2\frac{\theta}{2} - 2s\cos\frac{\theta}{2}\left(1+\sin^2\frac{\theta}{2}\right)\right]$$

$$\beta_{11} = \frac{1}{4}(1-2s)^{-1}(1+s)^{-1}(2s^2+s+8)$$

$$\beta_{12} = \frac{2}{\delta_1}\cos\frac{\theta}{2}\left(1-2\cos\frac{\theta}{2}\right)\left[(1+s)\left(1+\sin^2\frac{\theta}{2}\right) - \cos^2\frac{\theta}{2}\left(1+4s\cos^2\frac{\theta}{2}\right)\right]$$
$$-2(1+s)\sin^2\frac{\theta}{2}\left[\sin^2\frac{\theta}{2}\left((1+4s\sin^2\frac{\theta}{2}\right)^{-1} - \frac{1}{\delta_1}\left\{\cos\frac{\theta}{\delta_2}\left(1+\sin^2\frac{\theta}{2}\right)\right.\right.$$
$$\left.\left.-2\cos^2\frac{\theta}{2}\right\}\right] - \frac{s}{2}(1+s)^{-1}(\sin^2\theta + 3\cos^2\theta) + 3\cos\theta + 1$$

$$\alpha_{11} = \frac{3}{4}(1+s)^{-2}(1-2s)^{-2}(4s^4+4s^3-9s^2+s-8)\cos\frac{\theta}{2}$$

$$\alpha_{12} = \frac{1}{8}(1+s)^{-2}(1-2s)^{-1}(s-1)(2s^2+s+8)\cos\frac{\theta}{2}$$

$$\lambda_{11} = \frac{1}{2}(\delta_2+\delta_3) - \frac{1}{2}(1+3s)\left(1+4s\sin^2\frac{\theta}{2}\right)^{-1}\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)$$

$$-\frac{1}{2\delta_1}(1+s)\cos\frac{\theta}{2}\left(3\cos\frac{\theta}{2}+4\right)\left(1+\sin^2\frac{\theta}{2}-2\cos\frac{\theta}{2}\right)$$

$$-(1+s)\left(1+4s\sin^2\frac{\theta}{2}\right)^{-1}\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\left(1-\cos\frac{\theta}{2}\right)$$

$$-\frac{1}{2}(1+s)^{-1}(1+3s)\cos\frac{\theta}{2}\left(5-7\cos\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)$$

$$+\frac{1}{4}\cos\frac{\theta}{2}\left(2\cos^3\frac{\theta}{2}-1\right)+s(1+s)^{-1}\cos\frac{\theta}{2}(2\cos^2\theta+4\cos\theta+3)$$

$$\lambda_{12}=\frac{1}{2}(\delta_2+\delta_3)-\frac{1}{2\delta_1}(1+s)\cos\frac{\theta}{2}\left(\cos^2\frac{\theta}{2}+2\cos\frac{\theta}{2}+4\right)\left(1+\sin^2\frac{\theta}{2}\right.$$

$$\left.-2\cos\frac{\theta}{2}\right)+s(1+s)^{-1}\cos\frac{\theta}{2}(2\cos^2\theta+4\cos\theta+3)$$

$$+\frac{1}{4}\cos\frac{\theta}{2}\left(2\cos^3\frac{\theta}{2}-2\cos^2\frac{\theta}{2}-7\right)-\frac{1}{2}(1+s)^{-1}(1+3s)$$

$$\times\cos\frac{\theta}{2}\left(5-7\cos\frac{\theta}{2}-2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\right)-\frac{1}{2}(1+3s)$$

$$\times\left(1+4s\sin^2\frac{\theta}{2}\right)^{-1}\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\left(5-2\cos^2\frac{\theta}{2}\right)$$

$$\lambda_{13}=\frac{1}{2}(1+s)\left(1-4s\sin^2\frac{\theta}{2}\right)^{-1}\cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-2\sin\frac{\theta}{2}-1\right)$$

$$-2(1+3s)\sin^2\frac{\theta}{2}\cos^3\frac{\theta}{2}\left(1+4s\sin^2\frac{\theta}{2}\right)^{-1}+\frac{1}{4}\cos\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-5\right)$$

$$+(1+s)^{-1}\cos\frac{\theta}{2}\left[s(2\cos^2\theta+1)-(1+3s)\left(1-2\cos\frac{\theta}{2}\right.\right.$$

$$\left.\left.+\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}\right)\right]+\frac{1}{2\delta_1}\cos\frac{\theta}{2}\left(1+\sin^2\frac{\theta}{2}-2\cos\frac{\theta}{2}\right)$$

$$\times\left[(1+s)\cos\theta\cos\frac{\theta}{2}-2(1+3s)\left(2\cos^3\frac{\theta}{2}-\cos\frac{\theta}{2}+2\right)\right]$$

$$\mu=2\cos^2\frac{\theta}{2}$$

where δ_1, δ_2, δ_3 appearing in the above expressions are given by

$$\delta_1 = 2(1+s) - \cos\frac{\theta}{2}\left(1 + 4s\cos^2\frac{\theta}{2}\right)$$

$$\delta_2 = \frac{1}{\delta_1}(1+s)\left(1-2\cos^2\frac{\theta}{2} - 2\cos\frac{\theta}{2}\right)\left[\frac{1}{\delta_1}(1+3s)\cos\frac{\theta}{2}\left(1+\sin^2\frac{\theta}{2} - 2\cos\frac{\theta}{2}\right) + 2\cos\frac{\theta}{2}\left(1+\sin^2\frac{\theta}{2} + \cos\frac{\theta}{2}\sin^2\frac{\theta}{2}\right) - \left(1+\sin^2\frac{\theta}{2}\right)\right]$$

$$\delta_3 = \frac{1}{\delta_1}(1+s)\cos^2\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2} + 2\cos\frac{\theta}{2} - 1\right)\left[\frac{2}{\delta_1}(1+3s)\left(1+\sin^2\frac{\theta}{2} - 2\cos\frac{\theta}{2}\right) + \sin^2\frac{\theta}{2} - \cos\frac{\theta}{2} + 1\right]$$

Appendix B

Inserting (14) and (15) in (8) and (9) and linearising with respect to ζ_i', θ_i' $(i = 1,2)$ we get the following four equations

$$P_1\zeta_1' + P_2\theta_1' - \frac{1}{4}\left[\alpha_1^2(\alpha_{11}+\alpha_{12}) + \alpha_2^2\lambda_{11}\right]\frac{\partial\zeta_1'}{\partial\xi} - \frac{\alpha_2^2}{4}(\lambda_{12}+\lambda_{13})\frac{\partial\zeta_2'}{\partial\xi} = 0 \qquad \text{(C1)}$$

$$P_1\theta_1' - P_2\zeta_1' + \frac{\alpha_1^2}{2}\beta_{11}\zeta_1' + \frac{1}{4}\left[\alpha_1^2(\alpha_{12}-\alpha_{11}) - \alpha_2^2\lambda_{11}\right]\frac{\partial\theta_1'}{\partial\xi} + \frac{\alpha_2^2}{4}(\lambda_{13}-\lambda_{12})\frac{\partial\theta_2'}{\partial\xi}$$

$$+ \frac{\alpha_2^2\beta_{12}}{2}\zeta_2' + \frac{\alpha_1^2\mu}{2\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi'-\xi}\frac{\partial\zeta_1'}{\partial\xi'} + \frac{\alpha_2^2\mu}{2\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi'-\xi}\frac{\partial\zeta_2'}{\partial\xi'} = 0 \qquad \text{(C2)}$$

and

$$P_1\zeta_2' + P_2\theta_2' - \frac{1}{4}\left[\alpha_2^2(\alpha_{11}+\alpha_{12}) + \alpha_1^2\lambda_{11}\right]\frac{\partial\zeta_2'}{\partial\xi} - \frac{\alpha_1^2}{4}(\lambda_{12}+\lambda_{13})\frac{\partial\zeta_1'}{\partial\xi} = 0 \qquad \text{(C3)}$$

$$P_1\theta_2' - P_2\zeta_2' + \frac{\alpha_2^2}{2}\beta_{11}\zeta_2' + \frac{1}{4}\left[\alpha_2^2(\alpha_{12}-\alpha_{11}) - \alpha_1^2\lambda_{11}\right]\frac{\partial\theta_2'}{\partial\xi} + \frac{\alpha_1^2}{4}(\lambda_{13}-\lambda_{12})\frac{\partial\theta_1'}{\partial\xi}$$

$$+ \frac{\alpha_1^2\beta_{12}}{2}\zeta_1' + \frac{\alpha_2^2\mu}{2\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi'-\xi}\frac{\partial\zeta_2'}{\partial\xi'} + \frac{\alpha_1^2\mu}{2\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi'-\xi}\frac{\partial\zeta_1'}{\partial\xi'} \qquad \text{(C4)}$$

where

$$P_1 = \frac{\partial}{\partial \tau} + \gamma_{12} \frac{\partial^3}{\partial \xi^3}; \tag{C5}$$

$$P_2 = \gamma_{11} \frac{\partial^2}{\partial \xi^2}. \tag{C6}$$

Now, if we suppose the τ-dependence of ζ_i', θ_i' $(i = 1,2)$ is of the form $\exp(-i\,\Omega'\tau)$, then equations (C1)–(C4) remain the same as before but P_1 now stands for

$$P_1 = -i\,\Omega' + \gamma_{12} \frac{\partial^3}{\partial \xi^3}. \tag{C7}$$

Next taking the Fourier transform of equations (C1)–(C4) with respect to ξ defined by

$$(\bar{\zeta}_1', \bar{\theta}_1', \bar{\zeta}_2', \bar{\theta}_2') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\zeta_1', \theta_1', \zeta_2', \theta_2') \exp(-i\lambda\xi)\, d\xi, \tag{C8}$$

we get four linear algebraic equations for $\bar{\zeta}_i'$, $\bar{\theta}_i'$ $(i = 1, 2)$. The condition for the existence of a nontrivial solution of these four equations gives the nonlinear dispersion relation (16), where it is assumed that the amplitudes of the two modes are equal ($\alpha_1 = \alpha_2 = \alpha$).