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STEADY FLOW OF A NON-NEWTONIAN FLUID WITH HEAT TRANSFER IN A WAVY CYLINDRICAL TUBE

R. K. BHATNAGAR AND M. N. MATHUR
(Department of Applied Mathematics, Indian Institute of Science, Bangalore-12)

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ABSTRACT

The problem of heat transfer in a steady flow of a Rivlin-Ericksen fluid inside a wavy cylindrical tube is considered. Taking the deformation of the boundary to be small, the equations of momentum and energy have been solved using the perturbation technique. The solution for the velocity field is then employed to study the nature of the temperature field, the boundary of the tube being maintained at a constant temperature.

It is interesting to note that the velocity field for non-Newtonian and Newtonian fluids is the same while the pressure is modified. The stream lines near the boundary of the tube proceed parallel to it and the deformation of these lines goes on decreasing as the axis is approached where they become straight. However, the deformity of the isotherms goes on increasing towards the axis so much so that between $z=\pi$ and $z=2\pi$ they form closed loops. In this respect also, the Newtonian and the non-Newtonian fluids are found to be the same. These features have also been found to exist in the case of flow between two wavy walls.

Recently Citron[1] has studied the slow motion of an incompressible viscous fluid between two rough circular cylinders rotating about their common axis, taking the roughness in the form of a sinusoidal deformation of the boundaries extending upto infinity. His assumption of azimuthal velocity vanishing at infinity does not seem to be correct. The corresponding problem for a non-Newtonian visco-inelastic fluid was studied by Bhatnagar (P. L.) and Rao[2] employing the Fourier series instead of Fourier Transforms in order to avoid the explicit reference to the conditions at infinity. Recently we have investigated[3] the steady flow of a non-Newtonian Rivlin-Ericksen fluid with heat transfer between two wavy walls situated symmetrically about a mid-plane.

In the present note we discuss the steady flow of a non-Newtonian fluid characterised by the Rivlin-Ericksen contitutive equation

$$T_{ij} = -p\,\delta_{ij} + \phi_1 E_{ij} + \phi_2 D_{ij} + \phi_3 E_{im} E_{mj},$$

inside a wavy cylindrical tube, the small deformation being represented by a Fourier series in axial coordinate z given in [1.5] later.

Here
$$E_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i},$$

is the rate of strain tensor, and

$$D_{ij} = \frac{\partial a_i}{\partial x_j} + \frac{\partial a_j}{\partial x_i} + 2\frac{\partial u_m}{\partial x_i} \cdot \frac{\partial u_m}{\partial x_j}$$

is the acceleration gradient tensor, while ϕ_1, ϕ_2, ϕ_3 are respectively the coefficients of viscocity, visco-elastisity and cross-viscocity. We indicate the flow pattern by drawing the stream lines in Fig. I. After determining the flow pattern, we discuss the problem of heat transfer when the wavy boundary of the tube is kept at a constant temperature. We have drawn the isotherms to indicate the temperature distribution in the flow field in Fig. II.

We mention that we have obtained the solutions to the first power of small deformation in the boundary and correct to the fourth power of the Reynolds number defined with respect to radius of the tube as the characteristic length and the velocity at the axis with the neglect of deformation as the characteristic velocity.

We find that, to our approximation, the velocity profiles are not affected by the presence of cross-viscosity and visco-elasticity of the fluid so that the stream lines are the same as in the case of ordinary Newtonian viscous fluids. However, these non-Newtonian effects contribute to the isotropic pressure and temperature distribution. The contribution of cross-viscocity and visco-elasticity of the fluid to the temperature field is quite small to the order of approximation considered so that the isotherms for inelastic and elastic fluids differ very slightly from those of Newtonian fluids and therefore we have not drawn isotherms for these fluids. However, in Table 3, we have recorded the values of temperature at $z = 0$, $\pi/2$, π, $3\pi/2$ and 2π for values of r between $r = 0$ and $r = 1$.

1. EQUATIONS OF THE PROBLEM

Rendering the physical and dynamical quantities dimensionless with the help of radius 'a' of the tube as the characteristic length, U the velocity at the axis in the absence of deformation as the characteristic velocity, characteristic pressure ρU^2, ρ being the density of the fiuid and characteristic temperature T_0, the constant temperature at the surface of the tube, the equations governing the solution of the problem are:

Continuity equation:

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0, \tag{1.1}$$

Momentum equations:

$$\begin{aligned}
u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}&=-\frac{\partial p}{\partial r}+\frac{1}{R}\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\right]\\
&+K\left[10\frac{\partial u}{\partial r}\cdot\frac{\partial^2 u}{\partial r^2}+2w\frac{\partial^3 u}{\partial r^2\partial z}+2u\frac{\partial^3 u}{\partial r^3}+5\frac{\partial w}{\partial r}\cdot\frac{\partial^2 u}{\partial z\,\partial r}+3\frac{\partial u}{\partial z}\cdot\frac{\partial^2 w}{\partial r^2}\right.\\
&+4\frac{\partial w}{\partial r}\cdot\frac{\partial^2 w}{\partial r^2}+u\frac{\partial^3 u}{\partial r\,\partial z^2}+4\frac{\partial u}{\partial z}\cdot\frac{\partial^2 u}{\partial r\,\partial z}+3\frac{\partial u}{\partial r}\cdot\frac{\partial^2 u}{\partial z^2}+w\frac{\partial^3 u}{\partial z^3}\\
&+2\frac{\partial^2 u}{\partial z^2}\cdot\frac{\partial w}{\partial z}+\frac{\partial u}{\partial z}\cdot\frac{\partial^2 w}{\partial z^2}+\frac{2u}{r}\cdot\frac{\partial^2 u}{\partial r^2}+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2+\frac{2w}{r}\cdot\frac{\partial^2 u}{\partial r\,\partial z}\\
&+\frac{2}{r}\cdot\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}-\frac{2u}{r^2}\cdot\frac{\partial u}{\partial r}-\frac{2w}{r^2}\cdot\frac{\partial u}{\partial z}+u\frac{\partial^3 w}{\partial r^2\partial z}+\frac{\partial u}{\partial r}\cdot\frac{\partial^2 w}{\partial r\,\partial z}\\
&\left.+4\frac{\partial w}{\partial z}\cdot\frac{\partial^2 w}{\partial r\,\partial z}+3\frac{\partial w}{\partial r}\cdot\frac{\partial^2 w}{\partial z^2}+w\frac{\partial^3 w}{\partial r\,\partial z^2}+\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^2-\frac{2u^2}{r^3}\right]\\
&+S\left[8\frac{\partial u}{\partial r}\cdot\frac{\partial^2 u}{\partial r^2}-\frac{2u}{r}\cdot\frac{\partial^2 w}{\partial r\,\partial z}-\frac{2u}{r}\cdot\frac{\partial^2 u}{\partial z^2}-\frac{1}{r}\left(\frac{\partial u}{\partial z}\right)^2+\frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^2\right.\\
&\left.+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2-\frac{4u^2}{r^3}+2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)\left(\frac{\partial^2 u}{\partial r\,\partial z}+\frac{\partial^2 w}{\partial r^2}\right)\right],
\end{aligned} \tag{1.2}$$

$$\begin{aligned}
u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}&=-\frac{\partial p}{\partial z}+\frac{1}{R}\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right]\\
&+K\left[10\frac{\partial w}{\partial z}\cdot\frac{\partial^2 w}{\partial z^2}+2w\frac{\partial^3 w}{\partial z^3}+u\frac{\partial^3 w}{\partial r^3}+w\frac{\partial^3 w}{\partial r^2\partial z}+2u\frac{\partial^3 w}{\partial r\,\partial z^2}\right.
\end{aligned}$$

$$+ 2\frac{\partial u}{\partial r}\cdot\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{\partial r}\cdot\frac{\partial^2 u}{\partial r^2} + 3\frac{\partial^2 w}{\partial r^2}\cdot\frac{\partial w}{\partial z} + 4\frac{\partial w}{\partial r}\cdot\frac{\partial^2 w}{\partial r\,\partial z} + 5\frac{\partial u}{\partial z}\cdot\frac{\partial^2 w}{\partial r\,\partial z}$$

$$+ 3\frac{\partial^2 u}{\partial z^2}\cdot\frac{\partial w}{\partial r} + \frac{u}{r}\cdot\frac{\partial^2 w}{\partial r^2} + 3\frac{\partial u}{\partial z}\cdot\frac{\partial^2 u}{\partial r^2} + 4\frac{\partial u}{\partial r}\cdot\frac{\partial^2 u}{\partial r\,\partial z}$$

$$+ \frac{\partial w}{\partial z}\cdot\frac{\partial^2 u}{\partial r\,\partial z} + w\frac{\partial^3 u}{\partial r\,\partial z^2} + \frac{u}{r}\cdot\frac{\partial^2 u}{\partial r\,\partial z} + \frac{w}{r}\cdot\frac{\partial^2 u}{\partial z^2}$$

$$+ 4\frac{\partial u}{\partial z}\cdot\frac{\partial^2 u}{\partial z^2} + \frac{w}{r}\cdot\frac{\partial^2 w}{\partial r\,\partial z} + u\frac{\partial^3 u}{\partial r^2\,\partial z} + \frac{1}{r}\cdot\frac{\partial u}{\partial r}\cdot\frac{\partial w}{\partial r}$$

$$+ \frac{3}{r}\cdot\frac{\partial w}{\partial r}\cdot\frac{\partial w}{\partial z} + \frac{3}{r}\cdot\frac{\partial u}{\partial r}\cdot\frac{\partial u}{\partial z} + \frac{1}{r}\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial w}{\partial z}\Bigg]$$

$$+ S\Bigg[\,8\,\frac{\partial w}{\partial z}\cdot\frac{\partial^2 w}{\partial z^2} - \frac{2u}{r^2}\cdot\frac{\partial^2 u}{\partial r\,\partial z} - \frac{2\,u}{r}\cdot\frac{\partial^2 w}{\partial\, r^2}$$

$$- \frac{2}{r}\cdot\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) + 2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial r\,\partial z}\right)\Bigg], \qquad [13]$$

Energy equation :

$$u\,\frac{\partial T}{\partial r} + w\,\frac{\partial T}{\partial z} = \frac{1}{R\sigma}\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right]$$

$$+ \frac{E}{R}\left[2\left(\frac{\partial u}{\partial r}\right)^2 + 2\left(\frac{u}{r}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right)^2\right]$$

$$+ KE\Bigg[2\,\frac{\partial u}{\partial r}\Bigg\{u\,\frac{\partial^2 u}{\partial r^2} + 2\left(\frac{\partial u}{\partial r}\right)^2 + w\,\frac{\partial^2 u}{\partial r\partial z} + \frac{\partial w}{\partial r}\cdot\frac{\partial u}{\partial z}$$

$$+ \left(\frac{\partial w}{\partial r}\right)^2\Bigg\} + 2\,\frac{\partial w}{\partial z}\Bigg\{u\,\frac{\partial^2 w}{\partial r\partial z} + \frac{\partial u}{\partial z}\cdot\frac{\partial w}{\partial r}$$

$$+ w\,\frac{\partial^2 w}{\partial z^2} + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\Bigg\} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\Bigg\{u\,\frac{\partial^2 u}{\partial r\,\partial z}$$

$$+3\frac{\partial u}{\partial r}\cdot\frac{\partial u}{\partial z}+w\frac{\partial^2 u}{\partial z^2}+\frac{\partial u}{\partial z}\cdot\frac{\partial w}{\partial z}+u\frac{\partial^2 w}{\partial r^2}+\frac{\partial u}{\partial r}\cdot\frac{\partial w}{\partial r}$$

$$+w\frac{\partial^2 w}{\partial r\partial z}+3\frac{\partial w}{\partial r}\cdot\frac{\partial w}{\partial z}\Big\}+\frac{2u}{r}\left\{\frac{u}{r}\cdot\frac{\partial u}{\partial r}+\frac{w}{r}\cdot\frac{\partial u}{\partial z}+\frac{u^2}{r^2}\right\}\Big]$$

$$+SE\left[\frac{\partial u}{\partial r}\left\{4\left(\frac{\partial u}{\partial r}\right)^2+\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)^2\right\}+4\frac{u^3}{r^3}\right.$$

$$\left.+\frac{\partial w}{\partial z}\left\{4\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)^2\right\}-\frac{2u}{r}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\right], \qquad [1.4]$$

where u and w are the radial and axial velocities respectively and

(i) $R=\dfrac{aU\rho}{\phi_1}$ is the Reynolds number,

(ii) $K=\dfrac{\phi_2}{\rho a^2}$, $S=\dfrac{\phi_3}{\rho a^2}$ are the dimensionless parameters characterising

the visco-elasticity and cross-viscosity of the fluid,

(iii) $E=\dfrac{U^2}{C_p T_0}$ is the Eckert number,

and

(iv) $\sigma=\dfrac{\phi_1 C_p}{k}$ is the Prandtl number, k being the conductivity.

Boundary conditions:

$$\left.\begin{aligned}&u=0,\ w=0,\ \text{on } r=1+\epsilon f(z)\\ &u=0,\ \frac{\partial w}{\partial r}=0 \text{ on } r=0\end{aligned}\right\} \qquad [1.5]$$

where
$$f(z)=\sum_{n=1}^{\infty}\left(a_n\cos\frac{nz}{h}+b_n\sin\frac{nz}{h}\right),$$

ϵ being assumed to be a small quantity. The amplitude of the deformation of the boundary can be made as small as we please by properly choosing ϵ and the wave length of the periodic deformation can be adjusted by properly choosing h.

2. PERTURBATION EQUATIONS AND THEIR SOLUTIONS

We now introduce the stream function ψ so as to satisfy the equation of continuity identically, namely

$$u = -\frac{1}{r}\cdot\frac{\partial\psi}{\partial z}, \quad w = \frac{1}{r}\cdot\frac{\partial\psi}{\partial r} \qquad [2.1]$$

and set

$$\psi = \psi_0 + \epsilon\,\psi_1. \qquad [2.2]$$

Solution of the zeroth order equations

The zero order flow is simply the flow in a circular pipe for which ψ_0 is purely a function of radial distance r. Thus setting

$$\psi_{0,z} = 0, \qquad [2.3]$$

the zeroth order equations reduce to

$$p_{0,r} = K\left[\frac{4}{r^2}\psi_{0,rr}\psi_{0,rrr} - \frac{4}{r^3}\psi_{0,r}\psi_{0,rrr} + \frac{12}{r^4}\psi_{0,r}\psi_{0,rr} - \frac{6}{r^5}\psi_{0,r}^2 - \frac{6}{r^3}\psi_{0,rr}^2\right] + S\left[-\frac{3}{r^5}\psi_{0,r}^2 - \frac{3}{r^3}\psi_{0,rr}^2 + \frac{6}{r^4}\psi_{0,r}\psi_{0,rr} + \frac{2}{r^2}\psi_{0,rr}\psi_{0,rrr} - \frac{2}{r^3}\psi_{0,r}\psi_{0,rrr}\right] \qquad [2.4]$$

and

$$p_{0,z} = \frac{1}{R}\left[\frac{1}{r}\psi_{0,rrr} - \frac{1}{r^2}\psi_{0,rr} + \frac{1}{r^3}\psi_{0,r}\right], \qquad [2.5]$$

which have to be solved under the boundary conditions:

$$\left.\begin{array}{l}\psi_{0,z}(1,z) = \psi_{0,r}(1,z) = 0\\ \psi_{0,z}(0,z) = \psi_{0,r}(0,z) = 0\end{array}\right\} \qquad [2.6]$$

The equations [2.4], [2.5] and [2.6] give

$$p_{0,z} = \text{constant}, \quad p_0 = z\,p_{0,z} + \tfrac{3}{8}R^2 p_{0,z}^2 (2K+S)\,r^2, \qquad [2.7]$$

and

$$\psi_0 = -\frac{R\,p_{0,z}}{8}\left(r^2 - \frac{r^4}{2}\right). \qquad [2.8]$$

We notice that in the absence of wavyness, the velocity profiles for Newtonian and non-Newtonian fluids are the same for the axial flow through a pipe. This illustrates the general result of paper referred in reference 4 in the sense that the equation [3.15] of this paper becomes independent of cross-viscosity and visco-elasticity by putting $\psi_{0,z} = 0$.

Solution of first-order equations

Setting $\psi_{0z} = 0$, the first-order equations reduce to

$$-\frac{1}{r^2}\psi_{0,r}\,\psi_{1,zz} = -p_{1,r} + \frac{1}{R}\left[-\frac{1}{r}\psi_{1,rrz} + \frac{1}{r^2}\psi_{1,rz} - \frac{1}{r}\psi_{1,zzz}\right]$$

$$+K\left[-\frac{1}{r^2}\psi_{0,r}\,\psi_{1,rrzz} - \frac{2}{r^2}\psi_{0,rr}\,\psi_{1,rzz} - \frac{3}{r^2}\psi_{1,zz}\,\psi_{0,rrr}\right.$$

$$+\frac{4}{r^2}(\psi_{0,rr}\,\psi_{1,rrr} + \psi_{1,rr}\psi_{0,rrr}) - \frac{1}{r^2}\psi_{0,r}\,\psi_{1,zzzz}$$

$$+\frac{3}{r^3}\psi_{0,r}\,\psi_{1,rzz} + \frac{9}{r^3}\psi_{0,rr}\,\psi_{1,zz} - \frac{4}{r^3}(\psi_{0,r}\,\psi_{1,rrr} + \psi_{1,r}\,\psi_{0,rrr})$$

$$-\frac{9}{r^4}\psi_{0,r}\,\psi_{1,zz} + \frac{12}{r^4}(\psi_{0,r}\,\psi_{1,rr} + \psi_{1,r}\,\psi_{0,rr})$$

$$\left.-\frac{12}{r^5}\psi_{0,r}\,\psi_{1,r} - \frac{12}{r^3}\psi_{0,rr}\,\psi_{1,rr}\right]$$

$$+S\left[-\frac{6}{r^5}\psi_{0,r}\,\psi_{1,r} - \frac{6}{r^3}\psi_{0,rr}\,\psi_{1,rr}\right.$$

$$+\frac{6}{r^4}(\psi_{0,r}\,\psi_{1,rr} + \psi_{1,r}\,\psi_{0,rr}) - \frac{2}{r^2}\psi_{0,rrr}\,\psi_{1,zz}$$

$$+\frac{6}{r^3}\psi_{0,rr}\,\psi_{1,zz} - \frac{6}{r^4}\psi_{0,r}\,\psi_{1,zz} - \frac{2}{r^2}\psi_{0,rr}\,\psi_{1,rzz}$$

$$+\frac{2}{r^3}\psi_{0,r}\,\psi_{1,rzz} + \frac{2}{r^2}(\psi_{0,rr}\,\psi_{1,rrr} + \psi_{1,rr}\,\psi_{0,rrr})$$

$$\left.-\frac{2}{r^3}(\psi_{0,r}\,\psi_{1,rrr} + \psi_{1,r}\,\psi_{0,rrr})\right], \qquad [2.9]$$

and

$$\frac{1}{r^2}\psi_{0,r}\,\psi_{1,rz} - \frac{1}{r^2}\psi_{0,rr}\,\psi_{1,z} + \frac{1}{r^3}\psi_{0,r}\,\psi_{1,z}$$

$$= -p_{1,z} + \frac{1}{R}\left[\frac{1}{r}\psi_{1,rrr} - \frac{1}{r^2}\psi_{1,rr} + \frac{1}{r^3}\psi_{1,r} + \frac{1}{r}\psi_{1,rzz}\right]$$

$$+K\left[\frac{1}{r^2}\psi_{0,r}\,\psi_{1,rzzz}-\frac{1}{r^2}\psi_{0,rrrr}\,\psi_{1,z}+\frac{1}{r^2}\psi_{0,r}\,\psi_{1,rrrz}\right.$$

$$+\frac{1}{r^2}\psi_{0,rrr}\,\psi_{1,rz}+\frac{3}{r^2}\psi_{0,rr}\,\psi_{1,rrz}-\frac{3}{r^2}\psi_{0,rr}\,\psi_{1,zzz}$$

$$+\frac{4}{r^3}\psi_{0,rrr}\,\psi_{1,z}-\frac{4}{r^3}\psi_{0,r}\,\psi_{1,rrz}-\frac{2}{r^3}\psi_{0,rr}\,\psi_{1,rz}$$

$$\left.+\frac{3}{r^3}\psi_{0,r}\,\psi_{1,zzz}-\frac{9}{r^4}\psi_{0,rr}\,\psi_{1,z}+\frac{9}{r^5}\psi_{0,r}\,\psi_{1,z}+\frac{3}{r^4}\psi_{0,r}\,\psi_{1,rz}\right]$$

$$+S\left[\frac{2}{r^3}\psi_{0,rrr}\,\psi_{1,z}-\frac{6}{r^4}\psi_{0,rr}\,\psi_{1,z}+\frac{6}{r^5}\psi_{0,r}\,\psi_{1,z}-\frac{2}{r^2}\psi_{0,rr}\,\psi_{1,zzz}\right.$$

$$\left.+\frac{2}{r^3}\psi_{0,r}\,\psi_{1,zzz}+\frac{2}{r^2}\psi_{0,rr}\,\psi_{1,rrz}-\frac{2}{r^3}\psi_{0,r}\,\psi_{1,rrz}\right], \qquad [2.10]$$

which have to be solved under the following set of boundary conditions:

$$\left.\begin{aligned}&\psi_{1,z}(1,z)=0,\ \psi_{1,z}(0,z)=0,\ \psi_{1,r}(0,z)=0\\&\psi_{1,r}(1,z)=-\frac{R\,p_{0,z}}{2}\sum_{n=1}^{\infty}\left(a_n\cos\frac{nz}{h}+b\,n\sin\frac{nz}{h}\right)\end{aligned}\right\}. \qquad [2.11]$$

These conditions suggest that we must choose $\psi_1(r,z)$ in the form

$$\psi_1(r,z)=-\frac{R\,p_{0,z}}{2}\sum_{n=1}^{\infty}\left[A_n(r)\cos\frac{nz}{h}+B_n(r)\sin\frac{nz}{h}\right] \qquad [2.12]$$

and then the boundary conditions [2.11] take the form

$$\left.\begin{aligned}&A_n(0)=A_n'(0)=A_n(1)=0,\ A_n'(1)=a_n\\&B_n(0)=B_n'(0)=B_n(1)=0,\ B_n'(1)=b_n\end{aligned}\right\}. \qquad [2.13]$$

Eliminating p_1 from the equations [2.9] and [2.10] we get

$$\psi_{0,r}\left[\frac{1}{r^2}\psi_{1,rrz}+\frac{1}{r^2}\psi_{1,zzz}-\frac{1}{r^3}\psi_{1,rz}-\frac{3}{r^4}\psi_{1,z}\right]+\frac{3}{r^3}\psi_{0,rr}\psi_{1,z}-\frac{1}{r^2}\psi_{1,z}\,\psi_{0,rrr}$$

$$=\frac{1}{R}\left[\frac{1}{r}\psi_{1,rrrr}+\frac{1}{r}\psi_{1,zzzz}-\frac{2}{r^2}\psi_{1,rrr}-\frac{2}{r^2}\psi_{1,rzz}+\frac{3}{r^3}\psi_{1,rr}\right.$$

$$-\frac{3}{r^4}\psi_{1,r}+\frac{2}{r}\psi_{1,rrzz}\Big]+K\Big[\psi_{0,r}\Big\{\frac{2}{r^2}\psi_{1,rrzzz}$$

$$+\frac{1}{r^2}\psi_{1,rrrrz}-\frac{2}{r^3}\psi_{1,rrrz}+\frac{9}{r^5}\psi_{1,rz}-\frac{45}{r^6}\psi_{1,z}$$

$$+\frac{3}{r^4}\psi_{1,rrz}+\frac{1}{r^2}\psi_{1,zzzzz}-\frac{2}{r^3}\psi_{1,rzzz}\Big\}+\psi_{0,rr}\Big\{\frac{45}{r^5}\psi_{1,z}-\frac{12}{r^4}\psi_{1,rz}\Big\}$$

$$+\psi_{0,rrr}\Big\{-\frac{21}{r^4}\psi_{1,z}+\frac{4}{r^3}\psi_{1,rz}\Big\}+\frac{6}{r^3}\psi_{0,rrrr}\psi_{1,z}\Big]$$

$$+S\Big[\psi_{0,r}\Big\{\frac{12}{r^5}\psi_{1,rz}-\frac{30}{r^6}\psi_{1,z}\Big\}+\psi_{0,rr}\Big\{-\frac{12}{r^4}\psi_{1,rz}+\frac{30}{r^5}\psi_{1,z}\Big\}$$

$$+\psi_{0,rrr}\Big\{\frac{4}{r^3}\psi_{1,rz}-\frac{12}{r^4}\psi_{1,z}\Big\}+\frac{2}{r^3}\psi_{0,rrrr}\psi_{1,z}\Big] \qquad [2.14]$$

Substituting from [2.8] for ψ_0 and from [2.12] for ψ_1 and equating the coefficients of $\cos(nz/h)$ and $\sin(nz/h)$ on both sides of [2.14] we get:

$$\frac{1}{2r}A_n^{iv}-\frac{1}{r^2}A_n'''+\left(\frac{3}{2r^3}-\frac{1}{r}\cdot\frac{n^2}{h^2}\right)A_n''-\left(\frac{3}{2r^4}-\frac{1}{r^2}\cdot\frac{n^2}{h^2}\right)A_n'+\frac{1}{2r}\cdot\frac{n^4}{h^4}A_n$$

$$=-R^2p_{0,z}\left[\frac{n}{8h}\left(\frac{1}{r}-r\right)B_n''-\frac{n}{8h}\left(\frac{1}{r^2}-1\right)B_n'\right.$$

$$\left.-\frac{n^3}{8h^3}\left(\frac{1}{r}-r\right)B_n\right]+KR^2p_{0,z}\left[\frac{n}{8h}\left(\frac{1}{r}-r\right)B_n^{iv}\right.$$

$$-\frac{n}{4h}\left(\frac{1}{r^2}-1\right)B_n'''+\left\{\frac{3n}{8h}\left(\frac{1}{r^3}-\frac{1}{r}\right)-\frac{n^3}{4h^3}\left(\frac{1}{r}-r\right)\right\}B_n''$$

$$\left.+\left\{\frac{n^3}{4h^3}\left(\frac{1}{r^2}-1\right)+\frac{3n}{h}\left(-\frac{1}{8r^4}+\frac{1}{8r^2}\right)\right\}B_n'+\frac{n^5}{8h^5}\left(\frac{1}{r}-r\right)B_n\right], \qquad [2.15]$$

and

$$\frac{1}{2r}B_n^{iv}-\frac{1}{r^2}B_n'''+\left(\frac{3}{2r^3}-\frac{1}{r}\cdot\frac{n^2}{h^2}\right)B_n''-\left(\frac{3}{2r^4}-\frac{1}{r^2}\cdot\frac{n^2}{h^2}\right)B_n'+\frac{1}{2r}\cdot\frac{n^4}{h^4}B_n$$

$$=R^2p_{0,z}\left[\frac{n}{8h}\left(\frac{1}{r}-r\right)A_n''-\frac{n}{8h}\left(\frac{1}{r^2}-1\right)A_n'\right.$$

$$- \frac{n^3}{8h^3}\left(\frac{1}{r} - r\right)A_n\Bigg] - K R^2 p_{0,z}\Bigg[\frac{n}{8h}\left(\frac{1}{r} - r\right)A_n^{iv} - \frac{n}{4h}\left(\frac{1}{r^2} - 1\right)A_n'''$$

$$+ \left\{\frac{3n}{8h}\left(\frac{1}{r^3} - \frac{1}{r}\right) - \frac{n^3}{4h^3}\left(\frac{1}{r} - r\right)\right\}A_n'' + \left\{\frac{n^3}{4h^3}\left(\frac{1}{r^2} - 1\right)\right.$$

$$\left. + \frac{3n}{h}\left(-\frac{1}{8r^4} + \frac{1}{8r^2}\right)\right\} A_n' + \frac{n^5}{8h^5}\left(\frac{1}{r} - r\right)A_n\Bigg], \qquad [2.16]$$

dash denoting differentiation with respect to r. We note that the contribution of cross-viscosity terms gets out from the equations [2.15] and [2.16] so that even in the presence of wavyness, cross viscosity of the fluid does not affect the velocity profiles.

We make now another assumption namely $R^2 p_{0,z}$ is small to avoid cumbersome mathematical analysis and set

$$\left.\begin{aligned} A_n &= A_{1,n} + R^2 p_{0,z}\, A_{2,n} + R^4 p_{0,z}^2\, A_{3,n} \\ B_n &= B_{1,n} + R^2 p_{0,z}\, B_{2,n} + R^4 p_{0,z}^2\, A_{3,n} \end{aligned}\right\} \qquad [2.17]$$

so that

$$\psi_1(r, z) = -\frac{R p_{0,z}}{2}\sum_{n=1}^{\infty}\Bigg[(A_{1,n} + R^2 p_{0,z}\, A_{2,n} + R^4 p_{0,z}^2 A_{3,n})\cos\frac{nz}{h}$$

$$+ (B_{1,n} + R^2 p_{0,z}\, B_{2,n} + R^4 p_{0,z}^2\, B_{3,n})\sin\frac{nz}{h}\Bigg]. \qquad [2.18]$$

The equations determining $A_{1,n}, B_{1,n}$; $A_{2,n}, B_{2,n}$; and $A_{3,n}, B_{3,n}$ are:

$$r^3 A_{1,n}^{iv} - 2r^2 A_{1,n}''' + r(3 - 2\alpha^2 r^2)A_{1,n}'' - (3 - 2\alpha^2 r^2)A_{1,n}' + \alpha^4 r^3 A_{1,n} = 0, \qquad [2.19]$$

$$r^3 B_{1,n}^{iv} - 2r^2 B_{1,n}''' + r(3 - 2\alpha^2 r^2)B_{1,n}'' - (3 - 2\alpha^2 r^2)B_{1,n}' + \alpha^4 r^3 B_{1,n} = 0, \qquad [2.20]$$

$$r^3 A_{2,n}^{iv} - 2r^2 A_{2,n}''' + r(3 - 2\alpha^2 r^2)A_{2,n}'' - (3 - 2\alpha^2 r^2)A_{2,n}' + \alpha^4 r^3 A_{2,n}$$

$$= -\left[\frac{\alpha}{4}(r^3 - r^5)B_{1,n}'' - \frac{\alpha}{4}(r^2 - r^4)B_{1,n}' - \frac{\alpha^3}{4}(r^3 - r^5)B_{1,n}\right], \qquad [2.21]$$

$$r^3 B_{2,n}^{iv} - 2r^2 B_{2,n}''' + r(3 - 2\alpha^2 r^2)B_{2,n}'' - (3 - 2\alpha^2 r^2)B_{2,n}' + \alpha^4 r^3 B_{2,n}$$

$$= \left[\frac{\alpha}{4}(r^3 - r^5)A_{1,n}'' - \frac{\alpha}{4}(r^2 - r^4)A_{1,n}' - \frac{\alpha^3}{4}(r^3 - r^5)A_{1,n}\right], \qquad [2.22]$$

$$r^3 A^{iv}_{3,n} - 2r^2 A'''_{3,n} + r(3 - 2\alpha^2 r^2) A''_{3,n} - (3 - 2\alpha^2 r^2) A'_{3,n} + \alpha^4 r^3 A_{3,n}$$
$$= -(\alpha/4) r^2 (1 - r^2) [r B''_{2,n} - B'_{2,n} - \alpha^2 r B_{2,n}]$$
$$+ (K\alpha^2/16) r^2 (1 - r^2)^2 [r A''_{1,n} - A'_{1,n} - \alpha^2 r A_{1,n}], \qquad [2.23]$$

and
$$r^3 B^{iv}_{3,n} - 2r^2 B'''_{3,n} + r(3 - 2\alpha^2 r^2) B''_{3,n} - (3 - 2\alpha^2 r^2) B'_{3,n} + \alpha^4 r^3 B_{3,n}$$
$$= (\alpha/4) r^2 (1 - r^2) [r A''_{2,n} - A'_{2,n} - \alpha^2 r A_{2,n}]$$
$$+ [K\alpha^2/16) r^2 (1 - r^2)^2 [r B''_{1,n} - B'_{1,n} - \alpha^2 r B_{1,n}]. \qquad [2.24]$$

where $\alpha = (n/h)$ and we have utilised [2.19], [2.20] in writing [2.21], [2.22] and [2.21], [2.22] in writing [2.23] and [2.24]. The boundary conditions [2.13] reduce to

$$\left.\begin{aligned} A_{1,n}(0) = A'_{1,n}(0) = A_{1,n}(1) = 0,\ A'_{1,n}(1) = a_n \\ B_{1,n}(0) = B'_{1,n}(0) = B_{1,n}(1) = 0,\ B'_{1\cdot n}(1) = b_n \end{aligned}\right\}, \qquad [2.25]$$

$$\left.\begin{aligned} A_{2,n}(0) = A'_{2,n}(0) = A_{2,n}(1) = A'_{2,n}(1) = 0 \\ B_{2,n}(0) = B'_{2,n}(0) = B_{2,n}(1) = B'_{2,n}(1) = 0 \end{aligned}\right\}, \qquad [2.26]$$

and
$$\left.\begin{aligned} A_{3,n}(0) = A'_{3,n}(0) = A_{3\cdot n}(1) = A'_{3,n}(1) = 0 \\ B_{3,n}(0) = B'_{3,n}(0) = B_{3,n}(1) = B'_{3,n}(1) = 0 \end{aligned}\right\}. \qquad [2.27]$$

We note that since it is not possible to solve the equations [2.19] – [2.24] in a closed form for general value of $\alpha = (n/h)$, we utilise the procedure of numerical integration of two point boundary value problems for special case when the boundary of tube has a sinusodial deformation defined by

(i) $a_n = 0$ for all n (ii) $b_n = 0$ for $n > 1$ and $b_1 = 1$ [2.28]

Further, we choose $\alpha = 1/h = 1$ so that the wavelength of the periodic deformation is 2π. In such a scheme, we note that $A_{1,1}$, $B_{2,1}$, $A_{3,1}$ and $B_{3,1}$ are zero at every point of the interval for r between 0 and 1 for the numerical case considered, so that the velocity profiles for Newtonian and non-Newtonian fluids are the same even in the presence of sinusodial deformation in the tube upto the order of approximation considered in [2.17]. The table I below gives the values of $B_{1,1}$ and $A_{2,1}$ at a regular interval of 0.1 for r between 0 and 1.

The stream function ψ is given by

$$\psi = -\frac{R p_{0,z}}{8}\left(r^2 - \frac{r^4}{2}\right) + \epsilon\left[-\frac{R p_{0,z}}{2}\left\{R^2 p_{0,z} A_{2,1} \cos z + B_{1,1} \sin z\right\}\right]. \qquad [2.29]$$

TABLE 1

r	$B_{1,1}$	$A_{2,1}$
0	0	0
0.1	− 0.004628	− 0.000027
0.2	− 0.018512	− 0.000108
0.3	− 0.039915	− 0.000214
0.4	− 0.065971	− 0.000308
0.5	− 0.092672	− 0.000357
0.6	− 0.114778	− 0.000341
0.7	− 0.125670	− 0.000261
0.8	− 0.117165	− 0.000143
0.9	− 0.079291	− 0.000036
1.0	0	0

In Fig. I, we have drawn the stream lines for particular values $\epsilon = 0.05$, and $R^2 p_{0,z} = 0.05$, writing [2.29] in the form

$$\psi' = \frac{\psi}{-R p_{0,z}/(8)} = r^2 - \frac{r^4}{2} + 4\epsilon' \{R^2 p_{0,z} A_{2,1} \cos z + B_{1,1} \sin z\}. \qquad [2.30]$$

We note that the stream lines near the boundary of the tube run parallel to it. The deformity of the stream lines goes on decreasing as they approach the axis where they are just straight as expected in view of the symmetry about the axis of the tube. This behaviour is similar to the flow between two wavy walls situated symmetrical about a mid-plane.

3. SOLUTION OF ENERGY EQUATION

Let us now consider the energy equation [1.4]. We shall solve this equation under the boundary condition that the wavy boundary of the cylindrical tube is maintained at a constant temperature, *i.e.*

$$T = 1, \text{ on } r = 1 + \epsilon \sum_{n=1}^{\infty} (a_n \cos \alpha z + b_n \sin \alpha z) \qquad [3.1]$$

and

$$\frac{\partial T}{\partial r} = 0, \text{ on } r = 0. \qquad [3.2]$$

Setting $T = T_0 + \epsilon T_1$ the zeroth and first order equations in terms of ψ are:

$$\frac{1}{r} \psi_{0,r} T_{0,z} = \frac{1}{R\sigma}\left[T_{0,rr} + \frac{1}{r} T_{0,r} + T_{0,zz}\right] + \frac{E}{R}\left[\left(\frac{1}{r}\psi_{0,rr} - \frac{1}{r^2}\psi_{0,r}\right)^2\right], \qquad [3.3]$$

and

$$-\frac{1}{r}\psi_{1,z}T_{0,r}+\frac{1}{r}\psi_{0,r}T_{1,z}$$

$$=\frac{1}{R\sigma}\left[T_{1,rr}+\frac{1}{r}T_{1,r}+T_{1,zz}\right]+\frac{E}{R}\left[\psi_{0,rr}\left\{\frac{2}{r^2}\psi_{1,rr}-\frac{2}{r^3}\psi_{1,r}-\frac{2}{r^2}\psi_{1,zz}\right\}\right.$$

$$\left.-\psi_{0,r}\left\{\frac{2}{r^3}\psi_{1,rr}-\frac{2}{r^4}\psi_{1,r}-\frac{2}{r^3}\psi_{1,zz}\right\}\right]+KE\left[\frac{1}{r^2}\psi_{1,z}\left\{7\left(\frac{1}{r}\psi_{0,rr}-\frac{1}{r^2}\psi_{0,r}\right)^2\right.\right.$$

$$\left.-\psi_{0,rrr}\left(\frac{1}{r}\psi_{0,rr}-\frac{1}{r^2}\psi_{0,r}\right)\right\}-\frac{1}{r}\psi_{1,rz}\left\{2\left(\frac{1}{r}\psi_{0,rr}-\frac{1}{r^2}\psi_{0,r}\right)^2\right.$$

$$\left.+\frac{1}{r^2}\psi_{0,r}\left(\frac{1}{r}\psi_{0,rr}-\frac{1}{r^2}\psi_{0,r}\right)\right\}+\left(\frac{1}{r^2}\psi_{0,r}\psi_{1,rrz}-\frac{3}{r^2}\psi_{0,r}\psi_{1,zzz}\right)$$

$$\left.\times\left(\frac{1}{r}\psi_{0,rr}-\frac{1}{r^2}\psi_{0,r}\right)\right]+SE\left[\frac{4}{r^2}\psi_{1,z}\left(\frac{1}{r}\psi_{0,rr}-\frac{1}{r^2}\psi_{0,r}\right)^2\right]. \qquad [3.4]$$

The boundary conditions [3.1] and [3.2] reduce to

$$T_0(1)=1,\ \left(\frac{\partial T_0}{\partial r}\right)_{r=0}=0 \qquad [3.5]$$

$$\left.\begin{aligned} T_1(1,z) &= -\left(\frac{\partial T_0}{\partial r}\right)_{r=1}\sum_{n=1}^{\infty}(a_n\cos\alpha z+b_n\sin\alpha z)\\ \left(\frac{\partial T_1}{\partial r}\right)_{r=0} &= 0\end{aligned}\right\} \qquad [3.6]$$

Solution of zeroth order equations:

Since the zeroth order flow is same as the flow in a circular tube, setting $T_{0,z}=0$, [3.3] and [3.5] give

$$T_0(r)=1+\frac{E\sigma R^2 p_{0,z}^2}{64}(1-r^4). \qquad [3.7]$$

Solution of first-order equations:

The boundary conditions [3.6] reduce to

$$T_1(1,z)=\frac{E\sigma R^2 p_{0,z}^2}{16}\sum_{n=1}^{\infty}(a_n\cos\alpha z+b_n\sin\alpha z) \text{ and } \left(\frac{\partial T_1}{\partial r}\right)_{r=0}=0. \qquad [3.8]$$

These conditions suggest that $T_1(r, z)$ should be chosen in the form

$$T_1(r,z) = \frac{E\sigma\, R^2 p_{0,z}^2}{16} \sum_{n=1}^{\infty} \left[C_n(r)\cos\alpha z + D_n(r)\sin\alpha z\right] \tag{3.9}$$

and then

$$\left.\begin{aligned} C_n(1) &= a_n, \quad D_n(1) = b_n \\ C_n'(0) &= 0, \quad D_n'(0) = 0 \end{aligned}\right\} \tag{3.10}$$

Taking as before,

$$\left.\begin{aligned} C_n &= C_{1,n} + R^2\sigma\, p_{0,z}\, C_{2,n} \\ D_n &= D_{1,n} + R^2\sigma\, p_{0,z}\, D_{2,n} \end{aligned}\right\} \tag{3.11}$$

so that

$$\begin{aligned} T_1(r,z) = \frac{E\sigma\, R^2 p_{0,z}^2}{16} \sum_{n=1}^{\infty} &\left[\{C_{1,n} + R^2\sigma\, p_{0,}z\, C_{2,n}\}\cos\alpha z\right. \\ &\left. + \{D_{1,n} + R^2\sigma\, p_{0,}z\, D_{2,n}\}\sin\alpha z\right]. \end{aligned} \tag{3.12}$$

The equations determining $C_{1,n}$, $D_{1,n}$; $C_{2,n}$, $D_{2,n}$ are:

$$C_{1,n}'' + \frac{1}{r} C_{1,n}' - \alpha^2 C_{1,n} = 8\left[A_{1,n}'' - \frac{1}{r} A_{1,n}' + \alpha^2 A_{1,n}\right], \tag{3.13}$$

$$D_{1,n}'' + \frac{1}{r} D_{1,n}' - \alpha^2 D_{1,n} = 8\left[B_{1,n}'' - \frac{1}{r} B_{1,n}' + \alpha^2 B_{1,n}\right], \tag{3.14}$$

$$\begin{aligned} C_{2,n}'' + \frac{1}{r} C_{2,n}' - \alpha^2 C_{2,n} = &\frac{8}{\sigma}\left[A_{2,n}'' - \frac{1}{r} A_{2,n}' + \alpha^2 A_{2,n}\right] \\ &- \left[\frac{r^2}{2} B_{1,n} + \frac{1-r^2}{4} D_{1,n}\right] - \alpha K\left[(r^2-1) B_{1,n}''\right. \\ &\left. + (5r - 1/r) B_{1,n}' + \{3\alpha^2(r^2-1) + 8\} B_{1,n}\right] \\ &- 8 S\alpha\, B_{1,n} \end{aligned} \tag{3.15}$$

$$\begin{aligned} D_{1,n}'' + \frac{1}{r} D_{1\,n}' - \alpha^2 D_{1,n} = &\frac{8}{\sigma}\left[B_{2,n}'' - \frac{1}{r} B_{2,n}' + \alpha^2 B_{2,n}\right] \\ &+ \left[\frac{r^2}{2} A_{1,n} + \frac{1-r^2}{4} C_{1,n}\right] + \alpha K\left[(r^2-1) A_{1,n}''\right. \\ &\left. + (5r - 1/r) A_{1,n}' + \{3(r^2-1)\alpha^2 + 8\} A_{1,n}\right] \\ &+ 8 S\alpha\, A_{1,n}. \end{aligned} \tag{3.16}$$

The equations [3.13]—[3.16] are to be solved under the boundary conditions

$$C_{1,n}(1) = a_n,\ D_{1,n}(1) = b_n,\ C'_{1,n}(0) = D'_{1,n}(0) = 0 \qquad [3.17]$$

and

$$C_{2,n}(1) = D_{2,n}(1) = 0,\ C'_{2,n}(0) = D'_{2,n}(0) = 0 \qquad [3.18]$$

To visualise the temperature field. we solve the equations [3.13]—[3.16] through the procedure of numerical integration of two point boundary value problems by changing these equations into difference equations for the special case when the boundary of the tube has sinusoidal deformation defined by [2.27]. Choosing the wavelength of the periodic deformation to be 2π, we note that $C_{1,1}$ and $D_{2,1}$ are zero at all points of the interval for r between 0 and 1. Table II gives the values of $D_{1,1}$ and $C_{2,1}$ at a regular interval of 0.1 for r and for particular values $K = S = 0$, $K = 0.2$, $S = 0$, and $K = 0, S = 0.2$.

TABLE II

Values of $D_{1,1}$ and $C_{2,1}$ for Newtonian, Visco-inelastic and Visco-elastic fluids

r	$C_{2,1}$ (K=S=0)	$C_{2,1}$ (K=0.2, S=0)	$C_{2,1}$ (S=0.2, K=0)	$D_{1,1}$
0	0.009178	0.249031	0.021116	−0.356790
0.1	0.001557	0.034787	0.003333	−0.363600
0.2	−0.002950	−0.109883	−0.007784	−0.384030
0.3	−0.005600	−0.219561	−0.014984	−0.407107
0.4	−0.006843	−0.304976	−0.019117	−0.421248
0.5	−0.006883	−0.366129	−0.020444	−0.411337
0.6	−0.005932	−0.385820	−0.019136	−0.356997
0.7	−0.004344	−0.370663	−0.015576	−0.230879
0.8	−0.002557	−0.299957	−0.010436	+0.003299
0.9	−0.000956	−0.169761	−0.004708	+0.393045
1.0	0	0	0	1

The temperature $T(r, z)$ is given by

$$T(r,z) = 1 + \frac{E\sigma R^2 p_{0,z}^2}{64}(1 - r^4) + \epsilon \frac{E\sigma R^2 p_{0,z}^2}{16} \times [D_{1,1} \sin z + R^2 \sigma p_{0,z} C_{2,1} \cos z]. \qquad [3.19]$$

In Fig. II, we have drawn the isotherms for Newtonian fluids for particular values $E = 0.5$, $\epsilon = 0.05$ and $R^2 \sigma p_{0,z} = 0.05$, writing [3.19] in the form

$$T'(r,z) = 64\frac{T(r,z) - 1}{p_{0,z}} = E\sigma R^2 p_{0,z} [(1 - r^4) + 4\epsilon\{D_{1,1} \sin z + R^2 \sigma p_{0,z} C_{2,1} \cos z\}]. \qquad [3.20]$$

TABLE III

Temperature distribution for Newtonian fluids (K=S=0), Visco-elastic fluids (K=0.2) and Visco-inelastic fluids (S=0.2)

r/z	$T(r,z)$				
	0	$\pi/2$	π	$3\pi/2$	2π
0	0.025002	0.023216	0.024998	0.026784	0.025002
	0.025062	0.023216	0.024938	0.026784	0.025062
	0.025053	0.023216	0.024947	0.026784	0.025053
0.1	0.024998	0.023180	0.024998	0.026816	0.024998
	0.025007	0.023180	0.024989	0.026816	0.025007
	0.024999	0.023180	0.024997	0.026816	0.024999
0.2	0.024959	0.023040	0.024961	0.026880	0.024959
	0.024935	0.023040	0.024985	0.026880	0.024935
	0.024958	0.023040	0.024962	0.026880	0.024958
0.3	0.024797	0.022762	0.024799	0.026834	0.024797
	0.024743	0.022762	0.024853	0.026834	0 024743
	0.024794	0.022762	0.024802	0.026834	0.024794
0.4	0.024358	0.022254	0.024362	0.026466	0.024358
	0.024284	0.022254	0.024436	0.026466	0.024284
	0.024355	0.022254	0.024365	0.026466	0.024355
0.5	0.023436	0.021381	0.023440	0.025495	0.023436
	0.023346	0.021381	0.023530	0.025495	0.023346
	0.023433	0.021381	0.023443	0.025495	0.023433
0.6	0 021758	0.019975	0.021762	0.023545	0.021758
	0.021664	0.019975	0.021856	0.023545	0.021664
	0.021755	0.019975	0.021765	0.023545	0.021755
0.7	0.018997	0 017844	0.018999	0.020152	0.018997
	0.018905	0.017844	0.019091	0.020152	0.018905
	0.018994	0.017844	0.019002	0.020152	0.018994
0.8	0 014759	0.014777	0.014761	0.014743	0.014759
	0.014685	0.014777	0.014835	0.014743	0.014685
	0.014757	0.014777	0.014763	0.014743	0.014757
0.9	0.008598	0.010563	0.008598	0.006633	0.008598
	0.008556	0.010563	0.008640	0.006633	0.008556
	0.008597	0 010563	0.008599	0.006633	0.008597

N.B. The first entry in each column corresponds to Newtonian fluids, second entry corresponds to visco-elastic fluids and third corresponds to visco-inelastic fluids.

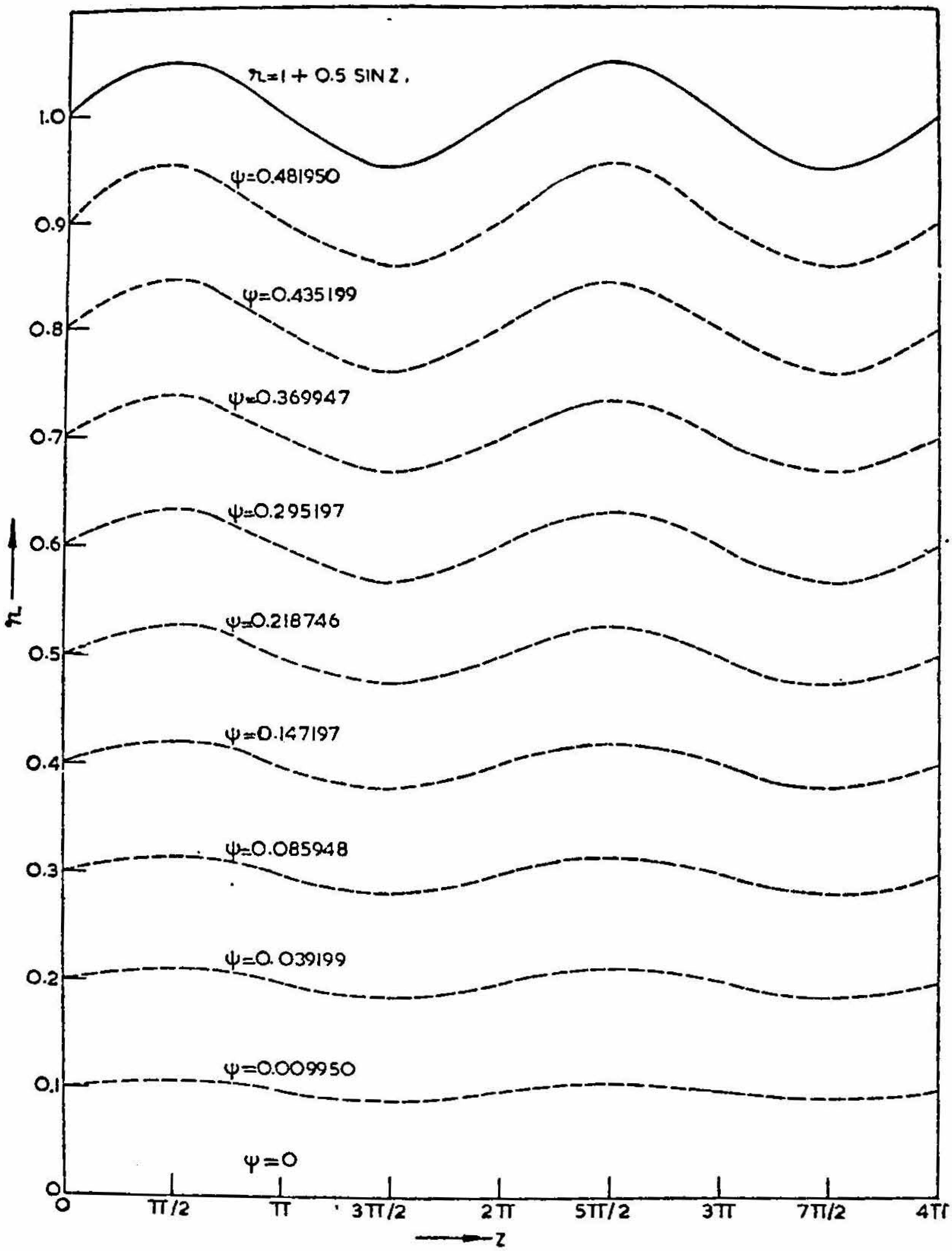

FIG. I

Stream lines for particular values $\epsilon=0.05$, $R^2 p_{0,z}=0.05$

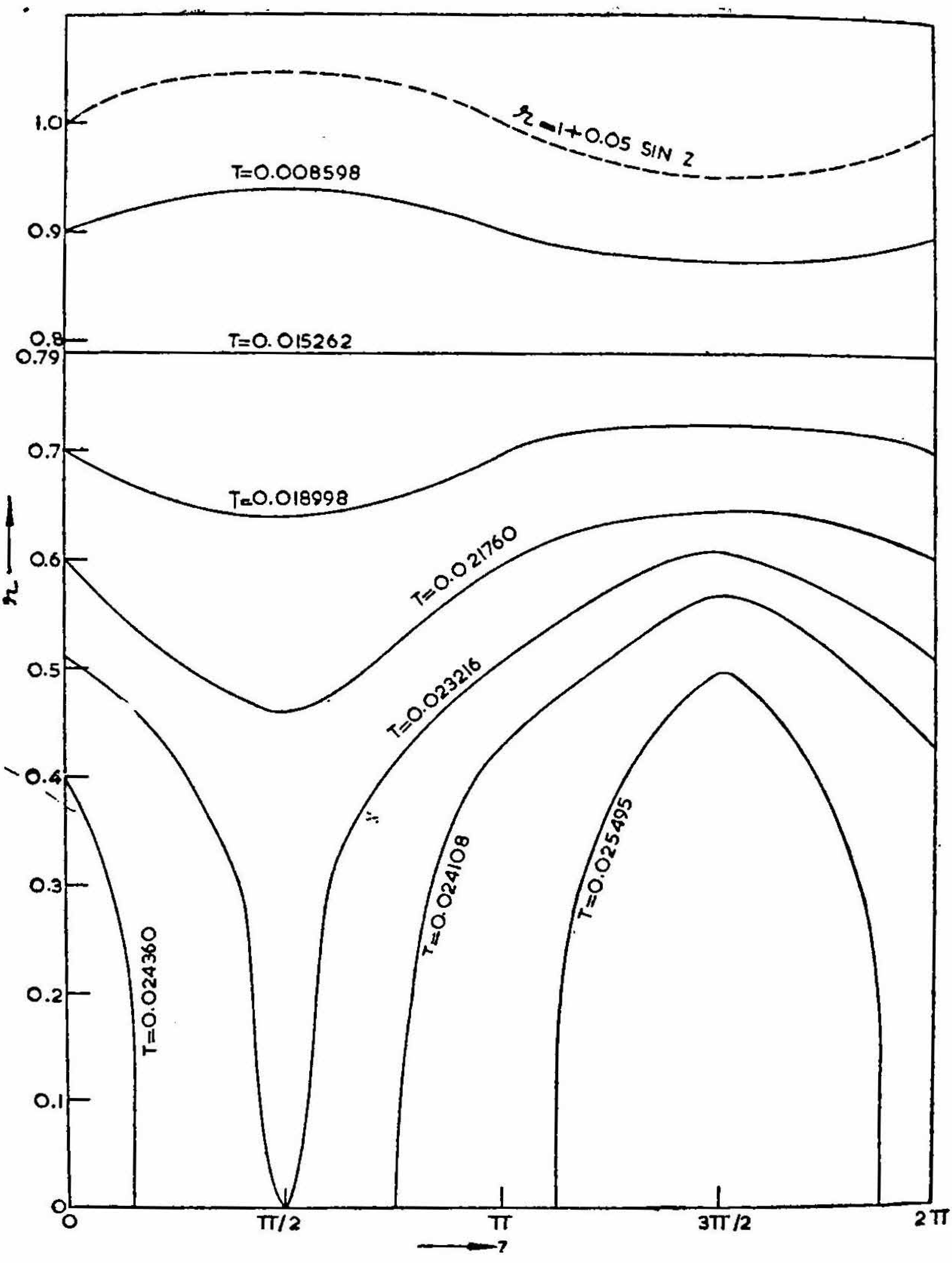

FIG. 11

Isotherms for particular values $\epsilon=0.05$, $R^2 \sigma p_{0,z}=0.05$, $E=0.5$.

We note that the isotherms near the boundary of the tube proceed more or less parallel to it. At a certain distance, $r = 0.79$ (for the above chosen values for E, ϵ and $R^2 \sigma p_0, z$) the isotherm is just a straight line parallel to the axis of the tube. The deformity of the isotherms between $r = 0.79$ and the axis of the tube goes on increasing as we proceed towards the axis so much so that the the isotherm $T = 0\ 023216$ divides the temperature field into two domains and between $z = \pi$ and $z = 2\pi$ the isotherms form closed loops. This phenomenon is found to be similar in nature to the flow between wavy walls. We have not drawn the isotherms for the non-Newtonian inelastic and elastic fluids as their position differs very slightly from the position of isotherms for Newtonian fluids as is evident from the values of temperature tabulated in Table III.

To sum up, we record the following important points:

(1) The non-Newtonian parameters do not effect the velocity field but modify the isotropic pressure.

(2) The stream lines near the boundary of the tube run parallel to it and deformity goes on decreasing as they approach the axis where they become straight.

(3) When the boundary of the tube is kept at a constant temperature, the isotherms near it run more or less parallel to the boundary. At a certain distance from the axis depending on the wavelength of deformation, Reynolds number, Prandtl number and Eckert number, the isotherm becomes straight. As we approach the axis, the deformity of the isotherms becomes more and more pronounced so much so that between $z = \pi$ and $z = 2\pi$ they form closed loops. The isotherms for non-Newtonian fluids are more or less same as for the Newtonian fluids to the order of approximation considered for ϵ and R.

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