



Synthetic Gauge Fields for Ultra Cold Atoms: A Primer

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Abstract | We start by reviewing the concept of gauge invariance in quantum mechanics, for abelian and non-abelian cases. Then we describe how the various gauge potential and field can be associated with the geometrical phase acquired by a quantum mechanical wave function while adiabatically evolving in a parameter space. Subsequently we show how this concept is exploited to generate light induced gauge field for neutral ultra cold bosonic atoms. As an example of such light induced Abelian and Non Abelian gauge field for ultra cold atoms we discuss ultra cold atoms in a rotating trap and creation of synthetic spin orbit coupling for ultra cold atomic systems using Raman lasers.

1 Introduction

“Quantum Simulation with ultra cold atoms”¹ is a much discussed and investigated topic nowadays. Such quantum simulations imply that certain model hamiltonians that were originally proposed to explain exotic quantum behavior in condensed matter systems and sometimes in high energy physics can be realized with ultra cold atoms in various types of optical potentials. In a typical condensed matter or high energy systems, the actual system is far more complex than the one described by such model hamiltonians, and that leaves a lot of issues less than verified. Ultra cold atomic systems on the other hand, are much more controlled, and model behavior can be studied almost exactly. In nature, electromagnetic fields are known as Abelian gauge fields, whereas more complicated non Abelian gauge fields are the ones that are responsible for strong and weak interaction. Gauge fields are therefore known and confirmed to define three fundamental interactions, and form the very basis of our understanding of the microscopic world.² Given this crucial role of gauge fields in fundamental physics, the recent success in simulation of Abelian and Non Abelian gauge fields for ultra cold neutral atoms³⁻⁷ is one of the most significant achievement in the field of ultra cold atomic research.

Apart from allowing us to study the fields responsible for fundamental interaction in a non-trivial context, this development also opens up the

possibility of studying the model hamiltonians that predicts Quantum Hall Effect,⁸ Topological Insulators and Superconductors,⁹ Interesting vortex phases in superconductor¹⁰ and other interesting magnetic field dependent phenomena in electronic system. There have been already a number of excellent and detailed review articles written by some of the most prominent experts in this field. We are not going to repeat the topic of their review. We describe the basic physics associated with the processes that create such synthetic abelian and non abelian gauge fields for ultra-cold atomic system, and a systematic comparison with true gauge fields that occurs in nature is presented, which is responsible for the fundamental interactions with the aim to analyze their “fundamental”-ness.

To that purpose we shall first begin with the gauge invariance of Schrödinger Equation (for example see [11]) for Abelian gauge theory in Sec. 2. In Section 3 we shall introduce the Non Abelian gauge theory. Rather than taking a field theory route we shall do that in a theoretical background of Quantum Mechanics.¹² In Section 4 we shall show how such gauge fields described in the two preceding sections can arise in ordinary Quantum Mechanics purely for geometrical reasons. In Section 5 we present how the rotating Bose Einstein Condensate such an abelian gauge field can be simulated. Section 6 describes a general scheme of producing Abelian and Non Abelian gauge potential for multilevel ultra cold atoms by

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using laser induced coupling between different hyperfine states. In the next section we shall try to describe how both the geometric gauge potentials are related to a more fundamental geometrical concept of parallel transport. We shall also briefly describe in this section why inspite of this connection the gauge fields for ultra cold atomic systems are called “synthetic”. In Section 7, as a specific example of synthetic non-abelian gauge field, we shall explain how synthetic spin-orbit coupling for ultra cold gases. We shall finally conclude.

2 Gauge Invariance of Schrödinger Equation in Electromagnetic Field

In this section we shall start with the gauge invariance properties of the ordinary Schrödinger Equation in Quantum Mechanics, namely how this equation transforms under the gauge transformation and how it compares with our knowledge about gauge invariance from classical physics. Though any standard book on “Quantum Mechanics” may be consulted for more detail, we shall closely follow the discourse given in the book by J. J. Sakurai.¹¹

In quantum mechanics our description of a physical system starts with Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = H\psi$$

The solution of this equation can be written as

$$\Psi(\mathbf{x}, t) = \mathcal{U}(t)\Psi(\mathbf{x}, 0)$$

where the time evolution operator $\mathcal{U}(t) = \exp(-i\frac{\hat{H}t}{\hbar})$. We shall consider the quantum mechanical motion of a charged particle in presence of electric and magnetic field. Such a situation is very common to many solid state electronic systems where one can subject the free electrons that carry electric current to such forces. According to the law of electrodynamics, such electric and magnetic field can be given in terms of scalar potential ϕ and vector potential \mathbf{B} such that

$$\mathbf{E} = -\nabla\phi; \mathbf{B} = \nabla \times \mathbf{A}$$

For the discussion below we shall only concentrate on the vector potentials alone, which can be easily generalized to scalar potential. This is because according to the convention of Special Relativity, scalar and vector potentials are just the time and space like components of a Four Potential.

Since,

$$\begin{aligned} \nabla \times \nabla\Lambda = 0 &\Rightarrow \mathbf{B} = \nabla \times \mathbf{A}' \\ &= \nabla \times (\mathbf{A} + \nabla\Lambda), \end{aligned} \quad (1)$$

Similarly if

$$V' = V - \frac{1}{c} \frac{\partial \Lambda}{\partial t} \text{ and } \mathbf{A}' = \mathbf{A} + \nabla\Lambda$$

Then

$$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\nabla V' - \frac{1}{c} \frac{\partial \mathbf{A}'}{\partial t} = \mathbf{E}' \quad (2)$$

Therefore, classical electrodynamics tells us that upto a gauge transformations the vector and scalar potentials are arbitrary. This means that two vector potentials differ from each other by a gradient of a well behaved scalar function Λ , and will lead to the same magnetic field which is clearly a measurable quantity. This non uniqueness of the vector potential for a given magnetic field however, does not create a problem in describing the motion of charged particle in classical mechanics under Newton’s Laws. This is because the force on such classical charged particle, Lorenz force is given by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$

Therefore, in classical physics the fundamental quantity is magnetic field and vector potential is more like mathematical quantity that defines such magnetic field upto a gauge transformation. By this statement, all the physically measurable quantities must depend on the magnetic field and not on the vector potential.

We shall first construct the definition of gauge transformation from the preceeding discussion and then continue to improve over it in this section before going to ultra cold atoms. Summarizing the above description, such a transformation is the one that links a vector potential with another with both yielding the same magnetic field. Magnetic field is therefore a gauge invariant quantity, and only this appears in the classical equation of motion. This is why all measurable dynamical variables in particle mechanics are gauge invariant. Please note at this stage we did not mention anything to the field part of the Lagrangian which is required for a complete description of the problem. This is done keeping in mind the special situation in ultra cold atoms which we will analyze in this particular review. One can therefore expect at the level of Quantum mechanics where we only quantize the motion of the particle and use the same field as used in the classical mechanics, such gauge invariance should be respected.

The Hamiltonian in presence of such magnetic field is given as

$$H = \frac{1}{2} \left(\mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2.$$

In the above expression, the canonical momentum is \mathbf{p} is distinguished from the mechanical momentum

$$\mathbf{\Pi} = \mathbf{p} - \frac{e}{c} \mathbf{A}$$

However, the previously proposed idea about gauge invariance now needs a careful scrutiny since the vector potential now appears directly in the Hamiltonian. It can be checked that the commutation relations between different components of the mechanical momentum does not vanish,

$$[\Pi_i, \Pi_j] = \frac{i\hbar e}{c} \varepsilon_{ijk} B_k$$

However, the commutator is indeed gauge invariant. Using the above commutator and the fact that

$$H = \frac{\mathbf{\Pi}^2}{2m}$$

It can be straightforwardly shown that the spectrum is given by the so called one dimensional harmonic oscillator like Landau levels

$$E_n = \left(n + \frac{1}{2} \right) \hbar \omega_c; \quad \omega_c = \frac{eB}{mc}$$

The energy spectrum is thus a gauge invariant quantity. However, the gauge invariant form of the energy and the basic commutation relation does not necessarily ensure that the relevant physical quantities in quantum mechanics, such as the transition matrix elements two different states under the action of a given operator, are necessarily gauge invariant.

To understand this issue better, let us recall the Ehrenfest theorem, which states that expectation values of the observables in quantum mechanics behave in the same way as the classical quantities. Therefore, we can expect them to transform in the same gauge invariant way like classical quantities. As one can see this is not trivially satisfied, since what appears in the dynamical variable like Hamiltonian is \mathbf{A} and not \mathbf{B} . This tells us that under a gauge transformation the operators indeed get affected. To see how the gauge invariance of expectation values can be ensured, let us define a state ket $|\alpha\rangle$ in presence of vector potential \mathbf{A} and the corresponding state ket $|\alpha'\rangle$ for the

same magnetic field with a different vector potential $\mathbf{A}' = \mathbf{A} + \nabla\Lambda$. Our basic requirement for gauge invariance is

$$\begin{aligned} \langle \alpha | \mathbf{x} | \alpha \rangle &= \langle \alpha' | \mathbf{x} | \alpha' \rangle \\ \langle \alpha | \left(\mathbf{p} - \frac{e}{c} \mathbf{A} \right) | \alpha \rangle &= \langle \alpha' | \left(\mathbf{p} - \frac{e}{c} \mathbf{A}' \right) | \alpha' \rangle, \end{aligned} \quad (3)$$

apart from the normality of each ket. Now, since both kets are normalized there must be a unitary operator such that

$$|\alpha'\rangle = \mathcal{G} |\alpha\rangle \quad (4)$$

The invariance of position and momentum expectation value then demands

$$\begin{aligned} \mathcal{G}^\dagger \mathbf{x} \mathcal{G} &= \mathbf{x} \\ \mathcal{G}^\dagger \left(\mathbf{p} - \frac{e}{c} \mathbf{A} - \frac{e}{c} \nabla \Lambda \right) \mathcal{G} &= \mathbf{p} - \frac{e}{c} \mathbf{A} \end{aligned}$$

One can immediately see that the unitary operator that does the job is

$$\mathcal{G} = \exp \left[\frac{ie}{\hbar c} \Lambda(\mathbf{r}) \right] \quad (5)$$

This is actually the generator of $U(1)$ gauge transformation and is same as a phase transformation. This is also the simplest gauge transformation. The moral of the above story is that in Quantum Mechanics, in order to keep dynamical variables $U(1)$ gauge invariant, the wavefunction also needs to change under gauge transformation and acquire an additional phase.

This has highly non trivial consequences, such as the Aharonov-Bohm effect. However, to stay focussed on our topic in the next section we shall not discuss this issue further. The function Λ that appeared as the exponent and implements the gauge transformation is a function of local coordinates. Since all such functions commute with each other, such a gauge transformation is called Abelian. In the next section we shall consider more complicated gauge transformations that are non-Abelian.

3 Non-Abelian Phases

We shall now introduce Non Abelian gauge field using the language of quantum mechanics rather than quantum field theory. To this purpose we shall follow the treatment given in Ref. [12]. Yang and Mills in 1954 generalized this gauge (phase) invariance properties of the Schrödinger Equation for multicomponent wave function, namely, the wavefunction has internal degrees of freedom apart from the co-ordinate space or orbital degrees of freedom. As we know when the wavefunction

has such internal degrees of freedom, the wavefunction is a complex vector defined at each point in space time rather than a complex number. In such cases

$$\Psi(\mathbf{r}, t) = \begin{bmatrix} \Psi_1(\mathbf{r}, t) \\ \Psi_2(\mathbf{r}, t) \\ \dots \\ \Psi_{N-1}(\mathbf{r}, t) \\ \Psi_N(\mathbf{r}, t) \end{bmatrix}$$

We shall now extend the concept of gauge invariance for a scalar Schrödinger equation in the earlier section, to the case of vector Schrödinger equation that will be satisfied by such multicomponent wave function and analyze the consequences. However, unlike in the previous case, here we pretend that initially we have no idea about the type of “Electromagnetic field” will demand the resulting gauge invariance. So we start by demanding generalization of such gauge invariance for a multicomponent Schrödinger equation and wait for the outcome.

Since here the wavefunction is multicomponent the generalization of $\exp(i\frac{e}{\hbar c}\Lambda(x))$ will be an unitary matrix, where the condition for unitarity comes from the constraint that the norm of the wavefunction

$$|\Psi(\mathbf{r}, t)|^2 = |\Psi_1(\mathbf{r}, t)|^2 + |\Psi_2(\mathbf{r}, t)|^2 + \dots + |\Psi_N(\mathbf{r}, t)|^2 \quad (6)$$

should remain invariant under this transformation

$$\Psi'(\mathbf{r}, t) = U\Psi(\mathbf{r}, t)$$

If we apply this transformation to the Schrödinger equation for the multicomponent wavefunction, which is (here we assume that all operators are correctly multiplied by suitable matrices so that they have the correct dimension),

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2\Psi(\mathbf{r}, t) + eV\Psi(\mathbf{r}, t), \quad (7)$$

it can be written as

$$i\hbar\frac{\partial}{\partial t}U^{-1}\Psi'(\mathbf{r}, t) = \left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2U^{-1}\Psi'(\mathbf{r}, t) + eV(\mathbf{r})U^{-1}\Psi'(\mathbf{r}, t).$$

We multiply both sides of the above equation from left by U and expand the covariant momentum operator inside bracket and finally obtain

$$i\hbar\frac{\partial}{\partial t}\Psi'(\mathbf{r}, t) = \left(-i\hbar\nabla - \frac{e}{c}U\mathbf{A}U^{-1} - i\hbar U\nabla U^{-1}\right)^2\Psi'(\mathbf{r}, t) + eUVU^{-1}\Psi'(\mathbf{r}, t) - i\hbar U\frac{\partial}{\partial t}U^{-1}\Psi'(\mathbf{r}, t) \quad (8)$$

as the Schrödinger equation for transformed wave function Ψ' . Equation (8) will have the same form as Eq. (7) if we define

$$A' = UAU^{-1} + i\frac{\hbar c}{e}U\nabla U^{-1} \\ V' = UVU^{-1} - i\frac{\hbar}{e}U\frac{\partial}{\partial t}U^{-1} \quad (9)$$

The above set of transformations define the rules for gauge transformations for a multicomponent wave function. The gauge transformations defined in this way through the unitary matrix U is a generalization of the one done for scalar wave function in Sec. 2, but now A and V are matrices. To see this connection explicitly let us note that any unitary matrix U can be written as

$$U = e^{i\mathcal{H}},$$

where \mathcal{H} is a Hermitian matrix. In case of a scalar function, $\mathcal{H} = \frac{e}{\hbar c}\Lambda(\mathbf{r})$. Since Hermitian matrices in general do not commute, the gauge fields that transform according to the transformations defined in (9) are called Non-Abelian gauge fields, whereas for the scalar wave functions they are abelian.

What will be the corresponding gauge field for such non-abelian gauge potential? To find out, that let us note that the principle we adopted in finding out the electromagnetic field is that under the gauge transformation

$$A_\mu(x) \rightarrow A_\mu + \partial_\mu\Lambda(x)$$

This is basically the same equations written earlier in Eq. (2) and Eq. (1), but written in a more compact way using relativistic notation. The field strength should remain invariant under such gauge transformation. This implies that the field strength can be given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (10)$$

It can be readily checked that under the abelian gauge transformation this is indeed gauge invariant. Also in (10) the terms that only contain the spatial derivative of gauge potential, combines to give

$$\nabla \times \mathbf{A} = \mathbf{B}.$$

For Non-Abelian gauge potential under the transformation defined in Eq. (9), it can be shown

that

$$\begin{aligned}
F_{\mu\nu} &\rightarrow \partial_\mu[U^{-1}A_\nu A + iU^{-1}(\partial_\nu U)] \\
&\quad - \partial_\nu[U^{-1}A_\mu A + iU^{-1}(\partial_\mu U)] \\
&= U^{-1}(\partial_\mu A_\nu - \partial_\nu A_\mu)U \\
&\quad + i[(\partial_\mu U^{-1})(\partial_\nu U) - (\partial_\nu U^{-1})(\partial_\mu U)] \\
&\quad + (\partial_\mu U^{-1})A_\nu U + U^{-1}A_\nu(\partial_\mu U) \\
&\quad - (\partial_\nu U^{-1})A_\mu U - U^{-1}A_\mu(\partial_\nu U) \quad (11)
\end{aligned}$$

It is clear the same expression does not transform covariantly. To make it covariant let us note that under the same gauge transformation,

$$\begin{aligned}
-i[A_\mu, A_\nu] &\rightarrow -iU^{-1}[A_\mu, A_\nu]U \\
&\quad - i[(\partial_\mu U^{-1})(\partial_\nu U) - (\partial_\nu U^{-1})(\partial_\mu U)] \\
&\quad - (\partial_\mu U^{-1})A_\nu U - U^{-1}A_\nu(\partial_\mu U) \\
&\quad + (\partial_\nu U^{-1})A_\mu U + U^{-1}A_\mu(\partial_\nu U) \quad (12)
\end{aligned}$$

This means the additional terms that appear in the expression of $F_{\mu\nu}$, also appear in the transformed commutator of the Non Abelian gauge potentials, however with the opposite sign. Thus, suggest that if we define

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \quad (13)$$

then it transforms covariantly under the gauge transformation. Here the quantity g depends on the nature of the coupling with the gauge potential. Equation (13) defines the non abelian field strength.

Non Abelian gauge fields do appear in nature, and fields transforming according to the rules given in Eq. (9) are actually responsible for weak and strong interactions that happen inside nucleus. This is one of the dominant topic in quantum field theory.^{2,25} But as emphasised above, these may appear in ordinary quantum mechanics also. This is what we are going to discuss further in the next section. One of the motivation for that it shares a close connection with synthetic gauge field for ultra cold atoms.

4 Geometric Phase in Quantum Mechanics and the Related Gauge Fields

Abelian and Non Abelian gauge fields are fundamental to our understanding of the nature since it is known that the three fundamental interactions Electromagnetic, Weak and Strong are due to the existence of such gauge fields. However, the Abelian gauge transformations are equivalent to phase transformations and Non Abelian gauge transformations are their higher dimensional generalization. One can therefore naturally ask the

question whether such a transformation arises in the other domain of Quantum Mechanics, and if the answer is yes what are the possible realizations. This question was answered in the clearest form by M. V. Berry in his seminal work¹³ based on a number of other works which already indicated the existence of such different type of gauge fields in a number of physical phenomena, that spans optic,¹⁴ Chemistry,¹⁵ Atomic and Molecular Physics¹⁶ etc. A selected collection of papers on this related topic is available in Shapere and Wilczek edited book.¹⁷ Here we follow a pedagogical account of the key argument given in Ref. [18].

To this purpose we shall consider the phase change in a quantum mechanical wavefunction under an adiabatic change. The adiabatic theorem in quantum mechanics tells us that if the particle Hamiltonian is given by $H(R(t))$ where R is some external co-ordinate which changes sufficiently slowly (slower than the natural time scale set by the typical energy spacing in the unperturbed system) and appears parametrically in H , then the particle will sit in the n -th instantaneous eigenstate of $H(R(t))$ at the time t if it started out in the n -th eigenstate of $H(R(0))$.

The solution of the time dependent Schrödinger equation for this case is

$$|\psi(t)\rangle = c(t) \exp\left(-\frac{i}{\hbar} \int_0^t E_n(t') dt'\right) |n(t)\rangle \quad (14)$$

Here the exponential factor comes from the usual time evolution of an eigenstate of the Hamiltonian, after taking into account the fact that one is dealing with the instantaneous eigenstate of the time dependent Hamiltonian, which is changing with time. The other factor $c(t)$ is kept to check if there is any non-trivial additional time dependence due to the time evolution of the basis states. Substituting the state (14) in the time dependent Schrödinger equation and taking the inner product with the instantaneous $\langle n(t)|$ one gets

$$\frac{dc(t)}{dt} = -c(t) \left\langle n(t) \left| \frac{d}{dt} \right| n(t) \right\rangle$$

with the solution

$$c(t) = c(0) e^{i\gamma(t)}$$

with $\gamma(t) = i \int_0^t \langle n(t') | \frac{d}{dt'} | n(t') \rangle dt'$. The important thing here to notice that this phase is arising because the basis state $|n(t)\rangle$ is constantly changing with time. The instantaneous adiabatic state can therefore be written as

$$|n(R)(t)\rangle_a = e^{i\gamma(t)} |n(R(t))\rangle$$

where the subscript a is used to denote the difference with a time evolved state in the absence of such phase factor. We know that an extra phase factor in a quantum mechanical state may not have any measurable consequences. However, in the present case actually it does have. We shall explain it here. Let us rewrite

$$\begin{aligned} \exp\left(-\int_0^t \langle n(t') \left| \frac{d}{dt'} \right| n(t') \rangle dt'\right) &= \exp\left(\frac{i}{\hbar} \int_0^t \langle n(t') \left| \frac{d}{dt'} \right| n(t') \rangle dt'\right) \\ &= \exp\left(\frac{i}{\hbar} \int_0^t \langle n(R(t')) \left| \frac{d}{dR} \right| n(R(t')) \rangle \frac{dR}{dt'} dt'\right) \\ &= \exp\left(\frac{i}{\hbar} \int_0^t A^n(R) \frac{dR}{dt'} dt'\right) \end{aligned} \quad (15)$$

where

$$A^n(R) = i\hbar \left\langle n(R) \left| \frac{d}{dR} \right| n(R) \right\rangle$$

is known as the ‘‘Berry curvature’’. Now under a phase transformation on the state

$$|n(R)\rangle \rightarrow \exp(i\Phi(R))|n(R)\rangle = |n'(R)\rangle$$

$$\text{Berry Curvature transforms as } A^n(R) \Rightarrow A^n(R) - \hbar \frac{d\Phi(R)}{dR} \quad (16)$$

This is exactly the gauge invariance condition that was imposed on the vector potential in Eq. (1) in Section 2. The transformation of the wavefunction defined in (16) is same as the one defined in Eq. (4) for real electromagnetic field. What motivates a gauge invariant quantity in this case? To see that, consider a case where the adiabatic parameter R comes back to the same value after a time period T . This implies $R(T) = R(0)$ and $H(T) = H(0)$. Under that case the single valuedness of the wave function in the parameter (R) space demands that the line integral of the Berry curvature around the closed loop in the parameter space must be invariant under such gauge or phase transformation. This is exactly the same condition which states that two vector potential A and A' differing from each other through a gauge transformation when integrated over a closed contour will be same since this is equal to the flux enclosed by the area ($\int B \cdot dS$). Thus, the Berry curvature exactly plays the same role as the vector potential due to a real magnetic field under gauge transformation and its effect on the wave function, namely the integral of the vector potential around a close loop in the co-ordinate space, is gauge invariant as demanded by the single valuedness of the wavefunction.

A pertinent question at this point is whether such adiabatic evolution of the time (parameter)

dependent Hamiltonian will also generate a scalar potential along side a vector potential. This is important to establish a full analogy with electromagnetic theory, since vector and scalar potentials are space and time like component of the four

potential that appear in a relativistically invariant theory of Electromagnetism. It turns out that in this case there also exists a corresponding scalar potential which has the form

$$\begin{aligned} V(R) &= \frac{\hbar^2}{2m} \left[\left| \frac{d}{dR} |n(R)\rangle \right|^2 \right. \\ &\quad \left. - \left\langle \frac{d}{dR} |n(R)\rangle \left| n(R) \right\rangle \left\langle n(R) \left| \frac{d}{dR} \right| n(R) \right\rangle \right]. \end{aligned}$$

We again refer to ref. [18] for the detailed derivation of this accompanying scalar potential. The gauge potentials described in this section are known in literature as geometric gauge potential because of their origin.

The adiabatic parameter that appears here is not necessarily a scalar, it could be a vector as well, namely R , having a certain number of components. In that case a straightforward generalization of the above calculation will show in that case Berry Curvature will be a matrix and its different component will generate the Non Abelian counterpart of the Geometric gauge potential. The most significant impact of the concept of ‘‘Berry curvature’’ or ‘‘Geometric Vector Potential’’ is that it opens the possibility of identifying gauge potential and fields in a wide variety of quantum systems. In the following section we shall analyze how these concepts lead to the creation of synthetic gauge field for ultra cold atoms.

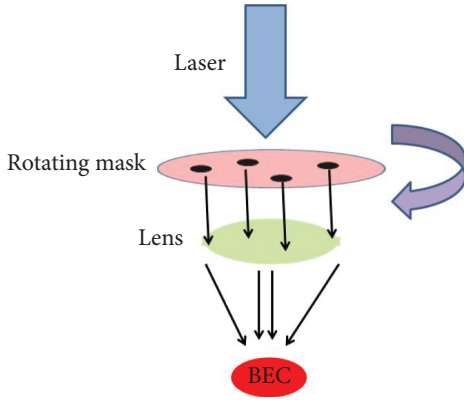


Figure 1: A typical configuration for rotated trap. Here the laser induced optical potential imprinted on BEC is rotated with the help of a rotating mask. In some experiments such a set up was used.

5 Rotating Bose-Einstein Condensates

The simplest example of implementing the artificial or synthetic gauge field for cold atomic condensates is through rotation. This was accomplished by ENS Group,³ MIT group⁴ and JILA group.⁵ This method exploits the equivalence between the Coriolis force in a rotating frame and Lorenz force acting on an electron in a uniform magnetic field.¹⁹ In this scheme the trap in which an ultra cold condensate is created is rotated by using a moving laser. A schematic figure for the set up is depicted in Fig. 1.

If the plane of this rotation is taken as $x - y$ plane and the symmetry axis as \hat{z} , the effect of such rotation on spatial co-ordinate is given by

$$R_z(\phi) \begin{bmatrix} x \\ y \end{bmatrix} R_z(\phi)^\dagger = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Here $R_z(\phi) = \exp(-i\frac{\phi\hat{L}_z}{\hbar})$ is the Rotation operator about \hat{z} axis. If the rotation is executed at an uniform angular velocity Ω , then $\phi = \Omega t$. We can immediately see the connection between $R_z(\phi)$ and the $U(1)$ gauge transformation defined in (5).

The time dependent Hamiltonian that describes a trapped boson in a rotating frame is given by

$$H(t) = R_z(\Omega t) \left[\frac{\mathbf{p}^2}{2m} + \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2) \right] R_z^\dagger(\Omega t)$$

Since the \mathbf{p}^2 remains invariant under the rotation it yields

$$H(t) = \frac{\mathbf{p}^2}{2m} + \frac{m}{2} [(\omega_x^2(x \cos \Omega t + y \sin \Omega t)^2 + \omega_y^2(-x \sin \Omega t + y \cos \Omega t)^2)] \quad (17)$$

One needs to solve the time dependent Schrödinger equation (TDSE) for such system which is

$$i\hbar \frac{\partial \psi}{\partial t} = H(t)\psi. \quad (18)$$

However, to do equilibrium thermodynamics of such bosonic systems, it is useful to go to the co-rotating frame where the Hamiltonian does not change with time. For that one needs to do a unitary transformation on the wave function by writing

$$\psi' = R_z^\dagger(\Omega t)\psi$$

This is again equivalent to the gauge transformation given in (4) where the ψ' is the wave function in the co-rotating frame. The transformed TDSE for ψ' looks like

$$i\hbar \frac{\partial \psi'}{\partial t} = \left[\frac{\mathbf{p}^2}{2m} + \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2) + \Omega L_z \right] \psi' \quad (19)$$

One can check the time independent Hamiltonian on the right hand side can be written as

$$H = \frac{(\mathbf{p} - m\mathbf{A})^2}{2m} + \frac{1}{2}m [\omega_x^2 x^2 + \omega_y^2 y^2 - \Omega^2 r^2] \quad (20)$$

Thus the unitary transformation that takes the wavefunction to the co-rotating frame, also induces a gauge potential in the stationary hamiltonian in the co-rotating frame. Comparing with our discussion in Section 2 we can comment that the unitary operator $R_z(\phi)$ defined here is mathematically same as unitary gauge transformation operator $U = \exp(i\frac{e}{\hbar c}\Lambda(x))$ defined. The transformation of the Hamiltonian operator introduces a gauge potential.

The gauge (vector) potential and the gauge field obtained in this way is of the form

$$\mathbf{A} = -\Omega y \hat{x} + \Omega x \hat{y} \\ \mathbf{B} = 2\Omega \hat{z}$$

This is however, not the only effect on the Hamiltonian by the unitary transformation. It will also introduce a scalar potential

$$V_R(\mathbf{r}) = -\frac{1}{2}m\Omega^2 r^2 \quad (21)$$

Thus the effective trap potential in the rotating frame gets reduced.

As we can see, equivalently, the creation of such “artificial” gauge field through rotation can also

be explained using the concept of Berry Curvature discussed in Ref. [13]. One can recognize here that the adiabatic parameter is the time dependent rotation angle Ωt and the Hamiltonian is a function of this parameter $R = \Omega t$. If the rotational frequency is ramped up adiabatically, then it can be ensured that system will always stay in the ground state of the rotated hamiltonian provided the initial system is in the ground state.

The ultra cold atomic Bose Einstein condensate we are going to talk about consists of interacting bosons. However, in the typical experimental condition, the system is described very well by the mean field Gross-Pitaevskii equation.²⁰ For simplicity we choose a two dimensional condensate that can be derived from a three dimensional model for a rotating cigar shaped condensate. The Gross-Pitaevskii equation is

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2) + g|\Psi|^2 \right) \Psi \quad (22)$$

Even though the Ψ that appeared in the above equation is the mean field superfluid order parameter of the N -boson condensate and not the quantum mechanical wavefunction of the corresponding many body Schrödinger equation, the above equation has the same mathematical structure of a usual one particle Schrödinger equation apart from the non-linear term which represents interaction.²⁰ It can be readily verified that the non-linear term is invariant under the action of the Unitary operator $R_z(\Omega t)$. Thus the entire previous discussion on the artificial gauge transformation of single boson Schrödinger equation can be applied here for the Gross-Pitaevskii equation also. In the early days of BEC this was the technique through which vortices and vortex lattice was created in ultra cold condensate. The entry of such vortices in a rotating condensates and the change of condensate profile due to this is shown in Fig. 2.

We shall not discuss the vortex physics in Ultra Cold BEC any further in this review since this has already been discussed in a number of earlier reviews. The early experiments in rotating ultra cold gases, vortices etc. was discussed in Refs. [21, 22]. An interesting regime is where the rotational frequency Ω is almost equal to the trap frequency in the transverse plane ω_\perp . This means that the trap potential becomes almost negligible. Because of the entry of the large number of vortices in the ultra cold condensate under this condensation, a number of interesting phases of large number of vortices appear in this regime. This is the regime of rapidly rotating ultra-cold gas, and has been reviewed extensively in Refs. [23, 24].

6 The Creation of Abelian and Non-Abelian Gauge Field for Ultra Cold Gases Using Berry Curvature

In the previous section we saw how synthetic Abelian field can be created for ultra cold atoms exploiting the similarity between the rotational operator about a particular axis with a $U(1)$ gauge transformation. The above scheme has a serious limitation. The additional deconfining potential in Eq. (21) destabilizes the trap in which condensate is created beyond a critical value of the rotational frequency, when $\Omega \rightarrow \omega_\perp$. This in turn limits the strength of the Abelian field that can be created in this method. Thus, the cold atom analogue of strongly correlated phases of two dimensional electronic systems in a perpendicular magnetic field with high value (of the order of several Tesla) such as Quantum Hall phases⁸ cannot be created in this set-up. This requires one to look for alternative schemes. We shall describe that in a very general way following the excellent review article²⁶ where one may look for further details.

6.1 Geometrically induced abelian gauge field

Consider a general model of a two level atom with $|g\rangle$ and $|e\rangle$ states being respectively its ground state and excited state and forms a two dimensional Hilbert space. They can be considered as the eigenstate of a simple Hamiltonian like

$$H_0 = \frac{P^2}{2m} \quad (23)$$

We consider the dynamics of the particle in space dependent external field that couples these two states (Fig. 3 (a)). One can recognize that a given laser with suitable parameters can accomplish this job through dipole interaction. For more details on laser-atom interaction refer to standard textbook on Quantum Optics such as ref. [27]. The general Hamiltonian of such coupled system can be written as

$$H_I = H_{gg}(\mathbf{r})|g\rangle\langle g| + H_{ee}(\mathbf{r})|e\rangle\langle e| + H_{ge}(\mathbf{r})|g\rangle\langle e| + H_{eg}(\mathbf{r})|e\rangle\langle g| \quad (24)$$

Since this is a two level system one can map this to spin $\frac{1}{2}$ system and rewrite this as

$$H_I = \frac{\hbar\Omega}{2} \mathbf{n} \cdot \boldsymbol{\sigma}$$

where \mathbf{n} is a three dimensional unit vector parameterized in terms of polar angle $\theta(\mathbf{r})$ and azimuthal angle $\phi(\mathbf{r})$. As one can see that the spatial dependence comes from the fact that the coupling between the states is assumed to be spatially

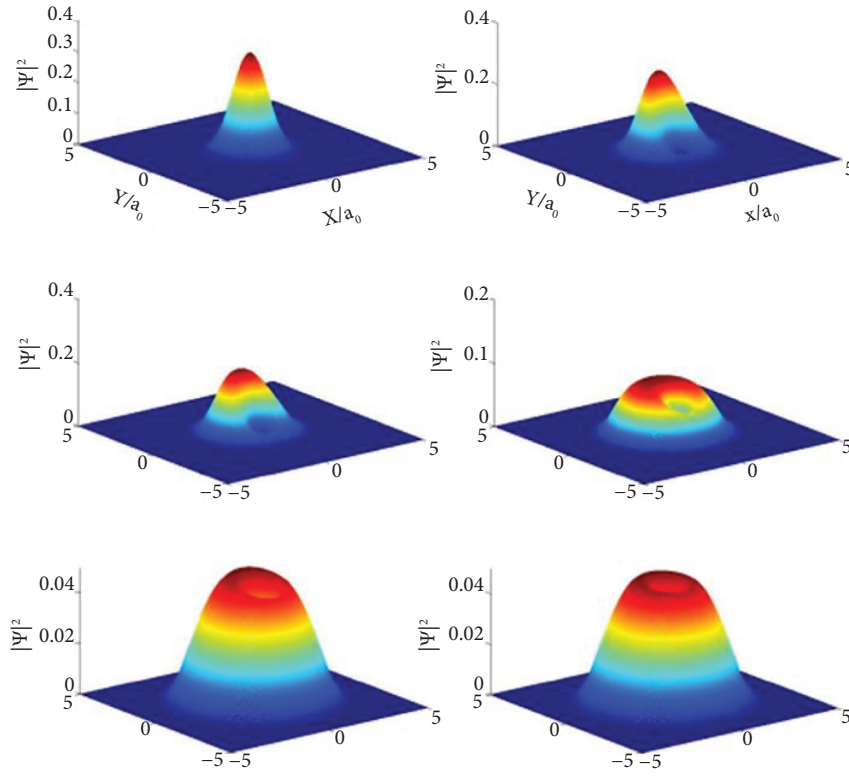


Figure 2: The entry of a vortex in a rotating condensate with increased rotational frequency and the change of condensate profile. The sequence of the plots are like $(1, 1) \rightarrow (1, 2) \rightarrow (2, 1) \rightarrow (2, 2) \rightarrow (3, 1) \rightarrow (3, 2)$. The numerical result is obtained by simulating Gross-Pitaevskii equation (22) in a co-rotating frame. The x and y axis is the condensate co-ordinates in dimensionless units. We also take $\omega_x = \omega_y = \omega_\perp$. The z axis is the superfluid density $|\Psi|^2$. At the position of the vortex the complex order parameter of the superfluid condensate ψ vanishes and the phase of complex order winds around the position of the vortex.

dependent since it will depend on electric field of the laser and the atomic wavefunction.

At the spatial point \mathbf{r} , the local eigenstates of H_I are now given by

$$\begin{aligned} |n_\uparrow(\mathbf{r})\rangle &= \begin{bmatrix} \cos \frac{\theta(\mathbf{r})}{2} \\ \sin \frac{\theta(\mathbf{r})}{2} e^{i\phi(\mathbf{r})} \end{bmatrix} \\ |n_\downarrow(\mathbf{r})\rangle &= \begin{bmatrix} -\sin \frac{\theta(\mathbf{r})}{2} e^{-i\phi(\mathbf{r})} \\ \cos \frac{\theta(\mathbf{r})}{2} \end{bmatrix} \end{aligned} \quad (25)$$

These two states form a local basis for the Hilbert space at each point in the co-ordinate space \mathbf{r} . In the language of quantum optics they are called dressed states. If the system evolves adiabatically through this space, this means that this local basis of the Hilbert space is also changing at every point in space. Following our discussion in Section 15 this adiabatic motion will generate Berry curvature. Further/moreover the quantity $i\langle n_\uparrow | \nabla n_\downarrow \rangle$ is real since $n_{\uparrow, \downarrow}$ forms an orthogonal basis. A general state in this Hilbert space at any point of time

can be written as

$$|\Psi(\mathbf{r}, t)\rangle = \psi_\uparrow(\mathbf{r}, t)|n_\uparrow(\mathbf{r})\rangle + \psi_\downarrow(\mathbf{r}, t)|n_\downarrow(\mathbf{r})\rangle \quad (26)$$

Since the basis vector is changing from one point to another in co-ordinate space,

$$\begin{aligned} \nabla(\psi_i(\mathbf{r})|n_i(\mathbf{r})\rangle) &= \nabla\psi_i(\mathbf{r})|n_i(\mathbf{r})\rangle \\ &+ \psi_i(\mathbf{r})|\nabla n_i(\mathbf{r})\rangle, i = \uparrow, \downarrow \end{aligned}$$

Given this relation when such state is operated by the momentum operator, one yields

$$\mathbf{p}|\Psi\rangle = \sum_{i,j=\uparrow}^{\downarrow} (\delta_{ij}\mathbf{p} - A_{ij})\psi_j|n_i\rangle$$

Now suppose an initial state the particle is in the state $|n_\downarrow\rangle$ and the motional state is such that it stays in this state all the time (the transition amplitude to the up-state is negligible). Under this condition we assume $\psi_\uparrow = 0$ and project the Schrödinger equation in the dressed state $|n(\mathbf{r})_\downarrow\rangle$.

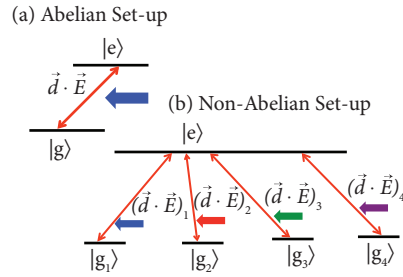


Figure 3: (a) A typical atom-laser configuration that can be used to generate Berry curvature of the form of Abelian gauge potential. The atom is modelled as two level system having states $|g\rangle$ and $|e\rangle$. For more details refer to the discussion in Sec. 6.1. (b) Generalization of the set up in (a) to produce Non Abelian gauge potential. For more details refer to the discussion in Sec. 6.2.

This gives us the following “gauged” Schrödinger equation for ψ_\downarrow .

$$i\hbar \frac{\partial \psi_\downarrow}{\partial t} = \left[\frac{(\mathbf{p} - \mathbf{A})^2}{2m} + \frac{\hbar\Omega}{2} + V \right] \psi_\downarrow \quad (27)$$

Here \mathbf{A} and V are essentially the vector potential and scalar potential that arise due to the geometric phase created by the slowly changing Hilbert space basis from one point to other. The resulting vector potential is given by in this case

$$\mathbf{A}(\mathbf{r}) = i\hbar \langle n_\downarrow(\mathbf{r}) | \nabla n_\downarrow(\mathbf{r}) \rangle = \frac{\hbar}{2} (\cos \theta - 1) \nabla \phi$$

The synthetic magnetic field that is created due to the vector potential is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\hbar}{2} \nabla \cos \theta \times \nabla \phi \quad (28)$$

The magnetic field and vector potential created in this way have geometric origin. It can be concluded from the preceding discussion on Berry’s phase that this gauge potential cannot be “gauged away” completely if the magnetic field (28) is non zero, since the line integral of this vector potential around a closed contour in a region where the magnetic field is non-zero should be equal to the flux enclosed by this region. This apart, the changing basis of the Hilbert space also introduces a

scalar potential

$$\begin{aligned} V(\mathbf{r}) &= \frac{\hbar^2}{2m} |\langle n_\uparrow(\mathbf{r}) | \nabla n_\downarrow \rangle|^2 \\ &= \frac{\hbar^2}{2m} [(\nabla \theta)^2 + \sin^2 \theta (\nabla \phi)^2] \end{aligned}$$

One of the practical advantages of generating a synthetic vector and scalar potential in this way is that if we consider the Hamiltonian of a trapped system, the scalar potential does not offset the trap potential. The scalar potential in this case also can be repulsive and attractive.²⁹ Also given the fact that there are various ways of coupling different hyperfine states of ultra cold atomic multiplets, such a scheme, provides one an wide range of possibilities to create such geometrically induced synthetic gauge potential and gauge field. But as we see in the next section that one of the most interesting aspect of this scheme is the fact that it can be easily generalized to create a Non Abelian-gauge field.

6.2 Geometrically induced non-abelian gauge field

One of the earliest paper that introduces the concept of such non-abelian Geometric phases in a general context is the work by Wilczek and Zee.²⁸ Here we shall discuss a specific example involving ultra cold atoms and laser following.²⁶ Particularly, we shall show how the idea discussed in the preceding section can be easily generalized to create synthetic non-abelian gauge field when a $N + 1$ state atomic system with $N \geq 3$ is suitable configuration of laser beams. A prototype configuration is displayed in Fig. 3(b). The structure of the coupling matrix $U(\mathbf{r})$ will be

$$U(\mathbf{r}) = \begin{bmatrix} \langle 1|U(\mathbf{r})|1\rangle & \langle 1|U(\mathbf{r})|2\rangle & \cdots & \langle 1|U(\mathbf{r})|N+1\rangle \\ \langle 2|U(\mathbf{r})|1\rangle & \cdots & \cdots & \langle 2|U(\mathbf{r})|N+1\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle N+1|U(\mathbf{r})|1\rangle & \cdots & \cdots & \langle N+1|U(\mathbf{r})|N+1\rangle \end{bmatrix} \quad (29)$$

For a fixed position \mathbf{r} , the above matrix can be diagonalized to give $N + 1$ dressed states $|n_i(\mathbf{r})\rangle$ with energy eigenvalues $E_i(\mathbf{r})$ where i goes from 1 to $N + 1$. Under certain circumstances it happens that a subset Q out of this $N + 1$ states are either degenerate or quasi-degenerate and are well separated from the rest of states energetically. It is under this condition that it is possible to realize adiabatic motion in this low lying degenerate subspace \mathcal{H}_Q of dimension Q . Assuming that the motional states are such that there is almost no scattering from this low energy subspace \mathcal{H}_Q to $(N + 1) - Q$ higher energy state.

Again we can write the full wave function of the

$$|\Psi\rangle = \sum_{i=1}^{N+1} \psi_i(\mathbf{r}) |n_i(\mathbf{r})\rangle$$

and then we can project this Schrödinger equation to the reduced Hilbert space \mathcal{H}_Q to get an equation for the reduced spinorial wavefunction $\Psi_Q = (\psi_1, \dots, \psi_Q)^T$. We can straightforwardly extend the gauged Schrödinger equation given in Eq. (27) to its spinorial counterpart, namely

$$i\hbar \frac{\partial \Psi_Q}{\partial t} = \left[\frac{(\mathbf{P} - \mathbf{A})^2}{2m} + \epsilon + V \right] \Psi_Q \quad (30)$$

With the important differences that \mathbf{A} and V are now matrices with their matrix elements given by.

$$\begin{aligned} A_{i,j} &= i\hbar \langle n_i(\mathbf{r}) | \nabla n_j(\mathbf{r}) \rangle \\ V_{i,j} &= \frac{1}{2m} \sum_{l=Q+1}^{N+1} A_{il} \cdot A_{lj} \end{aligned} \quad (31)$$

With different component (x, y, z) these effective vector potentials being matrices will not generally commute with each other, and are therefore called non abelian vector potential. Here ϵ corresponds to the energy of the unperturbed atomic systems.

The above described atom-light interaction induced synthetic abelian or non-abelian gauge potential has been successfully implemented by I. B. Spielman's group^{6,7} in NIST by coupling atomic states with Raman lasers. They created synthetic magnetic field, electric field as well as SO coupling in ultra cold atomic systems. However, it may be noted inspite of the fact that NIST method was able to overcome some of the difficulties that was encountered in rotating an ultra cold atomic system, particularly in the process of creating high "synthetic" magnetic field, it has its own limitations. Here the highest possible value of the synthetic magnetic field is capped by the wavevector of the Raman Laser. A detailed discussion on these experiments appeared in Ref. [29] which one can see for more details. Very recently, an experimental success was achieved in creating optical flux lattices³¹ where it is possible to create much higher value of synthetic magnetic field. We are not covering this topic here and direct the reader to the Refs. [32, 33] and the refs. cited there for the same. Studying the effect of artificial gauge field in presence of optical lattice is another interesting topic which is also not covered in this article. A summary of the relevant work and discussion on some relevant issues for this topic is available in Ref. [34]. The other case of synthetic gauge field for cold

atoms that we shall discuss in some detail in subsequent Section 7 is the principle of creating synthetic Spin-orbit coupling for ultra cold bosonic atoms. However, before that here we shall provide a comprehensive analysis of the "synthetic-ness" of the gauge field created from the geometric phases.

6.3 Geometrical interpretation of the gauge potential: Parallel transport

In this section we shall briefly digress the origin of gauge field in Quantum Mechanics/Field Theory and discuss the existence of similar reason in the current case of synthetic gauge field for ultra cold atoms. We know from quantum mechanics^{11,18} that a spinorial (two-component) wave function transforms under a spin rotation as

$$\Psi' = \exp\left(\frac{i}{2} \boldsymbol{\sigma} \cdot \mathbf{n} \phi\right) \Psi$$

Here

$$[\sigma_i, \sigma_j] = \epsilon_{ijk} \sigma_k$$

obeys the standard commutation relation between the generators of the SU(2) rotation in the spin-space. Under such general SU(2) transformation a n -dimensional iso-spinor similarly transforms to

$$\Psi' = \exp(iM^\mu \Lambda^\mu) \Psi = \mathcal{U}(x) \Psi(x).$$

In field theory a system is said to have gauge invariance, if under such transformation the defining Lagrangian density remains invariant. The Lagrangian density involves the derivative of the field. Same is true for the wavefunction in Quantum Mechanics where the Hamiltonian involves the derivative of the wavefunction. For that it is important that the $\partial_\mu \Psi$, must change covariantly. Now it can be immediately checked that this is not the case for the usual derivative as

$$\partial_\mu \Psi' = \mathcal{U}(\partial_\mu \Psi) + (\partial_\mu \mathcal{U}) \Psi$$

This happens because under the generalized spin-rotation, the axes in the space of such isospin is getting at each point in space. Thus $\Psi(x)$ and $\Psi(x + dx)$ are measured in different co-ordinate systems. To make the derivative co-variant one should compare $\Psi(x + dx)$ with the modified value of $\Psi(x)$ if it were transported from x to $x + dx$ keeping the iso-spin axis fixed. This is known as parallel transport in isospace. This is depicted in Fig. 3. Under this condition the change in the Ψ will be different and this change $\delta\Psi$ can be written as

$$\delta\Psi = igM^a A_\mu^a dx^\mu \Psi$$

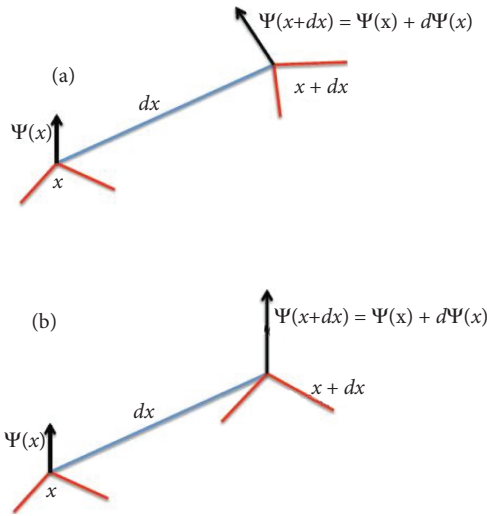


Figure 4: The concept of parallel transport is illustrated with the help of two figures adopted from ref. [25]. In (a) no parallel transport was done. In (b) the parallel transport was done.

Here, A_μ^a takes care of the change in the local co-ordinate axes in the isospace from one point to another. The derivative defined in this way becomes

$$\begin{aligned} D\Psi &= (\Psi + d\Psi) - (\Psi + \delta\Psi) \\ &= d\Psi - igM^a A_\mu^a dx^\mu \Psi \\ \frac{D\Psi}{dx^\mu} &= D_\mu \Psi = (\partial_\mu - igM^a A_\mu^a) \Psi \end{aligned} \quad (32)$$

It can be checked that the above modified derivative transforms covariantly under the gauge transformation and is therefore qualified to enter into the gauge invariant Lagrangian density. Here M^a are the generators of the rotation in the iso-spin space and their detail form depends on relevant group that represents the symmetry. The simplest of this case is $U(1)$ rotation where $M = 1$. In particular, it corresponds to our problem of a charged particle in electromagnetic field $g = e$. Under that situation

$$D_\mu = \partial_\mu + ieA_\mu.$$

The above discussion is available in a number of Field Theory books that discuss gauge field theory. We here mostly followed the notation and discussion of Ref. [25]. Now comparing this discussion with the discussion in the preceding sections (6.1 and 6.2), we can immediately recognise that there also it is the changing basis in the pseudospin space under adiabatic evolution. The adiabatic evolution of these basis states leads to the development of

synthetic gauge field or Berry Curvature, and has a similar geometrical interpretation like the true gauge field.

Finally this brings us to an important question, namely, why in spite of this similar geometrical origin, we call the Berry curvature related gauge field in ultra cold atoms as “synthetic”. The reason is in true gauge field theory there is purely gauge field dependent term in the Lagrangian density^{2,25} which stands for the Field energy. It is this term that gives the gauge field their independent dynamics in the complete matter-field Lagrangian density. Thus such gauge fields are dynamical. The corresponding gauge potentials appeared in the covariant momentum operator. However, in the atom-laser configuration the full Lagrangian density does not contain any such field energy term which is related to the Berry curvature generated due to adiabatic motion. The field part that appears here is just the electromagnetic field energy associated with the laser and the not one related to the geometric gauge field. This is why geometric gauge fields do not have any independent dynamics.

7 Synthetic Spin-Orbit Coupling for Ultra Cold Atomic Gases: Case of Non Abelian Gauge Field

The motivation behind creating synthetic spin-orbit coupling for ultra cold atoms primarily comes from the fact that spin orbit coupling plays a very important role in spintronics³⁰ and Topological Insulators,⁹ both of which have interesting practical applications. However, spin-orbit coupling also forms an interesting example for Non Abelian gauge potential which we shall describe in the following discussion.

7.1 Non abelian gauge potential and spin-orbit coupling

In our familiar notation a Non Abelian vector potential can be written as

$$\mathbf{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$

where A_x , A_y and A_z are now matrices. Field strength for such Non Abelian vector potential given by the expression (13) can be written as

$$\mathbf{B} = \nabla \times \mathbf{A} - \frac{i}{\hbar} \mathbf{A} \times \mathbf{A} \quad (33)$$

One can now easily identify that in the expression (33), the first part is a straight forward generalization of the relation between vector potential and magnetic field for the Abelian case, the second part is only non zero if the gauge potential is

Non Abelian. For Abelian cases, the second part is identically zero.

A major motivation of simulating synthetic magnetic field for ultra cold atoms is to observe Quantum Hall Effect like phenomenon which occurs when a two dimensional electron gas is subjected to an transverse uniform magnetic field.⁸ A related question can therefore be asked what are the gauge potentials that can create a uniform Non Abelian magnetic field. A detailed analysis of this problem was done in Ref. [35]. Here we provide a brief summary of the relevant results to discuss subsequently in some detail why and how one creates synthetic spin-orbit coupling for ultra cold atomic systems.

An important difference between Abelian and Non Abelian gauge field is that, whereas in the case of the former two vector potentials that create the same magnetic field are related to each other by a simple gauge transformations, in the case of Non Abelian field it is not so. By that one means that two non-equivalent Non-Abelian gauge potential can lead to the same Non-Abelian magnetic field. Following³⁵ we shall illustrate this case for the non abelian magnetic field

$$\mathbf{B} = 2\sigma_z \hat{z} = 2 \begin{bmatrix} \hat{z} & 0 \\ 0 & -\hat{z} \end{bmatrix} \quad (34)$$

The above magnetic field is uniform but its direction is opposite for spin-up and spin-down component of the wavefunction of the particle on which it is applied.

One type of vector potential that can give such uniform field is given by

$$\mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r} = y\sigma_z \hat{x} - x\sigma_z \hat{y} \quad (35)$$

This is again a straightforward generalization of vector potential in symmetric gauge for uniform magnetic field $\mathbf{B} = B\hat{z}$, and here the vector potential contributes to the magnetic field only through the first term (on R.H.S.) of the expression (33) for Non Abelian field strength. Even though all component of the vector potential are matrices, they are Abelian matrices. Thus this is also the case of Abelian gauge field. The single particle spectrum of Schrödinger Equation, in presence of such gauge potential, is a generalization of the Landau problem. Quantity like magnetic length, phenomenon like Aharonov Bohm effect etc. can be defined for such a problem.

Another type of vector potential that can also generate the same magnetic field is given by

$$\mathbf{A} = -\sigma_y \hat{x} + \sigma_x \hat{y}. \quad (36)$$

This is a uniform (does not depend on local coordinate) non commuting vector potential. This is indeed Non Abelian gauge potential. The contribution to the field purely comes from the second term in the expression (33). The single particle spectrum of Schrödinger Equation in presence of such gauge potential is very different from the Landau problem.³⁵ However, there is more interesting motivation for realizing such Non Abelian gauge potential for ultra cold atoms.

To see this let us recall the well known spin-orbit coupling (Thomas Term) which arises due to relativistic correction to the motion of a spin-1/2 electron obeying Schrödinger Equation, namely

$$H_{SO} = -\frac{e\hbar}{4m_e^2 c^2} \boldsymbol{\sigma} \cdot (\mathbf{E} \times \mathbf{p}) \quad (37)$$

Using the triple product rule the above hamiltonian can be rewritten as

$$H_{SO} = \frac{e\hbar}{4m_e^2 c^2} \mathbf{p} \cdot (\mathbf{E} \times \boldsymbol{\sigma})$$

For a uniform electric field along z -axis, $\mathbf{E} \times \boldsymbol{\sigma}$ is just the Non Abelian uniform vector potential defined in Eq. (36). Identifying this we can rewrite

$$H_{SO} \propto (\mathbf{p} \cdot \mathbf{A})$$

where \mathbf{A} corresponds to the vector potential defined in Eq. (36). Such a term also appear in the kinetic energy term of the Hamiltonian,

$$H_k = \frac{1}{2m} (\mathbf{p} - m\mathbf{A})^2$$

that describes a free particle in the presence of Non Abelian gauge field (36). Thus the simulation of such synthetic Non Abelian gauge field for ultra cold atoms is equivalent to creating synthetic spin-orbit (SO) coupling for such systems. SO coupling plays a crucial role in Spintronics³⁰ and Topological Insulator.⁹ With this background we shall now briefly discuss how such SO coupling is created experimentally for ultra cold BEC.

7.2 Principle of spin orbit coupling in ultra cold bosonic systems: NIST method

To generate SO coupling one considers the ⁸⁷Rb atoms whose ground state electronic structure is ²S_{1/2}, giving electron spin as $S = 1/2$, and nuclear spin as $I = 3/2$. Therefore, the total spin F can take value $F = 1$ and $F = 2$ due to hyperfine coupling. The low energy manifold therefore consists of three $F = 1$ states. Such states are characterized by state vectors $|F, m_F\rangle$ which represents simultaneous eigenstates of F^2 and F_z operators, and for

$F = 1$ they are given as $|1, 1\rangle$, $|1, 0\rangle$ and $|1, -1\rangle$ respectively. A schematic for the set-up is given in Fig. 5. In presence of Zeeman field all these three levels that have same energy will split into three different levels. The resultant system is exposed to two counter propagating Raman laser beams along the \hat{x} direction. The atom which is moving with velocity $\frac{\hbar k_x}{m}$ along \hat{x} direction will absorb a photon coming from the opposite direction of Laser I, and will have the momentum $\hbar(k_x - k_L)$. From this excited state it will emit a photon in the direction of laser II. As a result finally the momentum of the state along the x -direction will be $\hbar(k_x - 2k_L)$. Therefore, the state will finally be written as $|-1, k_x - 2k_L\rangle$. Here the first quantum number corresponds to the hyperfine quantum number m_F , whereas the second one gives the momentum along the \hat{x} direction. Similarly the atoms absorbing photon from laser II and emitting a photon in the direction of laser I will be finally in the state $|1, k_x + 2k_L\rangle$. The final outcome is to have the following three states

$$\begin{aligned} |1\rangle &= |1, k_x + 2k_L\rangle \\ |0\rangle &= |0, k_x\rangle \\ |-1\rangle &= |0, k_x - 2k_L\rangle \end{aligned} \quad (38)$$

It is possible to write down the effective hamiltonian now in 3×3 matrix form that includes contribution from atom, field (laser) and atom-laser interaction. However, shown explicitly in Ref. [36], it is possible to tune the Zeeman energy and the laser frequency in such a way, that the low energy subspace created by $|1, 0\rangle$ and $|1, -1\rangle$ states are well separated from the $|1, 1\rangle$ state. Under this situation it is possible to construct an effective Hamiltonian in this two dimensional Hilbert space spanned by hyperfine state $|0\rangle$ and $|-1\rangle$ which we shall respectively call as spin up $|\uparrow\rangle$ and spin-down $|\downarrow\rangle$ state. This is somewhat modified version of the scheme suggested in Section 6.2. The 2×2 effective hamiltonian becomes,

$$H = \begin{bmatrix} \frac{k_x^2}{2m} + \frac{\delta}{2} & \frac{\Omega}{2} e^{2ik_L x} \\ \frac{\Omega}{2} e^{-2ik_L x} & \frac{k_x^2}{2m} - \frac{\delta}{2} \end{bmatrix} \quad (39)$$

Here $2k_0$ is the momentum transfer due to the relative motion between the laser and the hyperfine state of the atom, and δ is the detuning between the Raman resonance and the energy difference between the spin up and spin-down level. We have also absorbed an overall \hbar factor in various terms. We refer to ref. [36] for the details about the derivation of the above hamiltonian.

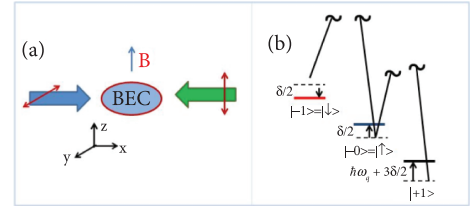


Figure 5: (a) Schematic set up by NIST set up for exposing a BEC to two counter-propagating laser beam. (b) The Raman coupling between three hyperfine states.

If one makes a unitary transformation on the two component wave function that will describe this system such that $\psi' = U\psi$, with

$$U = \begin{bmatrix} e^{-ik_L x} & 0 \\ 0 & e^{ik_L x} \end{bmatrix}$$

This changes the Hamiltonian from H to UHU^\dagger which is given by

$$H_{SO} = \begin{bmatrix} \frac{(k_x + k_L)^2}{2m} + \frac{\delta}{2} & \frac{\Omega}{2} \\ \frac{\Omega}{2} & \frac{(k_x - k_L)^2}{2m} - \frac{\delta}{2} \end{bmatrix} \quad (40)$$

As one can see the resulting Hamiltonian can be written as

$$H_{SO} = \frac{(k_x \parallel + k_L \sigma_z)^2}{2m} + \frac{\Omega}{2} \sigma_x + \frac{\delta}{2} \sigma_z \quad (41)$$

The above hamiltonian is the spin-orbit coupled hamiltonian realized in NIST experiment.⁶ Even though here the vector potential has only one component A_x , since that does not commute with the scalar potential $\frac{\Omega}{2} \sigma_x + \frac{\delta}{2} \sigma_z$, this is one of simplest realization of uniform non Abelian gauge potential. With a suitable spin rotation it can also be shown that the first term actually represents and linear combination of equal weight Rashba and Dresselhaus SO coupling. The energy eigenvalues are

$$\epsilon_k = \frac{k_x^2 + k_L^2}{2m} \pm \sqrt{\left(\frac{k_x k_L}{m} - \frac{\delta}{2}\right)^2 + \frac{\Omega^2}{4}} \quad (42)$$

which for zero detuning $\delta = 0$, the momentum dependent energy eigenvalues can be given as

$$\epsilon_k = \frac{k_x^2 + k_L^2}{2m} \pm \sqrt{\left(\frac{k_x k_L}{m}\right)^2 + \frac{\Omega^2}{4}} \quad (43)$$

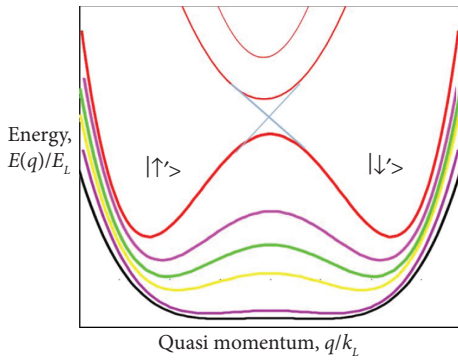


Figure 6: The transition from a single to double non-degenerate minima with the variation of detuning for a spin orbit coupled gas. The idea of the figure is taken from ref. [6].

The condition for the minima of the energy can now be obtained from $\frac{\partial \epsilon_k}{\partial k_x} = 0$ which gives

$$k_x = 0$$

$$\text{or } k_x = \pm k_0 \sqrt{1 - \left(\frac{\Omega}{4E_L}\right)^2} \quad (44)$$

Here $E_L = \frac{k_L^2}{2m}$ one can now check the following things from the above expression. If the detuning δ is 0, there will be distinctly one minima for $\Omega > 4E_L$ at $k_x = 0$ and another minima at $k_x = \pm k_0 \sqrt{1 - \left(\frac{\Omega}{4E_L}\right)^2}$ for $\Omega \leq 4E_L$. However, if the detuning δ is finite, there is still a single minima at $k_x = 0$ for $\Omega > 4E_L$, but there are also two non-degenerate minima at different height for $\Omega < 4E_L$. The height difference can be controlled by the detuning, and the transition from single to two minima is the way one can detect the spin-orbit coupling.⁶ For details of the experimental method one can look at Refs. [29, 36]. A schematic of the variation of the energy E as a function of the wavevector $q = k_x$ is given in Fig. 6.

8 Conclusion

In this introductory review we provided a careful comparison between the true gauge fields that is responsible for the fundamental interaction between elementary particles and the synthetic gauge field for ultra cold atoms. We analysed both Abelian and Non Abelian gauge field for this purpose. We have particularly show how both fundamental and synthetic gauge fields can be interpreted in a similar geometric way; however the latter does not have any independent dynamics and hence dubbed as synthetic. We also illustrate the examples of such synthetic gauge

field for ultra cold atoms by considering two specific cases: Abelian gauge field in rotating Bose Einstein condensates and Non Abelian gauge field in spin-orbit coupled Bose Einstein condensates. This primer by no way covered the large amount of exciting work that was done in the field of synthetic gauge field for ultra cold atoms. We referred to a number of excellent review articles to that purpose. We also cited only a limited number of mostly pedagogical articles and books on the relevant topic, and apologize for our inability to cite a large number of exciting and important and highly relevant work in this field. We hope the direction and information given in this review will be sufficient to direct the interested reader to more complete set of references on synthetic gauge field.

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