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Geometric Methods in Analysis and Control of Implicit Differential Systems

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Abstract | In this article, we discuss the application of differential geometric methods in analyzing the structure and designing control laws for implicit differential systems or differential algebraic equation (DAE) systems. While there have been several efforts toward numerical and quantitative analysis of DAE problems, the theoretical contributions especially in the case of nonlinear systems are scarce. We discuss two popular techniques from differential geometric control theory and bring out their merits in addressing implicit differential systems. In the first section, we review the theory of noninteracting control via input–output decoupling and its application in analyzing the intrinsic structure of DAE control problems. In particular, we focus on addressing the problem of well-posedness of DAE systems as well as feedback control design through a regularization process which allows one to solve the DAE by expressing the constraint variable as a dynamically dependent endogenous function of the states, inputs, and their derivatives. Further, extensions of these techniques to stochastic differential algebraic equations have been presented. In the second section, we review the theory of differential flatness and its applicability to feedback control design for a class of DAE systems. Here, the DAE system is expressed as a Cartan field on a manifold of jets of infinite order, and necessary and sufficient conditions for its equivalence to a linear, controllable system have been derived in order to design globally stabilizing nonlinear feedback laws. Examples from constrained mechanics have been presented in order to demonstrate the practical applicability of these methods.

1 Introduction

Differential algebraic equations (DAE) or implicit differential systems are of the form:

$$F(\dot{x}, x, u) = 0, \tag{1}$$

where $x \in X$ is a smooth manifold, $(x, \dot{x}) \in T(X)$the tangent bundle, and $u \in U$ is a space of admissible controls. Further, $\dim(X) = n$, $\dim(U) = m$ and $\operatorname{rank} \frac{\partial F}{\partial \dot{x}} = n - p$, in a suitable open dense subset of $T(X)$, with $0 \leq m, p \leq n$. A salient distinction from ordinary differential equation (ODE) systems is that F need not be invertible with respect to $\dot{x}$, i.e., $\frac{\partial F}{\partial \dot{x}}$ may not be of full rank. In case it is invertible, then (1) can be rewritten as an ODE, by applying the implicit function theorem to express $\dot{x}$ explicitly in terms of x and u as $\dot{x} = \hat{F}(x, u)$. The article by Petzold[41] further illustrates the distinctions between ODE and DAE systems with both analytical and numerical considerations. In particular, the following pertinent questions arise when addressing DAE problems:

- *Existence and uniqueness* Given an initial condition $(x_0, \dot{x}_0)$ satisfying (1) at $t = 0$, does there exist (at least locally) a unique trajectory $(x(t), \dot{x}(t))$ satisfying the DAE and initial conditions ?

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- *Stability* As in the case of ODEs, are there conditions to check whether an equilibrium point of the flow is locally or globally stable (in the sense of Lyapunov)?
- *Nature of solutions* Do the solutions exhibit bifurcations or impulses for certain initial conditions and parameters?
- *Numerical considerations* When designing a numerical integration algorithm to solve DAE systems, in addition to considering the nature of solutions, it is also essential to determine whether the initial condition of the integrator is consistent i.e., given $x(n)$, $u(n)$, does the numerical integrator generate a unique point $x(n+1), u(n+1)$ which is along the exponentiation of the flow of the original system along the constraint manifold?

Unlike ODEs, answering these questions is a non-trivial task, and in the general case they remain unanswered.

DAEs appear in several engineering problems which are described as follows:

- *Control systems with servo constraints* These problems are of the form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ h(x) - r(t) &= 0. \end{aligned} \tag{2}$$

  Here, $h(x)$ is an output function which is required to follow a reference trajectory $r(t)$. $g = [g_1, \ldots, g_m]$ such that $g_i : U \rightarrow T_x(M)$ are independent vector fields. This is a dynamic inversion problem, where a *feedforward* control $u_f(t)$ needs to be determined such that the servo constraint $h(t) = r(t)$ is satisfied, while solving the ODE. When $\dim(h(x)) = \dim(u) < n$, this may lead to a unique control u_f. When $\dim(h(x)) < \dim(u) < n$, there may be several control solutions and one may determine a solution that optimizes an appropriately defined cost. Works such as Blajer and Kołodziejczyk[6] and Brüls et al.[8] address this problem in the context of mechanical systems which are path constrained. In Baumgarte[5], in addition to solving the dynamic inversion problem, a feedback is applied to stabilize the servo constraints. However, these works assume certain structural conditions on the dynamics, thereby restricting the class of problems they address.

Mechanical systems: Mechanical systems with naturally occurring holonomic or nonholonomic constraints are regular.

- *Semi-implicit differential systems* These problems are of the form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + q(x)v \\ \phi(x) - \lambda(t) &= 0 \end{aligned} \tag{3}$$

$$y = h(x). \tag{4}$$

  Here $q = [q_1, \ldots, q_p]$ such that $q_i : V \rightarrow T_x(M)$ are independent vector fields. $v \in V$ is a set of *algebraic* variables which intrinsically evolve so as to satisfy the constraint $\phi(x) - \lambda(t) = 0$, where $\lambda(t)$ is a constrained trajectory. u are controls that may be appropriately varied to regulate an output y. As described in Krishnan and McClamroch[20], McClamroch[31], and Yim and Singh[61], an example of such a problem is a mechanical system with holonomic constraints. In these problems, the constraint manifold is described by $\phi(x) = 0$, and v is a set of *Lagrange multipliers* or constraint forces whose evolution is intrinsic. Another class of control problems in which such DAEs appear are those that involve feedback design based on the *immersion and invariance* principle[4], which is widely applicable in adaptive control design. Here, a suitable submanifold M of X is identified such that the controlled vector field, when restricted to $T(M)$, possesses certain desirable properties such as global exponential stabilizability and finite time reachability. The input space is then bifurcated into two components, one that achieves the desired system performance when restricted to M, and another that asymptotically drives the system toward M and renders M invariant. Here, the former can be considered as u and the latter as v.
- *Implicit forms of explicit control systems* Consider a control system

$$\dot{x} = f(x, u). \tag{5}$$

  If $\frac{\partial f}{\partial u}$ is of full rank, the implicit function theorem can be applied to express u as a function of x and $\dot{x}$, resulting in the implicit system

$$F(x, \dot{x}) = 0. \tag{6}$$

  Here, $\text{rank} \frac{\partial F}{\partial \dot{x}} = \dim(X) - \dim(U)$. This form is later discussed in this article in the context of differential flatness as presented in Levine[22], Fliess et al.[16,17], and Rouchon et al.[51].

Over the last two decades, deterministic differential algebraic equations have been studied in the following literature from the geometric viewpoint. For example, Reich[46], Brenan et al.[7], and Rheinbolt[50] used the analysis of differential equations on manifolds for a qualitative treatment of

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DAEs. More recently, Krishnan[20] and McClamroch[31] have studied the feedback stabilization of deterministic implicit control systems and have produced an explicit state space local realization under certain *regularity assumptions which will be discussed later. In Yim and Singh*[61], the problem of feedback linearization has been considered after extending the input trivially, under similar regularity conditions. Xiaoping[59] and Liu* and WC Ho[26], under slightly less restrictive assumptions, construct a static state feedback law for u so that the constraint is satisfied while simultaneously stabilizing certain outputs. In Xiaoping and Celikovsky[60] and Liu et al.[27], a dynamic state feedback for u is synthesized, which renders the system regular as described in Krishnan and McClamroch[20], enabling feedback control design. In da Silva et al.[13] and Pereira da Silva and Batista[40], state space realizations of implicit control systems have been studied by applying the dynamic extension algorithm as described in Nijmeijer and Van der Schaft[36] to the tracking output and constraints, in order to decompose the input into two components, one that tracks the output and another that independently satisfies the constraint. In Levine[22], Fliess et al.[17], Rouchon et al.[51], and Antritter and Lévine[1], a particular class of control problems whose implicit representation is *differentially flat* has been studied. The flatness property has been used to construct a dynamic feedback and diffeomorphism which transforms the system into a linear controllable canonical form, thereby enabling globally stable feedback design.

Regularity: Regularity implies that the relative degree with respect to algebraic variables is not larger than the relative degree with respect to controls.

In the first section of this article, we consider control systems described by nonlinear DAEs of the form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + q(x)v \\ \phi(x) - \lambda(t) &= 0, \end{aligned} \tag{7}$$

where $x \in X$ is an n-dimensional smooth manifold, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^p$, $f \in V^\infty(X)$, $g = [g_1, \ldots, g_m]$ where $g_i \in V^\infty(X)$, $q = [q_1, \ldots, q_p]$ where $q_i \in V^\infty(X)$, and $\sigma = [\sigma_1, \ldots, \sigma_d]$ where $\sigma_i \in V^\infty(X)$. $\phi(x) - \lambda(t)$ is an algebraic path constraint, describing a submanifold of the time-augmented manifold $X \times \mathbb{R}$, with $\phi(x) = [\phi_1, \ldots, \phi_m]^T : X \to \mathbb{R}^m$ and $\lambda(t) \in \mathbb{R}^m$ being smooth. We may further assume that these fields are analytic and Lipschitz. We refer to u as the control input, which is externally applied and v as the constraint input which is intrinsically determined so that the constraint is satisfied. It can be seen that the equations (7) are not in a state space form, rather they are in an implicit form due to the constraints. The problem is to express the intrinsic evolution $v(t)$ as an endogenous map of the states, and the externally applied input u, such that an explicit control system is obtained, for which the constraint $\phi - \lambda = 0$ is invariant. Equations of the form (7) represent an important class of mathematical control problems that have not been addressed in generality. (It will be shown that these equations generalize (1) using a dynamic extension.) We apply the geometric theory of input–output decoupling via dynamic extension to obtain a set of sufficient conditions on the DAE, in order to determine whether there exists an endogenous state feedback law for v which dynamically depends on x and u such that the DAE is satisfied for a restricted set of initial conditions and control inputs. We apply dynamic precompensators such as those obtained by the dynamic extension algorithm as described in Isidori[18], or the more general ones as in Martin[30] and Respondek[48]. We state sufficient conditions on the *Cartan prolongation*[22] of the unconstrained system such that the local existence and uniqueness of solutions are guaranteed. Further, the DAE admits a local explicit representation which is used for control design. By this, we mean a submanifold N of the level set $\phi(x) - \lambda(t) = 0$ and an explicit control system on N with a local description

$$\dot{\eta} = \bar{f}(t, \eta) + \bar{g}(t, \eta)\bar{u}. \tag{8}$$

Further, we consider the system (7) along with nonlinearly coupled white noise and extend the above result to obtain conditions on the noise fields such that the stochastic DAE admits an explicit state space form, thereby enabling one to apply classical methods of stochastic stabilization[19]. With the obtained local representations, questions about uniqueness, stability, nature of solutions, and consistency of numerical conditions can be answered with ease.

In the second section of the article, we review the theory of *differential flatness* of DAEs and apply it in designing globally stabilizing control laws for a class of systems. Consider the implicit form of a control system as described in (6). By differential flatness, we mean the existence of a *flat output* function $y = h(x, \dot{x}, \ldots, x^\nu)$, where $\dim(y) = m$, and a smooth map $x = \varphi(y, \dot{y}, \ddot{y}, \ldots, y^{(\mu)})$ such that

$$F(\varphi(y, \dot{y}, \ddot{y}, \ldots, y^{(\mu)}), \dot{\varphi}(y, \dot{y}, \ddot{y}, \ldots, y^{(\mu+1)}) = 0. \tag{9}$$

For this class of systems, we show that there exist a dynamic feedback

$$\begin{aligned} \dot{z} &= \alpha(x, z, v), \\ u &= \beta(x, z, v) \end{aligned} \tag{10}$$

and a diffeomorphism $Y = \Phi(x,z)$ such that the push-forward of the dynamics (5), along with the above dynamic feedback, is of the form:

$$\dot{Y} = \mathcal{A}Y + \mathcal{B}v, \tag{11}$$

where $(\mathcal{A}, \mathcal{B})$ is in the linear controllable canonical form. This enables globally stabilizing feedback control design for the nonlinear system. There have been three salient mathematical approaches to analyzing differential flatness of implicit systems, i.e., finite dimensional differential geometric approaches[9, 10, 52, 53, 56], differential algebra-based approaches[2, 51], and infinite dimensional differential geometry of jet bundles and prolongations[11, 12, 17, 22, 43, 44, 58].

2 Explicit State Space Representation

2.1 Regularity

Consider the deterministic system given by

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + q(x)v \\ \phi(x) &- \lambda(t) = 0. \end{aligned} \tag{12}$$

Definition The above system is said to be *regular*[31] at $x_0 \in X$ if the following conditions are satisfied in a neighborhood of x_0: (Let $L_f h(x)$ denote the Lie derivative of the function $h(x)$ with respect to the vector field f and $L_f^k h(x) = L_f(L_f^{k-1}h(x))$). $\exists$ positive integers $\gamma_1, \ldots, \gamma_p$ such that

- A1. $L_{q_i} L_f^k \phi_j(x) = 0, \ \forall k = 0, \ldots, \gamma_j - 2, \ \forall i, j = 1, \ldots, p$;
- A2. the $p \times p$ *decoupling* matrix $A(x)$ whose entries are given by $a_{ji} = L_{q_i} L_f^{\gamma_j - 1} \phi_j(x)$; is of full rank p at x_0
- A3. $L_{g_i} L_f^k \phi_j(x) = 0, \ \forall k = 0, \ldots, \gamma_j - 2, \ \forall i = 1, \ldots, m, j = 1, \ldots, p$.

The above definition essentially says that ϕ has a strict relative degree with respect to v, which is not larger than its relative degree with respect to u. With this assumption, it is clear[31, 36] that a (time-varying) state feedback law for the intrinsic evolution $v(t)$ can be obtained as

$$\begin{aligned} v = A(x)^{-1}&([\lambda_1(t)^{(\gamma_1)}, \ldots, \lambda_p(t)^{(\gamma_p)}]^T \\ &- [L_f^{\gamma_1}\phi_1(x), \ldots, L_f^{\gamma_p}\phi_p(x)]^T - \bar{A}(x)u(t)), \end{aligned} \tag{13}$$

where the $p \times m$ matrix $\bar{A}(x)$ is given as $\bar{a}_{ji} = L_{g_i} L_f^{\gamma_j - 1} \phi_j(x)$.

From Isidori[18] and Nijmeijer and Van der Schaft[36], we can show that the functions $\{L_f^k \phi_j(x), \ k = 0, \ldots, \gamma_j - 1, \ j = 1, \ldots, p\}$ are differentially independent and can therefore be extended to a diffeomorphism by a set of functions $\eta_1, \ldots, \eta_k$, where $k = n - \gamma_1 - \cdots - \gamma_p$. Define this diffeomorphism $(Y, \eta) = \Phi(x)$, where $Y_i^j = L_f^k \phi_j(x)$, the derivatives of $\phi_j(x)$. A direct computation shows that the push-forward of the dynamics (12) under the above diffeomorphism and feedback law can be obtained as

$$\begin{aligned} &Y_1^{\gamma_1} = 0, \\ &\vdots, \\ &Y_p^{\gamma_p} = 0, \\ &\dot{\eta} = P(\eta, Y) + Q(\eta, Y)u. \end{aligned} \tag{14}$$

The manifold on which the constraint dynamics evolve is given by

$$N = \{x : L_f^k \phi_j(x), \ k = 0, \ldots, \gamma_j - 1, \ j = 1, \ldots, p\}. \tag{15}$$

Therefore, the DAE system is locally well-posed and impulse free for any initial condition in N and any integrable $u(t)$. Further, the local stability and nature of the DAE solutions can be analyzed by analyzing the ODE:

$$\dot{\eta} = P(\eta, Y_r(t)) + Q(\eta, Y_r(t))u, \tag{16}$$

where $Y_r(t)$ is obtained by setting $y(t) = \lambda(t)$.

We consider the following model of a constrained robot system as discussed in Krishnan and McClamroch[20]:

Example 2.1

$$\begin{aligned} &M(q)\ddot{q} + H(q, \dot{q}) = J^T(q)v + u \\ &\phi(q) = 0, \end{aligned} \tag{17}$$

where $q \in \mathbb{R}^n$ is the parametrization of generalized displacement, $M(q)$ is the inertia matrix which is symmetric and positive definite, $H(q, \dot{q})$ is the vector of Coriolis, centripetal, and gravity forces, $u \in \mathbb{R}^m$ is a vector of generalized external forces, $\phi(q)$ is a constraint manifold, and $v \in \mathbb{R}^p$ are the intrinsically applied reaction forces that satisfy the constraint. $J = \frac{\partial \phi}{\partial q}$ is of full row rank p. Since M is positive definite and symmetric, we can write $M^{-1} = \hat{M}\hat{M}^T$, thereby obtaining the decoupling matrix as

$$A(q) = J(q)\hat{M}\hat{M}^T J(q)^T \tag{18}$$

which is of rank p uniformly. Thus, it can be verified that the regularity assumptions [A1], [A2], [A3] are uniformly satisfied. Therefore, (17) can be transformed into an explicit control system as given by (16).

An example of such a system is that of a simple pendulum subject to a control torque about the axis of rotation. The equations of motion are written in the constrained Euler–Lagrange form, where the Lagrange multiplier λ corresponding to the circular constraint denotes the tension in the rod which maintains the constraint. The control input is given by u.

$$\begin{aligned} m\ddot{x} &= 2\lambda x + yu \\ m\ddot{y} &= 2\lambda y - mg - xu \\ x^2 + y^2 - l &= 0 \end{aligned} \tag{19}$$

It can be observed that the above set of equations satisfies the assumptions [A1], [A2], [A3] with relative degree 2. Since the control force acts orthogonal to the constraint force, we can solve for λ independent of u as

$$\lambda(x, y, \dot{x}, \dot{y}) = \frac{mgy - m\dot{x}^2 - m\dot{y}^2}{2l}. \tag{20}$$

We refer the reader to Krishnan and McClamroch[20] and Yim and Singh[62] for the details of control design for such systems.

3 Dynamic Regularization of DAE Systems

We now generalize the above methods to systems which may not be regular.

Definition The implicit system (12) is said to be *dynamically regularizable* via a precompensation at x_0 if there exists a dynamic precompensator Π with state $z \in \mathbb{R}^{\nu}$:

$$\begin{aligned} \dot{z} &= \alpha(\bar{x}, z, \bar{v}) \\ v &= \beta(\bar{x}, z, \bar{v}), \end{aligned} \tag{21}$$

where $\bar{x} = (x, u, \dot{u}, \ldots, u^{(\mu-1)})$ contains a trivial input extension with μ being finite, such that the compensated system with state $(\bar{x}, z)^T$, constraint variable $\bar{v}$, and input $\bar{u} = u^{(\mu)}$ is regular at $(\bar{x}_0, z_0)$.

3.1 Cartan Prolongations

An important tool for characterizing the geometry of regularizability is Cartan Prolongations or Cartan Fields.

Definition For the explicit system given by (12) without the constraint, we define its Cartan prolongation as the smooth vector (Cartan) field on the infinite dimensional manifold $\mathcal{X} = X \times \mathbb{R}^m_\infty \times \mathbb{R}^p_\infty$ with coordinates $\zeta = (x, u^0, v^0, u^1, v^1, \ldots)$ as $F \in V^\infty(\mathcal{X})$, where

$$F(\zeta) = \sum_{i=1}^{n} \left(f_i(x) + \sum_{j=1}^{m} g^i_j(x) u^0_j + \sum_{k=1}^{p} q^i_k(x) v^0_k \right) \frac{\partial}{\partial x_i} + \sum_{l \geq 0} \left(u^{l+1} \frac{\partial}{\partial u^l} + v^{l+1} \frac{\partial}{\partial v^l} \right), \tag{22}$$

where $u^0 = u$, $v^0 = v$, and $u^1, v^1, \ldots$ are subsequent derivatives.

With this vector field, we can describe the following system on $\mathcal{X}$ as

$$\dot{\zeta} = F(\zeta) \tag{23}$$

which we will call Ξ, denoting the flow of the vector field F. Ξ is uniquely identified with the explicit system given by (12) without constraint and will be used interchangeably. It is well known that any smooth function on $\mathcal{X}$ depends locally on only finitely many coordinates (due to its cylinder topology). Hence, we can define the Lie derivative of a smooth function $L_F h(x, u^0, v^0)$ and subsequent iterated Lie derivatives $L^k_F h(x, u^0, v^0)$ as before (the reader is referred to Levine[22] for details).

Definition Given $\mu > 0$, define the restricted Cartan prolongation F_μ as

$$\begin{aligned} F_\mu &:= F/S(x, u^0, \ldots, u^{\mu-1}, v^0, v^1, \ldots) \\ &= \sum_{i=1}^{n} \left(f_i(x) + \sum_{j=1}^{m} g^i_j(x) u^0_j + \sum_{k=1}^{p} q^i_k(x) v^0_k \right) \frac{\partial}{\partial x_i} + \sum_{l \geq 0} \left(v^{l+1} \frac{\partial}{\partial v^l} \right) \\ &\quad + \sum_{s=0}^{\mu-2} \left(u^{s+1} \frac{\partial}{\partial u^s} \right). \end{aligned} \tag{24}$$

For example, the system Ξ_0 denotes the unconstrained system in (12) with zero control input and Ξ_1 denotes the unconstrained system in (12) with the input u being an arbitrary constant u^0, which is given as an explicit system on $X \times \mathbb{R}^m$

$$\begin{aligned} \dot{x} &= f(x) + g(x)u^0 + q(x)v \\ \dot{u}^0 &= 0. \end{aligned} \tag{25}$$

Explicit system: The explicit system need not be a submanifold of the original state space, but that of its associated higher order jet space.

Definition Following Martin[30] and Respondek[48], given the constrained output $\phi(x)$ and the restricted Cartan field F_μ, we define the following codistributions on $S \subset \mathcal{X}$:

$$\mathcal{E}_\mu^{-1} = \mathrm{span}\{\mathrm{d}\bar{x}\}$$
$$\mathcal{E}_\mu^{j} = \mathrm{span}\{\mathrm{d}\bar{x}, \mathrm{d}\phi, \ldots, \mathrm{d}L_{F_\mu}^{j}\phi\},\ j > 0, \tag{26}$$

where $\bar{x} = (x, u^0, \ldots, u^{\mu-1})$.

3.2 Dynamic Extension Algorithm

An important tool for constructing a regularizing feedback which we will use is the dynamic extension algorithm. There are various versions of this algorithm, and we will use the version given in Martin[49]. The equivalence with other versions can be studied in Di Benedetto et al.[14]. We will state the algorithm when applying to the system Ξ_μ in brief. The reader is referred to Respondek[49] for more details. Consider the system Ξ_μ, and denote the Cartan prolongation F_μ as F_μ^0. Let $\bar{x} = (x, u^0, \ldots, u^{\mu-1}), G^0 = \frac{\partial}{\partial v}$.

- *Step 1* Let ρ_i^0, $1 \le i \le p$, denote the smallest integer such that $L_{G^0}L_{F_\mu^0}^{\rho_i^0}\phi_i \neq 0$, denote $\rho^0 = (\rho_1^0, \ldots, \rho_p^0)$, $D^1(\bar{x}, v) = L_{G^0}L_{F_\mu^0}^{\rho_i^0}\phi$. Assume that $r_1(\bar{x}, v) = \mathrm{rank}\, D^1(\bar{x}, v)$ is constant around $(\bar{x}_0, v_0)$. Now reorder ϕ_i such that the first r_1 rows of D^1 are independent and $\rho_1^0 \le \cdots \le \rho_{r_1}^0$ is a minimal r_1-tuple among all such re-orderings.
- *Step 2* Using the implicit function theorem, apply the invertible transformation $v = \alpha^1(\bar{x}, w^1, \bar{w}^1)$ such that $L_{F_\mu^1}^{\rho^0}\phi = \begin{bmatrix} w^1 \\ \Psi^1(x, w^1) \end{bmatrix}$, where $F_\mu^1(\bar{x}, w^1, \bar{w}^1) = F_\mu^0(\bar{x}, \alpha^1(\bar{x}, w^1, \bar{w}^1))$.
- *Step 3* Denote $y = (y^1, \bar{y}^1) = (\phi^1, \bar{\phi}^1)$, where $\phi^1 = (\phi_1, \ldots, \phi_{r_1})$. Let ρ^1 denote the subindex of ρ^0 corresponding to ϕ^1. Change the V coordinates to W such that $\dot{w}^{1,i} = w^{1,i+1}$. Denote $G^1 = \frac{\partial}{\partial w^1}$ and $\bar{G}^1 = \frac{\partial}{\partial \bar{w}^1}$. (Here V and W denote the jet coordinates).
- *Step 4* Only consider the outputs $\bar{y}^1$ and analyze the dependence on $\bar{w}^1$. Let $\bar{\rho}_i^1$ denote, for every $i > r_1$, the smallest integer such that $L_{\bar{G}^1}L_{F_\mu^1}^{\bar{\rho}_i^1}\phi_i \neq 0$. Denote $\bar{\rho}^1 = (\bar{\rho}_{r_1+1}^1, \ldots, \bar{\rho}_p^1)$, $D^2(\bar{x}, W^1) = L_{\bar{G}^1}L_{F_\mu^1}^{\bar{\rho}^1}\bar{\phi}^1$. Define $r_2(\bar{x}, W^1) = \mathrm{rank}\, D^2(\bar{x}, W^1)$, where W^1 consists of w^1 and the time derivatives appearing in the D^2 expression.
- *Step 5* Assuming again that r_2 is constant around $(\bar{x}_0, V_0^1)$, reorder ϕ_i such that the first r_2 rows of D^2 are independent. Apply an invertible transformation $\bar{w}^1 = \alpha^2(\bar{x}, W^1, w^2, \bar{w}^2)$ such that $L_{F_\mu^2}^{\bar{\rho}^1}\bar{\phi}^1 = \begin{bmatrix} w^2 \\ \Psi^2(x, W^1, w^2), \end{bmatrix}$ where F_{μ^2} is the modification of F_{μ^1} under static feedback α. Denote $\bar{\phi}^1 = (\phi^2, \bar{\phi}^2)$, where $\phi^2 = (\phi_{r_1+1}, \ldots, \phi_{r_1+r_2})$, and correspondingly partition $\bar{y}^1 = (y^2, \bar{y}^2)$. Let ρ^2 denote the subindex of $\bar{\rho}^1$ corresponding to ϕ^2.
- *Step 6* Change the W coordinates such that $\dot{w}^{2,i} = w^{2,i+1}$, and let $G^2 = \frac{\partial}{\partial w^2}$ and $\bar{G}^2 = \frac{\partial}{\partial \bar{w}^2}$. The algorithm is iterated say k times until

$$L_{F_\mu^k}^{\rho^i}\phi_i = w^i,\ i = 1, \ldots, k. \tag{27}$$

Let $r = r_1 +, \ldots, +r_k$. If $r = p$, then the compensator obtained is regular. Since the algorithm terminates, $L_{F_\mu^k}^{j}\phi_i$ does not depend on $\bar{w}^k$ for any $j \ge 0$ and $i \ge r$. After k iterations, for the system which is affine in v, the compensator is of the form:

$$\begin{aligned} \dot{z} &= \alpha_1(\bar{x}, z) + \alpha_2(\bar{x}, z)w \\ v &= \beta_1(\bar{x}, z) + \beta_2(\bar{x}, z)w, \end{aligned} \tag{28}$$

and the closed loop system is of the form:

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{q}(\tilde{x})w, \tag{29}$$

where $\tilde{x} = (\bar{x}, z)$. The compensated system is denoted by $\Xi_\mu \circ \Pi_e$. A point $(\bar{x}_0, V_0)$ is called a regular point of the algorithm if the constant rank assumptions at each stage are valid in a neighborhood.

3.3 Conditions for Dynamic Regularizability

We are now ready to state the main result of the paper which states sufficient conditions for regularizability of (12).

Theorem 3.1 *Suppose there exists a finite μ, and the corresponding system Ξ_μ with restricted prolongation F_μ, and the corresponding codistributions satisfy the following conditions around $(\bar{x}_0, V_0)$:*

C1. $\mathcal{E}^{n+\mu m}$ *is of constant rank in a neighborhood of* $(\bar{x}_0, V_0)$.

C2. $\dim(\mathcal{E}^{n+\mu m+1}(\bar{x}, V)/\mathcal{E}^{n+\mu m}(\bar{x}, V)) = p$.

C3. Further if the point $(\bar{x}_0, V_0)$ *is a regular point of the extension algorithm, and at each* kth *iteration* $\forall 1 \le l \le p$

$$\left\{\frac{\partial}{\partial u_l^{\mu-1}}\right\} \text{inker} \left(\{\mathrm{d}L_{F_\mu^k}^{\gamma_i}\phi_i,\ 1 \le i \le k, 0 \le\ \gamma_i \le \rho_i - 1\}\right) + \mathrm{span}\{\tilde{q}_j,\ 1 \le j \le p\}, \tag{30}$$

then the implicit control system is dynamically regularizable with a precompensator derived from the extension algorithm.

Proof

Given μ, the original system Ξ can be written after a trivial input extension and precompensation (28) as

$$\begin{aligned} \dot{\bar{x}} &= \bar{f}(\bar{x}) + q(\bar{x})v + \bar{g}\bar{u} \\ \dot{z} &= \alpha_1(\bar{x}, z) + \alpha_2(\bar{x}, z)w \\ v &= \beta_1(\bar{x}, z) + \beta_2(\bar{x}, z)w, \end{aligned} \tag{31}$$

where

$$\bar{f} = f(x) + g(x)u^0 + \sum_{j=0}^{\mu-2} u^{j+1}\frac{\partial}{\partial u^j},$$

$$\bar{g} = \left[\frac{\partial}{\partial u_1^{\mu-1}}, \ldots, \frac{\partial}{\partial u_m^{\mu-1}}\right]$$

and $\bar{u} = u^\mu$. Hence, $\Xi \circ \Pi_e$ can be written as the system $\Xi_\mu \circ \Pi_e$ with an additional exogenous input $\bar{u}$ which is coupled to the system through the vector fields $\{\frac{\partial}{\partial u_i^{\mu-1}}, i = 1, \ldots, m\}$. At each iteration, we know that the one-forms $\mathrm{d}L^{\gamma_i}_{F^k_\mu}\phi_i$ $1 \le i \le k, 0 \le \gamma_i \le \rho_i - 1$ are independent[36] and hence can be completed to a basis, yielding the diffeomorphism $(Y, \eta) = \Phi(\bar{x}, z)$, where the partial coordinates Y are given by

$$Y = y_i^j = L^j_{F^k_\mu}\phi_i,\ 1 \le j \le \rho_i,\ 1 \le i \le r. \tag{32}$$

In these new coordinates, the push-forward of the dynamics through the diffeomorphism $\Phi(\bar{x}, z)$ can be obtained as

$$\begin{aligned} \dot{Y} &= AY + B\hat{w} + c(\eta, Y)\bar{u} \\ \dot{\eta} &= R_1(\eta, Y) + R_2(\eta, Y)w + \bar{c}(\eta, Y)(\bar{u}), \end{aligned} \tag{33}$$

where $\hat{w} = [w^1, \ldots, w^k]^T$ and (A, B) are in Brunovsky normal form as obtained in (27). Applying the push-forward through Φ to condition [C3.], we obtain (using the identity $\Phi^{-1*}dh = d(h \circ \Phi^{-1})$

$$\begin{aligned} \Phi_*\left(\frac{\partial}{\partial u_l^{\mu-1}}\right) &\in ker\{\mathrm{d}L^j_{F^k_\mu}\phi_i(\Phi^{-1}(Y, \eta))\} \\ &+ \mathrm{span}(\Phi_*\tilde{q}_j); \end{aligned} \tag{34}$$

however, observe that $\Phi_*\tilde{q}_j = B_j$ and $\Phi_*\left(\frac{\partial}{\partial u_l^{\mu-1}}\right) = [c_l^T, \bar{c}_l^T]^T$ and $\mathrm{d}L^j_{F^k_\mu}\phi_i(\Phi^{-1}(Y, \eta)) = \mathrm{d}y_i^j$. Hence we obtain

$$[c_l^T, \bar{c}_l^T]^T \in \mathrm{span}\left\{\frac{\partial}{\partial \eta}\right\} + \mathrm{span}\{B_j\}. \tag{35}$$

From this, the partial dynamics of Y can be obtained as

$$\begin{aligned} \dot{y}_i^1 &= y_i^2 \\ &\vdots \\ \dot{y}_i^{\rho_i - 1} &= w_i + c^i(Y, \eta)\bar{u},\ 1 \le i \le r. \end{aligned} \tag{36}$$

It is thus evident that the regularity conditions [A1.] and [A3.] are satisfied. Further, [C1.] and [C2.] applied to the system Ξ_μ are the intrinsic conditions for input (v) and output (ϕ) decoupling by dynamic feedback as given in Martin[30]. However, it is well known[18] that if the dynamic extension algorithm can be iterated at a regular point and any regularizing dynamic compensator exists, then the extension algorithm produces the one that is minimal. This implies that

$$r = r_1 + r_2 + \ldots + r_k = p. \tag{37}$$

Hence, the decoupling matrix $A(\tilde{x}) = Id_p$, thereby satisfying condition [A2]. The invariant manifold can be obtained as the integral submanifold of the distribution

$$N^* = \mathrm{span}\left\{\frac{\partial}{\partial \eta_i}\right\}, \tag{38}$$

i.e.

$$N = \{\tilde{x} : L^j_{F^k_\mu}\phi_i(\tilde{x}) = 0\}, \tag{39}$$

and the explicit system is given as

$$\begin{aligned} \dot{\eta} &= R_1(\eta, Y_\lambda(t)) + R_2(\eta, Y_\lambda(t))w(t, \eta, \bar{u}) + \bar{c}(\eta, Y_\lambda(t))(\bar{u}) \\ w(t, \eta, \bar{u}) &= \begin{bmatrix} \lambda_1^{(\rho_1)} - c^{\rho_1}(Y_\lambda(t), \eta)\bar{u} \\ \vdots \\ \lambda_p^{(\rho_p)} - c^{\rho_p}(Y_\lambda(t), \eta)\bar{u} \end{bmatrix} \end{aligned} \tag{40}$$

where $Y_\lambda(t) = \{\lambda_i^{(j)}(t)\}$. It is evident that the feedback control structure does not explicitly depend on the constraint variable v as the compensator variables (hence η) are completely recovered by integrating w. Any output $h(x)$ which is differentially independent of $\{y_i^j\}$ and is completely determined by η coordinates as $h \circ \Phi^{-1}(\eta)$ can be tracked with an appropriate feedback law $\bar{u}$ in (85). The tracking however is subject to the case that the chosen output is minimum phase. In this case, any of the classical nonlinear control approaches can be applied (such as those in Isidori[18]). □

Normal form: The normal form (14) expresses constrained outputs in the prime form, allowing them to be independently stabilized for any control input.

This theorem has an implication that the regularizing structure for a general implicit control

Theorem 3.1: Here (Theorem 3.1), controls are considered as disturbances with measurement feedthrough.

system may be obtained through higher derivatives of the control. The following simple example illustrates this fact:

Example 3.1

$$\begin{aligned}
\dot{x}_1 &= \sin(x_2^2 u) + \sin(x_3^2 u) \\
\dot{x}_2 &= \frac{x_4}{2\cos(x_2^2 u) x_2 u} \\
\dot{x}_3 &= \frac{-x_4}{2\cos(x_3^2 u) x_3 u} \\
\dot{x}_4 &= v \\
\phi(x) &= x_1 = 0
\end{aligned} \tag{41}$$

Conditions of Theorem 3.1: The conditions of Theorem 3.1 can be understood as follows: the structure of infinity with respect to algebraic variables is no larger than the relative degree with respect to controls.

It can be seen that the conditions of the theorem fail for the Cartan prolongations F_μ for $\mu = 0$ and $\mu = 1$. $\ddot{y} \equiv 0$ for the systems Ξ_0 and Ξ_1. However, for $\mu = 2$ the conditions are satisfied. This has an interesting implication that a DAE singularity is encountered when $\dot{u} = 0$. Also note that if the conditions of the theorem are satisfied for μ_0, then they are satisfied for any $\mu > \mu_0$.

3.4 Stochastic Implicit Control Systems

In this section, we will consider the problem of constructing a regularizing compensator for stochastic differential algebraic control systems of the form:

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u + q(x)v + \sigma(x)\xi \\
\phi(x) &- \lambda(t) = 0,
\end{aligned} \tag{42}$$

where ξ is a d-dimensional white noise and σ is appropriately defined similar to g and q. We will present two cases, i.e., *strong* and *weak* state space forms. Here, by *weak* we mean that the stochastic processes described by the solutions of the explicit stochastic system are to be interpreted as generalized (distribution valued) processes. It may be possible for some of the constraint variables v to be directly expressed in terms of white noise. A rigorous treatment of such processes using the *Wick* product can be found in the reference[15]. It is natural in several physical problems for such a situation to arise. For example, in the well-known DAE describing the simple pendulum[7], if an externally applied torque is modeled as a white noise process, then the constraint variable, which is the tension in the rod, directly depends on the externally applied torque, i.e., white noise. In the *strong* case, none of the variables will depend on the noise directly, and further we will show that the explicit feedback structure does not depend on the constraint variables. The solution processes in this case can be interpreted in the strong stochastic sense.

We will now extend Theorem (3.1) of the previous section to the stochastic case.

Theorem 3.2 *Given the implicit stochastic control system* (42), *suppose there exist a finite μ and the corresponding deterministic system Ξ_μ with restricted Cartan prolongation F_μ for which the conditions* [C1], [C2], *and* [C3] *of Theorem* (3.1) *hold at a regular point $(\bar{x}_0, V_0)$ of the extension algorithm applied to Ξ_μ, and moreover the following condition is satisfied around $(\bar{x}_0, V_0)$:*

$$\begin{aligned}
\sigma_l(x), \left(\frac{\partial \sigma_l(x)}{\partial x}\sigma_l(x)\right) \in ker(\{\mathrm{d}L_{F_\mu^k}^{\gamma_i}\phi_i,\ 1 \le i \le k, \\
\times\, 0 \le \ \gamma_i \le \rho_i - 1\}) \\
+ \operatorname{span}\{\tilde{q}_j,\ 1 \le j \le p\},
\end{aligned} \tag{43}$$

then the implicit stochastic control system is regularizable with a canonical precompensation obtained from the extension algorithm and has an (possibly weak) explicit state space form. Further, if the following is satisfied:

$$\begin{aligned}
\sigma_l(x), \left(\frac{\partial \sigma_l(x)}{\partial x}\sigma_l(x)\right) \in ker(\{\mathrm{d}L_{F_\mu^k}^{\gamma_i}\phi_i,\ 1 \le i \le k, \\
\times\, 0 \le \ \gamma_i \le \rho_i - 1\}),
\end{aligned} \tag{44}$$

then we obtain a strong explicit state space form.

Proof

On the lines of the proof of (3.1), *the system* (7) *along with a precompensator obtained after k steps of the extension algorithm can be written in the Stratonovich form as*

$$\begin{aligned}
\dot{\bar{x}} &= \bar{f}(\bar{x}) + q(\bar{x})v + \bar{g}\bar{u} \\
&\quad - \frac{1}{2}\sum_{i=1}^{d} \frac{\partial \sigma_i(x)}{\partial x}\sigma_i(x) + \sigma(x) \circ \xi \\
\dot{z} &= \alpha_1(\bar{x}, z) + \alpha_2(\bar{x}, z)w \\
v &= \beta_1(\bar{x}, z) + \beta_2(\bar{x}, z)w
\end{aligned} \tag{45}$$

which is (31) along with the noise ξ coupled with the noise fields in $\sigma(x)$, along with the Itô drift correction term[37]. The term $\sigma(x) \circ \xi$ is to be interpreted in the sense that

$$\int \sigma(x) \circ \xi \,\mathrm{d}t = \int \sigma(x) \circ \mathrm{d}W$$

which is a standard Stratonovich integral[37]. As before, we consider the push-forward of the dynamics under the diffeomorphism Φ which are given by the following equations:

$$\begin{aligned}\dot{Y} &= AY + B\hat{w} + c(\eta, Y)\bar{u} \\ &\quad + s_1(\eta, Y) + s_2(\eta, Y) \circ \xi \\ \dot{\eta} &= R_1(\eta, Y) + R_2(\eta, Y)w + \bar{c}(\eta, Y)(\bar{u}) \\ &\quad + \bar{s}_1(\eta, Y) + \bar{s}_2(\eta, Y) \circ \xi,\end{aligned} \tag{46}$$

where

$$\begin{bmatrix} s_2 \\ \bar{s}_2 \end{bmatrix} = \Phi_* \sigma, \quad \begin{bmatrix} s_1 \\ \bar{s}_1 \end{bmatrix} = \Phi_* \hat{\sigma}, \tag{47}$$

where $\hat{\sigma} = -\frac{1}{2}\sum_{i=1}^{d} \frac{\partial \sigma_i(x)}{\partial x}\sigma_i(x)$.

We again apply the push-forward of the map Φ to condition (43) obtaining $[s_1^T, \bar{s}_1^T]^T, \ [s_2^T, \bar{s}_2^T]^T \in \text{span}\left\{\frac{\partial}{\partial \eta}\right\} + \text{span}\{B_j\}$. Hence, we obtain the partial dynamics of Y as

$$\begin{aligned}\dot{y}_i^1 &= y_i^2 \\ &\vdots \\ \dot{y}_i^{\rho_i - 1} &= w_i + c^i(Y, \eta)\bar{u} + s_1^i(Y, \eta) \\ &\quad + s_2^i(Y, \eta) \circ \xi, \ 1 \le i \le r.\end{aligned} \tag{48}$$

Further, note that rewriting the stochastic dynamics in the Itô form with a corrected drift does not change the structure of the partial Y dynamics, i.e.

$$\begin{aligned}\dot{y}_i^1 &= y_i^2 \\ &\vdots \\ \dot{y}_i^{\rho_i - 1} &= w_i + c^i(Y, \eta)\bar{u} + \hat{s}_1^i(Y, \eta) \\ &\quad + s_2^i(Y, \eta)\xi, \ 1 \le i \le r,\end{aligned} \tag{49}$$

where $\hat{s_1}$ is s_1 along with the Itô drift correction. We have used the fact from Pan[38] that the drift correction commutes with the push-forward under a diffeomorphism. We now solve for the constraint variable as before to obtain

$$w(t, \eta, \bar{u}) = \begin{bmatrix} \lambda_1^{(\rho_1)} - c^{\rho_1}(Y_\lambda(t), \eta)\bar{u} - \hat{s}_1^{\rho_1}(Y, \eta) - s_2^{\rho_1}(Y, \eta)\xi \\ \vdots \\ \lambda_p^{(\rho_p)} - c^{\rho_p}(Y_\lambda(t), \eta)\bar{u} - \hat{s}_1^{\rho_p}(Y, \eta) - s_2^{\rho_p}(Y, \eta)\xi \end{bmatrix}. \tag{50}$$

Denote

$$S_1 = \begin{bmatrix} \lambda_1^{(\rho_1)} - c^{\rho_1}(Y_\lambda(t), \eta)\bar{u} - \hat{s}_1^{\rho_1}(Y, \eta) \\ \vdots \\ \lambda_p^{(\rho_p)} - c^{\rho_p}(Y_\lambda(t), \eta)\bar{u} - \hat{s}_1^{\rho_p}(Y, \eta) \end{bmatrix},$$

$$S_2 = \begin{bmatrix} -s_2^{\rho_1}(Y, \eta) \\ \vdots \\ -s_2^{\rho_p}(Y, \eta) \end{bmatrix}.$$

Finally, the explicit stochastic control system is obtained as an Itô stochastic differential equation after incorporating the drift correction in $\hat{\bar{s}}_1$ as

$$\begin{aligned}d\eta &= [R_1(\eta, Y) + R_2(\eta, Y)S_1 + \bar{c}(\eta, Y)\bar{u} + \hat{\bar{s}}_1(\eta, Y)]dt \\ &\quad + (R_2(\eta, Y)S_2 + \bar{s}_2(\eta, Y))dW.\end{aligned} \tag{51}$$

This is to be interpreted in the weak sense (as a generalized process) as the constraint variable now depends directly on the white noise[15]. It can be verified that if the stronger conditions (44) hold then $S_2 = 0$, and the stochastic dynamics are to be interpreted in the strong sense, further in this case the feedback structure is independent of the constraint variable as in the deterministic case. However, it is important to mention that this characterization is only local in a neighborhood where the regularity assumptions are valid. Moreover, in the stochastic case, the solutions are valid only up to an explosion time, further only till such time when the solutions stay within the regularity regions. We will not explicitly state the methods to stabilize the outputs of the system; however, once an explicit local state space form has been obtained, any of the classical stochastic control and stabilization methods can be applied to the outputs of the system (such as those described in Khasminskii[19], van Handel[57]). □

Example 3.2

Consider the following Itô stochastic differential algebraic control system:

$$\begin{aligned}dx_1 &= (x_3 + 2v_1 + u)dt \\ dx_2 &= (x_4 + v_1 + u)dt \\ dx_3 &= (x_1^2 + x_5)dt \\ dx_4 &= (x_2^2 + x_5)dt \\ dx_5 &= v_2 dt + \sin(x_3)dW \\ x_1 &= 0 \\ x_2 &= 0 \\ y &= h(x) = x_4,\end{aligned} \tag{52}$$

where v_1 and v_2 are the algebraic constraint variables, u is the control variable, $\phi_1(x) = x_1 = 0$ and $\phi_2 = x_2 = 0$ are the algebraic constraints, y is the tracking output, and W is a standard Brownian motion. It can be verified that the conditions of (3.1) are violated for $\mu = 0$ and $\mu = 1$. However, all the conditions of (3.1) and (3.2) are satisfied for $\mu = 2$ and higher. Hence, the following equations are appended to the dynamics (with $u^0 := u$):

$$
\begin{aligned}
du^0 &= u^1 dt \\
du^1 &= \bar{u} dt
\end{aligned} \tag{53}
$$

with $\bar{u}$ being the new control. And the following regularizing compensator is appended to the dynamics:

$$
\begin{aligned}
dz_1 &= z_2 dt \\
dz_2 &= w_1 dt \\
v_1 &= z_1, v_2 = w_2,
\end{aligned} \tag{54}
$$

where w_1 and w_2 are the new constraint variables which are determined using (50) as follows:

$$
\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -2x_1x_3 - 2x_1z_1 - 2x_1u^0 - \bar{u} - \xi \\ -2x_2x_4 - 2x_2z_1 - 2x_1u^0 - \bar{u} - \xi \end{bmatrix}. \tag{55}
$$

Substituting this equation together with (53) and (54) into (52) gives the explicit stochastic control system after using $\xi dt = dW$. The new coordinates and diffeomorphism $(Y, \eta) = \Phi(\bar{x}, z)$ are computed as

$$
\begin{aligned}
y_1^0 &= x_1 \\
y_2^0 &= x_2 \\
y_1^1 &= x_3 + 2z_1 + u^0 \\
y_2^1 &= x_4 + z_1 + u^0 \\
y_1^2 &= x_1^2 + x_5 + 2z_2 + u^1 \\
y_2^2 &= x_2^2 + x_5 + z_2 + u^1 \\
\eta_1 &= x_4 \\
\eta_2 &= u^0 \\
\eta_3 &= u^1.
\end{aligned} \tag{56}
$$

The controlled invariant manifold N is obtained as the integral submanifold of

$$
N^* = \operatorname{span}\left\{ \frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \eta_2}, \frac{\partial}{\partial \eta_3} \right\}. \tag{57}
$$

The dynamics on N is obtained as

$$
\begin{aligned}
d\eta_1 &= \eta_3 dt \\
d\eta_2 &= \eta_3 dt \\
d\eta_3 &= \bar{u}
\end{aligned} \tag{58}
$$

with the differentially independent output $y = \eta_1$ along a trajectory $y_d(t)$. A dynamic feedback control law for u is designed as $\bar{u} = \ddot{y_d} - (\eta_3 - \dot{y_d}) - (\eta_1 - y_d)$ and $\ddot{u} = \bar{u}$, which results in a closed loop stochastic DAE.

Stochastic DAE: Stochastic DAE systems are realizable if the relative degree with respect to diffusion fields is no smaller than the structure at infinity with respect to algebraic variables.

Some engineering examples of nonregular DAE control systems are as follows. A broad class of chemical processes modeled by high-index, irregular DAE systems consists of multiphase systems where the individual phases are in thermodynamic equilibrium. Kumar and Daoutidis[21] and Liu et al.[28] present an example of a two-phase vapor–liquid reaction system with the two phases in physical equilibrium. The irregular system is first regularized via a dynamic extension obtained through an application of extension algorithm, followed by feedback control design for stabilizing the state of the system on the constraint manifold given by the conservation law.

Lu and Liu[29] present an example of a two-link robotic manipulator with two flexible joints, where the end-effector is in contact with a constraint surface. They demonstrate that an application of the extension algorithm reveals all the "hidden" constraints corresponding to the algebraic variable, thereby regularizing the DAE system. They further show that the algorithm based on dynamic extension results in a feedback controller such that the closed loop system admits an explicit local representation, which is used to analyze the stability of the system.

4 Differential Flatness-Based Control

Consider the implicit nonlinear control system as in (12) with $\lambda(t) \equiv 0$ and $\dim(X) = n + p$:

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)u + q(x)v \\
\phi(x) &= 0
\end{aligned} \tag{59}
$$

such that $[g(x), q(x)]$ has full rank $m + p$ in a suitable open, dense subset. The implicit function theorem can be applied here to eliminate the controls and constraint variables by expressing them as $u = U(x, \dot{x})$ and $v = V(x, \dot{x})$ to yield the following implicit representation:

$$
\begin{aligned}
&\dot{x} - (f(x) + g(x)U(x, \dot{x}) + q(x)V(x, \dot{x})) \\
&\quad = F(x, \dot{x}) = 0.
\end{aligned} \tag{60}
$$

Since $m + p$ variables were eliminated, it can be verified that rank $\frac{\partial F}{\partial \dot{x}} = n - m$, therefore resulting in a fully implicit DAE system. For this system, we analyze whether there exist a smooth function $y = h(x, \dot{x}, \ldots, x^{\nu})$ and a smooth function $\varphi(.)$ with $\dim(y) = m$ such that the following is satisfied:

$$
F(\varphi(y, \dot{y}, \ddot{y}, \ldots, y^{(\mu)}), \dot{\varphi}(y, \dot{y}, \ddot{y}, \ldots, y^{(\mu+1)})) = 0. \tag{61}
$$

Here, y is called a *flat output*. While there have been various efforts toward analyzing this condition (as briefed in Sect. 1), the approach discussed here is that of the *generalized Euler–Lagrange operator* which was presented in Antritter and Lévine[1].

4.1 DAEs on Manifolds of Jets of Infinite Order

As defined earlier, consider the infinite dimensional manifold $\mathcal{X} = M \times \mathbb{R}^n_\infty$, where M is the submanifold $\phi(x) = 0$ which is of dimension $n + p - p = n$,

with local coordinates $\bar{x} = (x, \dot{x}, \ldots)$. To this, we associate a trivial Cartan field defined as

$$\tau_x = \sum_{i=1}^{n} \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial}{\partial x_i^{(j)}}. \tag{62}$$

It can be seen that the Cartan field acts on the coordinates as a shift to the right, and therefore $\mathcal{X}$ is called a manifold of jets of infinite order. We now formalize the definition of implicit DAEs as in Antritter and Lévine[1] and introduce the notion of equivalence between DAEs:

Definition A *regular implicit system* is defined as the triplet $(\mathcal{X}, \tau_x, F)$, such that $\mathcal{X}$ is as defined earlier, τ_x is its corresponding trivial Cartan field, and F is *meromorphic* with rank $\frac{\partial F}{\partial \dot{x}} = n - m$, in an open dense subset.

Definition The regular implicit system $(\mathcal{Y}, \tau_y, G)$ is Lie–Backlünd equivalent (or L–B equivalent) to $(\mathcal{X}, \tau_x, F)$ in a neighborhood $(\mathcal{X}_0, \mathcal{Y}_0)$ of the pair $(\bar{x}_0, \bar{y}_0) \in \mathcal{X} \times \mathcal{Y}$ if and only if

(1) there exists a one-to-one meromorphic mapping $\Phi = (\varphi, \dot{\varphi}, \ldots)$ from $\mathcal{Y}_0$ to $\mathcal{X}_0$ with $\Phi(\bar{y}_0) = \bar{x}_0$ such that $\Phi_* \tau_y = \tau_x$;

(2) there exists a one-to-one meromorphic mapping $\Psi = (\psi, \dot{\psi}, \ldots)$ from $\mathcal{X}_0$ to $\mathcal{Y}_0$ with $\Psi(\bar{x}_0) = \bar{y}_0$ such that $\Psi_* \tau_x = \tau_y$;

(3) for every $\bar{y}$ such that $L^k_{\tau_y} G(\bar{y}) = 0$, $\forall k \geq 0$, the image $\bar{x} = \Phi(\bar{y})$ satisfies $L^k_{\tau_x} F(\bar{x}) = 0$, $\forall k \geq 0$, and conversely. This essentially means that the maps Φ and Ψ preserve the flows of the implicit systems.

The mappings Φ and Ψ are called mutually inverse *Lie–Backlünd isomorphisms* at $(\bar{x}_0, \bar{y}_0)$. It can be seen that there exists a one-to-one correspondence between trajectories of any two systems which are L–B equivalent.

We are now ready to define differential flatness of implicit DAE systems and characterize it based on the machinery of infinite dimensional jets.

4.2 Differential Flatness and its Characterization

Definition The regular implicit system $(\mathcal{X}, \tau_x, F)$ is locally flat in a neighborhood of $(\bar{x}_0, \bar{y}_0) \in \mathcal{X}_0 \times \mathbb{R}^m_\infty$ if and only if it is locally L–B equivalent around $(\bar{x}_0, \bar{y}_0)$ to the *trivial* system $(\mathbb{R}^m_\infty, \tau_{\mathbb{R}^m_\infty}, 0)$. In this case, Φ and Ψ are called mutually inverse trivializations.

Let $\varphi(\bar{y})$ be the first component of the trivialization Φ. Since φ map is continuous, it locally depends only on finitely many coordinates (this can be concluded by considering the cylinder topology of infinite product spaces). Hence, at $(\bar{x}_0, \bar{y}_0)$ we can write $\varphi(\bar{y}) = \varphi(y, \dot{y}, \ldots, y^{(\mu)})$. From this, along the flows of the system (5) we obtain

$$\begin{aligned} x &= \varphi(y, \dot{y}, \ldots, y^{(\mu)}), \\ u &= U(\varphi(y, \dot{y}, \ldots, y^{(\mu)}), \dot{\varphi}(y, \dot{y}, \ldots, y^{(\mu+1)})). \end{aligned} \tag{63}$$

Theorem 4.1 *The regular implicit system $(\mathcal{X}, \tau_x, F)$ is locally flat at $(\bar{x}_0, \bar{y}_0) \in \mathcal{X}_0 \times \mathbb{R}^m_\infty$ if and only if there exists a local, meromorphic invertible map $\Phi : \mathbb{R}^m_\infty \to \mathcal{X}_0$, with meromorphic inverse, satisfying $\Phi(\bar{y}_0) = \bar{x}_0$ such that*

$$\Phi^* \mathrm{d}F = 0. \tag{64}$$

Another way of analyzing the equation (64) is to characterize the change of coordinates corresponding to the mapping Φ. More precisely, we can write

$$\begin{aligned} &\sum_{j=1}^{m} \sum_{k=0}^{r_j} \left(\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_j^{(k)}} \mathrm{d}y_j^{(k)} + \frac{\partial F}{\partial \dot{x}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \varphi}{\partial y_j^{(k)}} \right) \mathrm{d}y_j^{(k)} \right. \\ &\left. + \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(k)}} \mathrm{d}y_j^{(k+1)} \right) = 0. \end{aligned} \tag{65}$$

Since the one-forms $\{\mathrm{d}y_i^{k_i},\ k_i < r_i\}$ are independent, we obtain the following system of equations:

$$\frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(r_j)}} = 0 \tag{66}$$

$$\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_j^{(k)}} + \frac{\partial F}{\partial \dot{x}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \varphi}{\partial y_j^{(k)}} \right) + \frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(k-1)}} = 0 \tag{67}$$

$$\frac{\partial F}{\partial x} \frac{\partial \varphi}{\partial y_j} + \frac{\partial F}{\partial \dot{x}} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \varphi}{\partial y_j} \right) = 0. \tag{68}$$

Flatness: Flatness in general remains an open problem, as intrinsic conditions for the solvability of the Euler Lagrange PDEs is not known.

We define a generalized Euler–Lagrange operator ξ_F corresponding to F as

$$\xi_F = \frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}}\right). \tag{69}$$

This terminology is justified by the well-known fact that when $n - m = 1$, the curves that extremize the cost function $J = \int_0^T F(x, \dot{x})\mathrm{d}t$ are those that satisfy the Euler–Lagrange equation $\xi_F = 0$. The above equation results in the following theorem which states necessary and sufficient conditions for differential flatness of the implicit system.

Theorem 4.2 *A necessary and sufficient condition for the implicit system $F(x, \dot{x})$ to be differentially flat is that there exists $(r_1, \ldots, r_m)$ with $\sum_{i=1}^{m} r_i + m \geq n$ and a solution φ of the following triangular system of PDEs in an open dense subset of $\mathcal{X}$*

$$\frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(r_j)}} = 0,$$

$$\frac{\partial F}{\partial \dot{x}} \frac{\partial \varphi}{\partial y_j^{(l)}} = \sum_{k=0}^{r_j - l - 1} (-1)^{k+1} \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\xi_F \frac{\partial \varphi}{\partial y_j^{l+k+1}}\right),$$

$$\sum_{k=0}^{r_j} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\xi_F \frac{\partial \varphi}{\partial y_j^{(k)}}\right) = 0,\ \forall l = 0, \ldots, r_j - 1 \tag{70}$$

such that $\mathrm{d}\varphi_1 \wedge \cdots \wedge \mathrm{d}\varphi_n \neq 0$.

Consider the following example control system with three states, two inputs, and one constraint:

Example 4.1

$$\dot{x}_1 = \bar{u}_1 +,\ \dot{x}_2 = u_2,\ \dot{x}_3 = \sin\left(\frac{u_1 + x_4}{u_2}\right),\ \dot{x}_4 = v$$
$$\Phi(x) = x_4 = 0 \tag{71}$$

whose implicit representation is

$$F(\mathbf{x}, \dot{\mathbf{x}}) := \dot{x}_3 - \sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right) = 0. \tag{72}$$

We obtain

$$\frac{\partial F}{\partial \dot{x}} = \left(-\dot{x}_2^{-1}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right), \dot{x}_1\dot{x}_2^{-2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right), 1\right)$$

and $\xi_F = (\eta_1, \eta_2, 0)$, where

$$\eta_1 = -\frac{\ddot{x}_2}{\dot{x}_2^2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right) - \frac{\ddot{x}_1\dot{x}_2 - \dot{x}_1\ddot{x}_2}{\dot{x}_3^2}\sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right),$$

$$\eta_2 = (\ddot{x}_1\dot{x}_2 - 2\dot{x}_1\ddot{x}_2)\left(\frac{\dot{x}_1}{\dot{x}_2^4}\sin\left(\frac{\dot{x}_1}{\dot{x}_2}\right) - \frac{1}{\dot{x}_3^2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\right).$$

The first two equations of the PDE (70), with $r_1 = r_2 = 2$, read

$$\frac{\partial \varphi_3}{\partial \dot{y}_j} - \frac{1}{\dot{x}_2}\cos\left(\frac{\dot{x}_1}{\dot{x}_2}\right)\left(\frac{\partial \varphi_1}{\partial \dot{y}_j} - \frac{\dot{x}_1}{\dot{x}_2}\frac{\partial \varphi_2}{\partial \dot{y}_j}\right) = 0,\ j = 1, 2 \tag{73}$$

If we let $\frac{\partial \varphi_3}{\partial \ddot{y}_j} = \frac{\partial \varphi_3}{\partial \dot{y}_j} = 0$ and introduce the variable, $\psi = \frac{\dot{x}_1}{\dot{x}_2}$, and with $\frac{\partial \psi}{\partial \ddot{y}_j} = 0$, we obtain

$$\frac{\partial \varphi_1}{\partial \ddot{y}_j} - \psi\frac{\partial \varphi_2}{\ddot{y}_j} = \frac{\partial}{\partial \ddot{y}_j}(\varphi_1 - \psi\varphi_2) = 0. \tag{74}$$

Setting $\kappa(y, \dot{y}) = \varphi_1 - \psi\varphi_2$, we obtain

$$\dot{\kappa} = \dot{\varphi}_1 - \psi\dot{\varphi}_2 - \dot{\psi}\varphi_2 = -\dot{\psi}\varphi_2. \tag{75}$$

From the definition of κ, we thus obtain

$$\varphi_1 = \kappa - \frac{\dot{\kappa}\sqrt{1 - \dot{\varphi}_3}}{\ddot{\varphi}_3}\arcsin(\dot{\varphi}_3), \varphi_2 = -\frac{\dot{\kappa}}{\ddot{\varphi}_3}\sqrt{1 - \dot{\varphi}_3} \tag{76}$$

with $x_3 = \varphi_3 = y_2$. From the above equations, one can obtain y_1 and y_2, and it can be verified that the third equation in the PDE (70) is satisfied. This shows that the system is differentially flat.

As such, solvability conditions for the Euler Lagrange PDEs are still an open question. In the general case, there exists no set of sufficient and necessary conditions to characterize flatness and consequently no algorithm to construct flat outputs. In certain special cases however, there exist constructive results which are summarized as follows:

- For single-input systems, the popularly known conditions for feedback linearization as stated in Respondek[47] completely characterize flatness and also construct flat outputs as a solution of a first-order PDE.
- Rathinam and Murray[45] present conditions for configuration flatness of Lagrangian mechanical systems which are underactuated by one degree.
- In Nicolau and Respondek[34], flatness of mechanical systems with 3 degrees of freedom has been characterized.
- In Li and Respondek[23], flatness and flat outputs of two-input driftless control systems have been characterized.
- In Li et al.[25], flatness of control systems which are static state feedback equivalent to triangular forms has been analyzed.
- In Nicolau and Respondek[33], [35] and Respondek[32], normal forms and flatness of control systems linearizable via one- and two-fold prolongations have been characterized.

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- An extended notion of differential flatness, i.e., orbital flatness where the system is flat after a reparameterization of the trajectories via a state-dependent time-scale change, has been presented in Nicolau and Respondek[24]. Flatness of such systems via static state feedback has been characterized.

4.3 Control Design for Flat Systems

For the nonlinear system (5), assume that from the above theorem a set of flat outputs is obtained as $y_i = h_i(x)$ such that

$$\begin{aligned} x &= \varphi(y, \dot{y}, \ldots, y^{\mu}) \\ u &= U(\varphi(y, \dot{y}, \ldots, y^{(\mu)}), \dot{\varphi}(y, \dot{y}, \ldots, y^{(\mu+1)})). \end{aligned} \tag{77}$$

Further, suppose the system is L–B equivalent in an open dense subset, the following procedure enables one to design feedback laws that operate almost globally.

Consider the linear system

$$\dot{\mathcal{Y}} = \mathcal{A}\mathcal{Y} + \mathcal{B}v, \tag{78}$$

where $(\mathcal{A}, \mathcal{B})$ is in the Brunovsky form (i.e., trivial linear controllable form) and $\mathcal{Y}$ consists of y and its derivatives up to the maximum order as it appears in (77). It can be seen that the system (5) is L–B equivalent to (78), which means that their trajectories are in one-to-one correspondence. From this, a feedback law in order to stabilize the nonlinear system along a reference trajectory (x_r, u_r) can be obtained by designing a linear, globally stabilizing feedback law for v in order to stabilize the system (78) along the corresponding trajectory $(\mathcal{Y}_r, v_r)$ and obtaining the nonlinear feedback control law for u through the relation (77). This is illustrated in the following example of a thrust-vectored, ducted fan VTOL unmanned aerial vehicle which has been discussed in Pimlin et al.[42] and Simha et al.[55].

Example 4.2

Consider the following dynamics of a rigid body which is symmetric about a principal axis as shown in Fig. 1. We subject it to an external force at the terminal point of the axis and a torque about the same axis:

$$\begin{aligned} m\ddot{x} &= -mge_3 + RF \\ \dot{R} &= R\hat{\Omega} \\ J\dot{\Omega} &= J\Omega \times \Omega + hM, \end{aligned} \tag{79}$$

where $x \in \mathbb{R}^3$ is the position of the center of mass, $R \in SO(3)$ is the orientation of the body-fixed frame with respect to the inertial frame, $\hat{\Omega} \in \mathfrak{so}(3)$ is the body angular velocity, and $\Omega \in \mathbb{R}^3$ is its vector representation. $J = \mathrm{diag}[J_1, J_1, J_2]$ is the inertia matrix representing axis symmetry, m is the mass, g is the gravitational constant, and $\alpha > 0$ is a positive constant. $F = [F_x; F_y; F_z]$ is the applied control force and $M = [F_y; -F_x, M_z]$, where $M_z = \tau$ is the axial torque input and h is the moment arm corresponding to the forces F_x and F_y. We impose a servo constraint on the dynamics as

$$\Omega_3 = 0. \tag{80}$$

We will now demonstrate that the system is differentially flat when its dynamics are restricted to the manifold (80).

The implicit system on $\Omega_3 = 0$ is obtained by solving for the input variables as

$$\begin{aligned} \dot{R} &= R\hat{\Omega} \\ J\dot{\Omega} &= h\begin{bmatrix} e_2^T R^T(m\ddot{x} + mge_3) \\ -e_1^T R^T(m\ddot{x} + mge_3) \\ 0 \end{bmatrix}. \end{aligned} \tag{81}$$

Flat outputs and dynamic equivalence Define the variable $q = x + \frac{J_1}{hm}Re_3$ which denotes the Huygens center of oscillation of the rigid body[22], and $f = F_z - \frac{J_1}{hm}(\Omega_1^2 + \Omega_2^2)$ which is a virtual

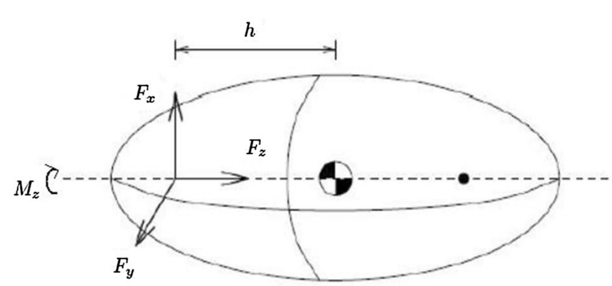

Figure 1: Axis-symmetric thrust-vectored rigid body.

thrust along the body z-axis which is applied on the Huygens center.

It can be verified via the Euler–Lagrange equations that when the system is restricted to the manifold $\Omega_3 = 0$, the variables $q_i,\ i = 1,2,3$ constitute a set of flat outputs. We will now establish an equivalence between the original system and the trivial linear system in the flat output variables q.

The system dynamics restricted to (80) in the variables q instead of x can be written (by normalizing $m = g = 1$) as

$$\begin{aligned} \ddot{q} &= -e_3 + fRe_3 \\ \dot{R} &= R\hat{\Omega} \\ J\dot{\Omega} &= J\Omega \times \Omega + hM \\ \ddot{f} &= T \\ \Omega_3 &= 0. \end{aligned} \tag{82}$$

We have added two more state variables by extending the virtual thrust input via two differentiations. The new control variables are now M and T. In order to satisfy the constraint, the moment M_z must be identically zero. In order to linearize the system, we further differentiate q until any of the inputs appear (as in Isidori[18]):

$$\begin{aligned} q^{(3)} &= \dot{f}Re_3 + fR\hat{\Omega}e_3 \\ q^{(4)} &= b(R,\Omega,f,\dot{f}) + A(R,\Omega,f)\bar{u}, \end{aligned} \tag{83}$$

where

$$\begin{aligned} b(R,\Omega,f,\dot{f}) &= 2\dot{f}R\hat{\Omega}e_3 + fR\hat{\Omega}^2e_3 \\ A(R,\Omega,f)\bar{u} &= [-fRe_1, -fRe_2, Re_3] \\ \bar{u} &= [F_x;\, F_y;\, T]. \end{aligned} \tag{84}$$

It can be verified that the matrix A is uniformly nonsingular as long as $f \neq 0$, which indicates that the net applied thrust must be strictly positive. This is quite practical as it includes all trajectories which avoid free fall. Hence, we can define a feedback law as

$$u = A^{-1}v - b. \tag{85}$$

Flat systems: Non flat systems, when constrained, may result in flat explicit systems in which case controls may be bifurcated after feedback, to independently stabilize constraints and track the restricted flat system.

Trajectory: Trajectory generation algorithms for nonminimum phase outputs may be unstable and hence constraints need to be modified or relaxed.

With this feedback, the dynamics in the flat output space can be written as

$$\begin{aligned} \dot{q} &= q^{(1)} \\ \dot{q}^{(1)} &= q^{(2)} \\ \dot{q}^{(2)} &= q^{(3)} \\ \dot{q}^{(3)} &= v, \end{aligned} \tag{86}$$

which is in the trivial controllable canonical form. Further, we can see that the functions $Q = [q, \dot{q}, \ddot{q}, q^{(3)}]$ are differentially independent and therefore constitute a diffeomorphism $Q = \phi(x, \dot{x}, R, \Omega, f, \dot{f})$. Hence, using standard linear control methods a stabilizing control law for v can be designed as $v = -K(Q - Q_r)$ and consequently a nonlinear dynamic feedback law can be designed for $\bar{u}$ as

$$\bar{u} = A(X)^{-1}(-K(\phi(X) - \phi(X_r))) - b(X), \tag{87}$$

where $X = (x, \dot{x}, R, \Omega, f, \dot{f})$. This control law will track any reference trajectory as long as $\Omega_3 = 0$. We have thus demonstrated that a system which may not be originally differentially flat can be made so when restricted to a submanifold. In practice, the control law for the torque can be chosen as $M_z = -k_\omega \Omega_3$ in order to stabilize the constraint manifold.

4.4 Trajectory Generation for Output Tracking

In several differentially flat systems, it may be necessary to track nonflat outputs asymptotically. In order to do this, it is first necessary to generate a corresponding state trajectory and consequently a flat output trajectory, subsequent to which the tracking control law can be designed based on the trivial controllable form (Fig. 2). More precisely, we consider the output-constrained DAE system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + q(x)v \\ \phi(x) &= 0 \\ h(x) - r(t) &= 0, \end{aligned} \tag{88}$$

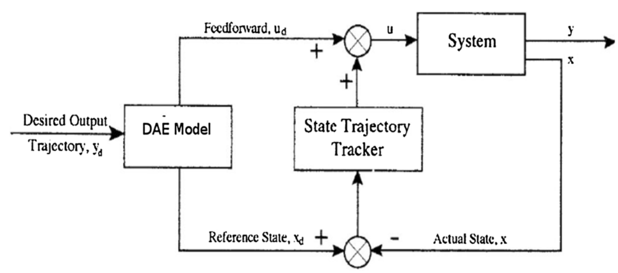

Figure 2: Feedforward–feedback control design.

where $h(x)$ denotes a set of tracking output variables and $r(t)$ is the corresponding output reference trajectory. The above system is numerically solved in order to obtain reference trajectories $x_r(t)$, $u_r(t)$, $v_r(t)$ corresponding to the output trajectory. It is generally preferred to compute the trajectories numerically because the expression for the explicit representation involves computing a set of residual coordinates η (as described in Sect. 2), which is in general a formidable task. Some of the numerical techniques to solve DAEs are outlined as follows. The reader is referred to Brenan et al.[7] for a comprehensive understanding of the computational techniques. The DAE is written in a fully explicit form as

$$F(t, x, \dot{x}) = 0. \tag{89}$$

- *Backward Euler* The backward Euler method to solve the DAE system is given by the set of nonlinear equations which are solved at each time step, of the form:

  $$F(t_n, y_n, (y_n - y_{n-1})/h) = 0, \tag{90}$$

  where h is the step size. Unfortunately, this method does not always work as there are simple high-index DAEs with well-defined solutions, for which this method is unstable or not even applicable. The reader is referred to Example 10.1 in Ascher and Petzold[3] for an example of such a situation.
- *BDF and general multistep methods* The constant step size BDF methods applied to the implicit DAE systems are given by the nonlinear system:

  $$F\left(t_n, y_n, \frac{1}{\beta_0 h}\sum_{j=0}^{k}\alpha_j y_{n-j}\right) = 0, \tag{91}$$

  where β_0 and α_j are the coefficients of the BDF method. The most available software based on BDF methods addresses the fully implicit index-1 system for which convergence results underlying the methods are a straightforward extension of the results of the backward Euler method. In particular, the k-step BDF method of fixed step size h for $k < 7$ converges to $O(h^k)$ if all the initial conditions are accurate to the same order and if the Newton iteration at each step is solved to an accuracy of one higher order. The convergence results can also be extended to index-2 DAEs which are in the semi-explicit form. The reader is referred to Section 10.1.2 of Ascher and Petzold[3] for further details.
- *Implicit Runge–Kutta methods* The s-stage implicit Runge–Kutta method applied to the implicit nonlinear DAE is given by

  $$\begin{aligned} 0 &= F(t_i, Y_i, K_i) \\ t_i &= t_{n-1} + c_i h,\ i = 1, 2, \ldots, s, \\ Y_i &= y_{n-1} + h\sum_{j=1}^{s} a_{ij}K_j, \\ y_n &= y_{n-1} + h\sum_{j=1}^{s} b_i K_i. \end{aligned} \tag{92}$$

  The coefficient matrix $A = [a_{ij}]$ is assumed to be nonsingular. The reader is referred to Ascher and Petzold[3] for stability and convergence analysis of this method.

The above-described BDF methods suffer from two main practical difficulties:

(1) *Ill conditioning of the iteration matrix* For explicit ODEs, as $h \to 0$, the iteration matrix tends to the identity. For index-1 and semi-explicit DAEs, the condition number of the iteration matrix is, in general, $O(h^{-p})$, where p is the index. The reader is referred to Ascher and Petzold[3] for a proof. Due to this, high-index problems are in general hard to integrate numerically, with the above methods.

(2) *Inconsistent initial conditions* For DAEs of high index, the initial conditions need to be restricted to a nontrivial submanifold of the constraint manifold, for example the integral manifold of the controlled invariant distribution N^* as described in Sect. 3. Due to this, the initial conditions which are outside (or close to) this submanifold could lead to impulsive (or stiff) behavior, thus destabilizing the numerical integration.

In order to compensate for these practical difficulties in implementing DAE integrators, the following regularized integration method has been proposed for semi-explicit DAEs.

- *Generalized-α method* The generalized-α method for numerical integration with regularization is given by

$$
\begin{aligned}
x_{n+1} &= x_n + \left(1 - \frac{\beta}{\gamma}\right) h_n f(t_n, x_n, u_n, v_n) \\
&\quad + \frac{\beta}{\gamma} h_n f(t_{n+1}, x_{n+1}, u_{n+1}, v_{n+1}) \\
&\quad + \left(1 - \frac{\beta}{\gamma}\right) h_n^2 a_n \\
a_{n+1} &= \frac{1}{h_n \gamma} (f(t_{n+1}, x_{n+1}, u_{n+1}, v_{n+1}) \\
&\quad - f(t_n, x_n, u_n, v_n)) + \left(1 - \frac{1}{\gamma}\right) a_n \\
\phi(x) &= 0 \\
h(x) &= 0,
\end{aligned} \tag{93}
$$

where

$$
\begin{aligned}
\gamma &= \frac{2}{\rho + 1} - \frac{1}{2} \\
\beta &= \frac{1}{(\rho + 1)^2}
\end{aligned} \tag{94}
$$

with $\rho \in [0, 1)$ being a user-selectable parameter and $h_n = t_{n+1} - t_n$ being the step size. The initial condition $a(0)$ is evaluated as $\frac{\mathrm{d}f}{\mathrm{d}t}|_{t=0}$. The extended state variable a acts as a regularizing compensator with parameter ρ. It has been shown that the local error of the generalized-α method discretization is $O(h^3)$. The reader is referred to Parida and Raha[39] for further details about the error and convergence analysis and performance analysis with varying regularization parameter. It has also been shown that the generalized-α method successfully integrates higher index DAEs, for which the BDF methods fail.

- *Baumgarte's method of constraint stabilization* In order to ensure that the numerical integration is free of impulsive or stiff behavior and to ensure that constraint errors are asymptotically stable, the constraint equation is replaced by its stabilizing counterpart as

$$
k_s \phi^{(s)}(x) + \cdots + k_1 \dot{\phi}(x) + k_0 \phi(x) = 0, \tag{95}
$$

where the coefficients k_i are chosen such that the polynomial is Hurwitz. This modified constraint relaxes the space of initial conditions and allows excursions which are asymptotically stabilized, as long as the trajectories are restricted to the domain corresponding to the above equation. This method however has no guarantee of stabilization, and it is in general difficult to determine the order of the modified constraint equation. We would like to point out here that the analysis presented in Sects. 2 and 3 provides important insights on deciding the form of the modified constraint, the order of which is nothing but the relative degree of the regularized dynamical system.

4.5 Numerical Simulation of Rigid Body Dynamics

The control law based on differential flatness has been implemented for the axis-symmetric, thrust-vectored rigid body as described in Example 4.2, and the closed loop system has been numerically simulated. The control objective is to track a prescribed trajectory for the center of mass x of the rigid body, which constitutes a set of regular outputs. The trajectory generation has been implemented using the generalized-α method after parameterizing the state space using

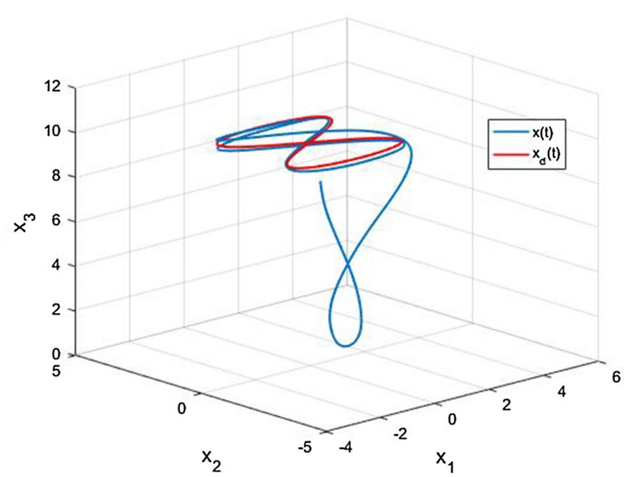

Figure 3: Trajectory tracking with flatness-based control law.

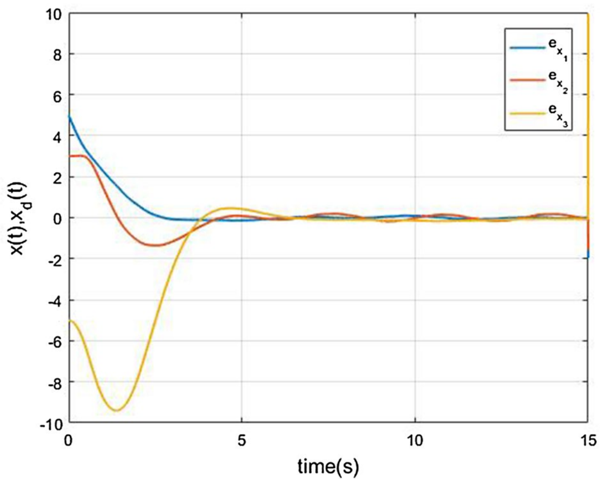

Figure 4: Position error during tracking maneuver.

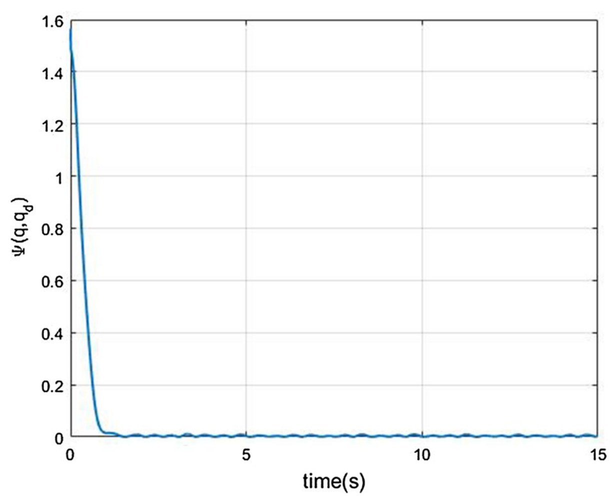

Figure 5: Reduced attitude error during tracking maneuver.

the *modified Rodriguez parameters* on $SO(3)$, which is discussed in Shuster[54]. The computation time of the feedforward system trajectories x_r, u_r, v_r is recorded to be 0.05 s on a standard workstation. The simulation results are presented as follows.

Figure 3 shows the tracking performance of the control law based on differential flatness and feedforward integration. The rigid body initially recovers from a downward-facing attitude and proceeds to track the reference trajectory. Figure 4 shows the position error during the tracking maneuver. The reference state trajectories were generated by choosing the initial conditions of the state and compensator to steer away from the singularities while maintaining overall stability.

Figure 5 shows the variation of the reduced attitude error, i.e., the angle between the actual and desired direction of the body z-axis, which is defined as $\Psi := 1 - z_b^T z_{b_d}$. It can be seen that the rigid body recovers from a large initial angle error. Figure 6 shows the variation of the net thrust and moments during the tracking maneuver. An additional constraint is imposed during tracking, in order to ensure that the net thrust is bounded away from zero, which is generally a practical constraint in most thrust-vectored aerial and submersible vehicles.

5 Concluding Remarks

DAEs have thus far been addressed from the numerical and analytical points of view; however, the progress of the field has been saturated due to the lack of comprehensive understanding of the qualitative and geometric behavior of

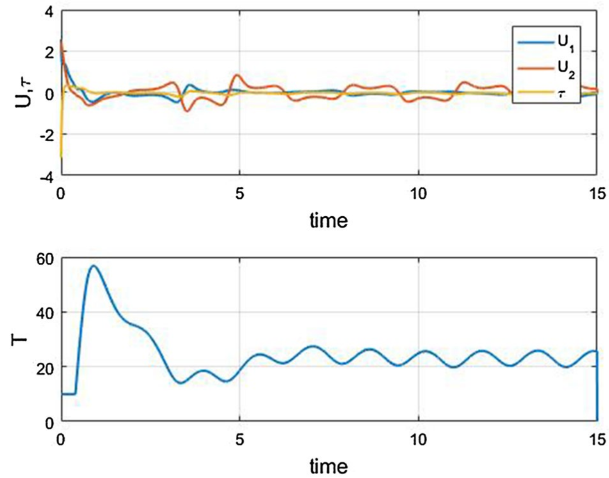

Figure 6: Control inputs during tracking maneuver.

the solutions. Further, most numerical methods and analytical results are restricted to low-index problems, of a specific nature. An early attempt to understand DAEs from the geometric point of view by considering them as ODEs with constraint manifolds also fails in the general case because, as demonstrated, the DAEs may have invariant manifolds that can be embedded only in a state space with a specifically defined extension. This article has provided insights into the local characterization of DAE systems which may be used to design control laws and numerical algorithms with a stabilizing and regularizing component, the performance of which does not depend on the index of the problem. However, the following open problems and avenues for further research still remain. Firstly, a global characterization of DAEs still does not exist, and the control laws depend on the local coordinatization of the invariant integral manifolds. A possible approach toward this is to consider the flow of the DAE system as a fibration over the state space where the fibers are locally isomorphic to the control and constraint variable space. Next, the behavior of the flows near singular points is yet to be understood comprehensively. At these points, the DAE may exhibit impulsive, discontinuous, or rough behavior. Additional mathematical structures may be necessary to characterize the flow of the DAE system at and around these points. A possible approach to study these two open problems is to construct a connection on the fiber bundle and perhaps relate the solvability of the DAE system to the flatness (i.e., vanishing curvature and torsion) of such a connection. In the high-index case, the fiber structure may also be extended to a suitably chosen jet space to capture dynamic precompensation variables, which may be essential in determining solvability. Another open area of research is to geometrically design and characterize numerical integration algorithms which solve DAEs. Thus far, the algorithms do not as such guarantee that the constraint manifold is invariant under the discretization map. Especially in high-index cases, there could be significant excursions from the constraint manifold, leading to highly oscillatory/stiff behavior, and in the very high-index case, leading to instability. In ODE problems, numerical methods which guarantee that the state space submanifold is invariant are developed using variational integrators. In the case of mechanical systems or any Riemannian manifold, the geometric properties of the metric structure may be exploited in designing numerical integration algorithms whose step evolution may be a geodesic exponentiation of the local tangent flow, subject to the constraint. Further, stability analysis may also be carried out in the same spirit. Coming to the problem of differentially flat DAE systems, the article provides a characterization of flatness based on the Euler–Lagrange PDE, whose solutions help construct flat outputs. However, as such there is no set of necessary and sufficient conditions for the solvability of these PDEs and the problem of characterizing flatness remains open. Although a local characterization is available for systems of a particular form such as SISO systems, triangular forms, and mechanical systems with one degree of underactuation, the general case remains an open problem. Further, in control design for flat systems, the feedforward component, which is numerically determined by solving a DAE system, may be unstable especially in the case of non-minimum phase systems. Numerical algorithms that provide stable solutions need to be investigated. A possible approach is to solve the DAE as a constrained boundary value problem which guarantees that the internal states of the DAE which starts on the unstable manifold reach the stable manifold in finite time. It is thus our general opinion that further advancements in DAE systems necessitate significant circumspection, and the geometric approach indeed provides the same, along with critical insights for both qualitative and numerical analysis and design.

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