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Stresses and voltage developed in a nonhomogeneous piezoelectric bar due to finite bending

AMARENDRA DAS AND KARTIK CHANDRA SANTRA
Faculty of Science, Burdwan University,
Hooghly Mohsin College, Chinsurah, 712 101, West Bengal, India.

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Abstract

Stress fields and electrostatic voltage developed in a nonhomogeneous piezoelectric bar subjected to bending moments at its ends are determined. This two-dimensional electro-mechanical problem of composite bar involving Maxwell's electro-magnetic equations, the equations of elasticity and the constitutive equations of piezoelectric quartz has been tackled effectively by using Seth's theory of finite deformation. Numerical results show wide differences in the voltages and the stresses of nonhomogeneous and homogeneous bars.

Key words: Bar, bending, nonhomogeneous, piezoelectricity.

1. Introduction

Piezoelectric effect was first discovered by the Curie brothers in 1880[1]. Piezoelectric properties of crystals are used to construct efficient transducers to work under different practical situations. For low-frequency operation bimorph, a composite transducing element is often used to reduce the mechanical impedance without lowering the output voltage[2]. However, the aim of the designers should be to achieve high output voltage and greater ruggedness with minimum weight in an electro-mechanical appliance. This requirement can be met in a thin layer made of quartz as a concrete aggregate[3]. Such a body should be considered nonhomogeneous. Nonhomogeneity is more pronounced if the bar is composed of such layers placed one after another having increasing or decreasing proportions of quartz crystals, following the bimorph principle. The present study on such a model bar shows wide differences in the output voltages of nonhomogeneous and homogeneous bars; ruggedness and weight of the body are also taken into account, since nonhomogeneity in elastic and piezoelectric parameters has been considered. Reports of some experimental results[4-7] published in the last decade may be cited in this context.

2. Formulation of the problem

We consider a uniform narrow rectangular cross-section of a curved bar (Fig. 1a, b)

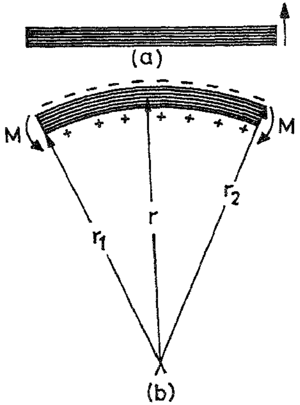

FIG. 1. Polarity of the voltage developed across the bar.

composed of different layers consisting of an aggregate of quartz–cement mixture[3]. Bending is effected in the plane of curvature by couple M at the ends of the bar, the bending moment in this case is constant along the length of the bar. It is natural to expect that the stress distribution is the same in all radial cross-sections, so that the stress components do not depend on θ, but are functions of r only. Due to this symmetry, the shearing stress does not exist.

The fundamental equations of this electro-mechanical problem consist of (i) electrostatic equations, (ii) equation of elasticity, and (iii) constitutive relations for the piezoelectric material[8,9].

From the electro-static equation Curl $\vec{E} = 0$ one can see $\vec{E}$ = grad ϕ, or $E_r = d\phi/dr$ for the present problem. If the upper and lower faces of the bar are coated with a conducting material, so that they are level surfaces of potential field ϕ, the electrical boundary condition becomes

$$\phi_2 - \phi_1 = \int_{r=r_1}^{r=r_2} E_r \cdot dr = V. \tag{1}$$

The integral is taken along the path from the lower face of the bar to its upper face boundary.

The stress equation of equilibrium (as there is no body force) is[10]

$$\frac{\delta \sigma_r}{\delta r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \cdot \tag{2}$$

For the present problem, the constitutive equations for the material can be taken as[7,9,11],

$$S_r = s_{11}\sigma_r + s_{12}\sigma_\theta + d_{11}E_r \tag{3a}$$

$$S_\theta = s_{12}\sigma_r + s_{11}\sigma_\theta - d_{11}E_r \tag{3b}$$

$$S_{r\theta} = 0 \tag{3c}$$

$$D_r = d_{11}\ (\sigma_r - \sigma_\theta) + \epsilon_{11}E_r \tag{3d}$$

$$D_\theta = 0. \tag{3e}$$

Assuming the bar having the width of the rectangular cross-section as unity, one can put the mechanical boundary conditions as

$$[\sigma_r]_{\ r=r_1} = 0 \tag{4a}$$

$$[\sigma_r]_{\ r=r_2} = 0 \tag{4b}$$

$$\int_{r_1}^{r_2} r\sigma_\theta dr = -\ M\cdot \tag{4c}$$

The nonhomogeneity of such a body may be characterized by the variations of elastic, piezoelectric and dielectric parameters from point to point in a static problem[12]. In particular, their variations, where radial symmetry is considered, may be of the form[13,14]

$$s_{ij} = c_{ij}\, f\,(r) \tag{5a}$$

$$d_{ij} = b_{ij}\, f\ (r) \tag{5b}$$

$$\epsilon_{ij} = \nu_{ij}\, f\ (r) \tag{5c}$$

$$i,\ j = 1,\ 2,\ 3.$$

In this problem, the nonhomogeneity parameter is defined by

$$f\,(r) = r^{2\beta} \tag{6}$$

to suit linear or parabolic or any other variation.

3. Method of solution

Gaussian electro-static divergence equation in two-dimensional polar coordinate stands as

$$\frac{1}{r}\frac{\delta(rD_r)}{\delta r}+\frac{1}{r}\frac{\delta D_\theta}{\delta\theta}=0. \tag{7}$$

Owing to the radial symmetry of the problem.

$$\frac{\delta D_\theta}{\delta\theta}=0,\ \text{eqn (7) yields,}$$

$$rD_r=D_0=\text{constant}. \tag{8}$$

Equation (3d) now becomes

$$E_r=\frac{D_0/r-d_{11}(\sigma_r-\sigma_\theta)}{\epsilon_{11}}\cdot \tag{9}$$

According to Seth's theory of finite deformation[15], one can take the radial and tangential components of displacements as $u=r\ (1-\psi)$ and $v=0$ where ψ is a function of r to be determined. The radial and circumferential strain components may now be written as

$$S_r=\frac{du}{dr}=1-\psi-r\frac{d\psi}{dr} \tag{10}$$

and
$$S_\theta=\frac{u}{r}=1-\psi\cdot \tag{11}$$

When the values of E_r, S_r and S_θ as per eqns (9)–(11) along with those of s_{ij}, d_{ij} and ϵ_{ij} given by eqns (5a–c) are inserted in eqns (3a–b), one gets

$$\lambda_1\,\sigma_r+\lambda_2\,\sigma_\theta=A_1 \tag{12a}$$

and
$$\lambda_2\,\sigma_r+\lambda_1\,\sigma_\theta=A_2 \tag{12b}$$

where
$$\lambda_1=c_{11}-\frac{b_{11}^2}{\nu_{11}}\,;\ \lambda_2=c_{12}+\frac{b_{11}^2}{\nu_{11}} \tag{13a}$$

$$A_1=\left(1-\psi-r\frac{d\psi}{dr.}-\frac{b_{11}}{\nu_{11}}\frac{D_0}{r}\right)/f(r) \tag{13b}$$

$$A_2=\left(1-\psi+\frac{b_{11}}{\nu_{11}}\frac{D_0}{r}\right)/f\ (r). \tag{13c}$$

Equations (12a–b) help to determine σ_r and σ_θ as

$$\sigma_r = \frac{(A_1\lambda_1 - A_2\lambda_2)}{(\lambda_1^2 - \lambda_2^2)}, \tag{14a}$$

$$\sigma_\theta = \frac{(A_2\lambda_1 - A_1\lambda_2)}{(\lambda_1^2 - \lambda_2^2)}. \tag{14b}$$

On the basis of eqns (14a–b) and (6), eqn (2) becomes

$$r^2 \frac{d^2\psi}{dr^2} + r\,(3 - 2\beta)\frac{d\psi}{dr} + 2\beta D'\psi$$

$$= 2\beta D' + \frac{b_{11}}{\nu_{11}}\frac{D_0}{r} \cdot D_1\,(2\beta - 1) \tag{15}$$

where $$D' = \frac{\lambda_2}{\lambda_1} - 1, \tag{16a}$$

$$D_1 = 1 + \frac{\lambda_2}{\lambda_1}. \tag{16b}$$

Now the dimensionless variable $\rho = \frac{r}{r_1}$ is introduced, r_1 being the inner radius of the bar. Using $\psi(r) = \xi(\rho)$ eqn (15) takes the form

$$\rho^2 \frac{d^2\xi}{d\rho^2} + \rho(3 - 2\beta)\frac{d\xi}{d\rho} + 2\beta D'\xi$$

$$= 2\beta D' + \frac{b_{11}}{\nu_{11}}\frac{D_0}{r_1\rho} D_1\,(2\beta - 1). \tag{17}$$

From (17)

$$\xi(\rho) = P\,\rho^{\delta_1} + Q\rho^{\delta_2} + \frac{2\beta D'}{\delta_1\,\delta_2} + \frac{b_{11}}{\nu_{11}}\frac{D_0 D_1}{r_1\rho}$$

$$\frac{(2\beta - 1)}{(1 + \delta_1)(1 + \delta_2)}\ (\delta_1\,,\ \delta_2 \neq -1) \tag{18}$$

and

$$\xi(\rho) = P\rho^{\delta_1} + Q\rho^{\delta_2} + \frac{2\beta D'}{\delta_1\delta_2} - \frac{b_{11}D_0D_1}{\nu_{11}r_1}\frac{(2\beta - 1)}{(1 + \delta_2)}\frac{ln\rho}{\rho} \tag{18a}$$

$$(\text{for } \delta_1 = -1,\ \delta_2 \neq -1)$$

where $$\delta_1, \delta_2 = (\beta - 1) \pm [(\beta - 1)^2 - 2\beta D']^{1/2} \tag{19}$$

and P and Q are arbitrary constants.

Combining eqns (13b), (18) and (6) with eqns (14a–b) the expression for σ_r (ρ) and σ_θ (ρ) becomes

$$\sigma_r(\rho) = l_{11}(\rho)\ P + l_{12}\ (\rho)\ Q + l_{13}\ (\rho)\ \frac{D_0}{r_1} + l_{14}(\rho) \tag{20a}$$

$$\sigma_\theta(\rho) = l_{21}(\rho)\ P + l_{22}\ (\rho)\ Q + l_{23}\ (\rho)\ \frac{D_0}{r_1} + l_{24}(\rho) \tag{20b}$$

where $$l_{11}(\rho) = (\lambda_2 - \lambda_1 - \lambda_1\delta_1)\ \rho^{\delta_1 - 2\beta}/(\lambda_1^2 - \lambda_2^2)\ r_1^{2\beta} \tag{21a}$$

$$l_{12}(\rho) = (\lambda_2 - \lambda_1 - \lambda_1\delta_2)\ \rho^{\delta_2 - 2\beta}/(\lambda_1^2 - \lambda_2^2)\ r_1^{2\beta} \tag{21b}$$

$$l_{13}(\rho) = \frac{b_{11}}{v_{11}}\left[(-\lambda_1 - \lambda_2) + \frac{\lambda_2 D_1(2\beta - 1)}{(1 + \delta_1)(1 + \delta_2)}\right] \Big/$$

$$(\lambda_1^2 - \lambda_2^2)\ r_1^{2\beta}\ \rho^{2\beta+1}(\delta_1,\ \delta_2 \neq -1)$$

or $$\frac{b_{11}}{v_{11}}\left[(-\lambda_1 - \lambda_2) - \frac{\lambda_2 D_1(2\beta - 1)}{(1 + \delta_2)}\ ln\rho\right] \Big/$$

$$(\lambda_1^2 - \lambda_2^2)\ r_1^{2\beta}\rho^{2\beta+1}\ (\delta_1 = -1,\ \delta_2 \neq -1) \tag{21c}$$

$$l_{14}(\rho) = 0\ (\text{ for } \beta \neq 0) = \frac{1}{\lambda_1 + \lambda_2}\ (\text{for } \beta = 0) \tag{21d}$$

$$l_{21}(\rho) = (\lambda_2 - \lambda_1 + \lambda_2\delta_1)\rho^{\delta_1 - 2\beta}/(\lambda_1^2 - \lambda_2^2)r_1^{2\beta} \tag{22a}$$

$$l_{22}(\rho) = (\lambda_2 - \lambda_1 + \lambda_2\delta_2)\ \rho^{\delta_2 - 2\beta}/(\lambda_1^2 - \lambda_2^2)\ r_1^{2\beta} \tag{22b}$$

$$l_{23}(\rho) = \frac{b_{11}}{v_{11}}\left[(\lambda_1 + \lambda_2) - \frac{\lambda_1 D_1(2\beta - 1)}{(1 + \delta_1)\ (1 + \delta_2)}\right] \Big/$$

$$(\lambda_1^2 - \lambda_2^2)\ r_1^{2\beta}\rho^{2\beta+1}\ (\delta_1,\ \delta_2 \neq -1)$$

or

$$\frac{b_{11}}{\nu_{11}}\left[(\lambda_1 + \lambda_2) + \frac{\lambda_1 D_1\ (2\beta - 1)\ ln\rho}{(1 + \delta_2)}\right] \bigg/ \ (\lambda_1^2 - \lambda_2^2) r_1^{2\beta} \rho^{2\beta + 1} \tag{22c}$$

$$(\delta_1 = -1,\ \delta_2 \neq -1)$$

$$l_{22}(\rho) = 0\ (\text{for } \beta \neq 0) = \frac{1}{\lambda_1 + \lambda_2}\ (\text{for } \beta = 0)\ \cdot \tag{22d}$$

Combining conditions (4a–c) with the expressions for $\sigma_r(\rho)$ and $\sigma_\theta(\rho)$, the following set of equations are obtained [for $\beta \neq 0$]

$$a_{11}P + a_{12}Q + a_{13}\frac{D_0}{r_1} = 0 \tag{23a}$$

$$a_{21}P + a_{22}Q + a_{23}\frac{D_0}{r_1} = 0 \tag{23b}$$

$$a_{31}P + a_{32}Q + a_{33}\frac{D_0}{r_1} = -\frac{M}{r_1^2} \tag{23c}$$

where

$$a_{11} = l_{11}\ (\rho_1) \tag{24a}$$

$$a_{12} = l_{12}\ (\rho_1) \tag{24b}$$

$$a_{13} = l_{13}\ (\rho_1) \tag{24c}$$

$$a_{21} = l_{11}\ (\rho_2) \tag{25a}$$

$$a_{22} = l_{12}\ (\rho_2) \tag{25b}$$

$$a_{23} = l_{13}\ (\rho_2) \tag{25c}$$

$$a_{31} = (\lambda_2 - \lambda_1 + \lambda_2\delta_1)(\rho_2^{\delta_1 - 2\beta + 2} - \rho_1^{\delta_1 - 2\beta + 2}) / r_1^{2\beta}(\lambda_1^2 - \lambda_2^2)\ (\delta_1 - 2\beta + 2) \tag{26a}$$

$$a_{32} = (\lambda_2 - \lambda_1 + \lambda_2\delta_2)(\rho_2^{\delta_2 - 2\beta + 2} - \rho_1^{\delta_2 - 2\beta + 2}) / r_1^{2\beta}(\lambda_1^2 - \lambda_2^2)(\delta_2 - 2\beta + 2) \tag{26b}$$

$$a_{33} = \frac{b_{11}}{\nu_{11}}\left[(\lambda_1 + \lambda_2) - \frac{\lambda_1 D_1 (2\beta - 1)}{(1 + \delta_1)\ (1 + \delta_2)}\right]$$

$$[\rho_2^{1-2\beta} - \rho_1^{1-2\beta}]/(\lambda_1^2 - \lambda_2^2).$$

$$r_1^{2\beta}\,(1-2\beta)\ (\text{for } \beta \neq 0{\cdot}5), \tag{26c}$$

or
$$= \frac{b_{11}}{\nu_{11}}\, ln(\rho_2/\rho_1)/(\lambda_1-\lambda_2)\ (\text{for } \beta = 0{\cdot}5)\cdot$$

From eqns (23a–c) the constants P, Q and $D_0\,/\,r_1$ can be found out from the following relations

$$P = \frac{\Delta_1}{\Delta}\,,\ Q = \frac{\Delta_2}{\Delta}\,,\ \frac{D_0}{r_1} = \frac{\Delta_3}{\Delta} \tag{27a,b,c}$$

where Δ is the non-singular value of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\,.$$

The values of Δ_i are obtained from those of Δ by replacing the ith column by 0, 0, $-\frac{M}{r_1^2}$. Knowing the values P, Q, and D_0/r_1 from eqns (27a–c) $\sigma_r\,(\rho)$ and $\sigma_\theta(\rho)$ can be found out from eqns (20a–b).

The electrical voltage developed between the upper and lower surfaces of the bar can be found out from the relation

$$V = \frac{D_0}{2\beta\nu_{11}r_1^{2\beta}}\,(\rho_1^{-2\beta}-\rho_2^{-2\beta}) - \frac{r_1 b_{11}}{\nu_{11}(\lambda_1-\lambda_2)}$$
$$\left[\frac{P\delta_1}{(\delta_1-2\beta+1)}\,(\rho_1^{\delta_1-2\beta+1}-\rho_2^{\delta_1-2\beta+1}) + \frac{Q\cdot\delta_2}{(\delta_2-2\beta+1)}\right.$$
$$(\rho_1^{\delta_2-2\beta+1}-\rho_2^{\delta_2-2\beta+1}) + \frac{b_{11}}{\nu_{11}}\frac{D_0}{r_1}$$
$$\left.\left\{-2+\frac{D_1(2\beta-1)(\rho_1^{-2\beta}-\rho_2^{-2\beta})}{2(1+\delta_1)(1+\delta_2)\beta}\right\}\right]\cdot \tag{28}$$

4. A special case for homogeneous bar

For a homogeneous piezoelectric bar, the elastic and piezoelectric parameters as well as the dielectric permittivity are independent of space position, *i.e.*, here the nonhomogeneity parameter $\beta = 0$. In this case, on the basis of eqns (13b–c, 14a–b, 18 and 19) the expressions for $\sigma_r(\rho)$ and $\sigma_\theta(\rho)$ become

$$\sigma_r(\rho) = -\frac{\bar{P}}{\lambda_1 + \lambda_2} + \frac{\bar{Q}}{(\lambda_1 - \lambda_2)\rho^2} + \frac{\bar{D}_0}{r_1\rho}\frac{b_{11}}{\nu_{11}}$$

$$\frac{(-\lambda_1 - \lambda_2 + \lambda_2 D_1)}{(\lambda_1^2 - \lambda_2^2)} + \frac{1}{\lambda_1 + \lambda_2} \tag{29a}$$

$$\sigma_\theta(\rho) = -\frac{\bar{P}}{\lambda_1 + \lambda_2} - \frac{\bar{Q}}{(\lambda_1 - \lambda_2)\rho^2} + \frac{\bar{D}_0}{r_1\rho}\frac{b_{11}}{\nu_{11}}$$

$$\frac{(\lambda_1 + \lambda_2 - \lambda_1 D_1)}{(\lambda_1^2 - \lambda_2^2)} + \frac{1}{\lambda_1 + \lambda_2} \tag{29b}$$

where $\bar{P}$, $\bar{Q}$ and $\dfrac{\bar{D}_0}{r_1}$ are the arbitrary constants for homogeneous case. The constants $\bar{P}$, $\bar{Q}$, and $\dfrac{\bar{D}_0}{r_1}$ can now be evaluated using the corresponding mechanical boundary conditions (4a–c).

$$a_1\bar{P} + b_1\bar{Q} + C_1\frac{\bar{D}_0}{r_1} = a_1 \tag{30a}$$

$$a_1\bar{P} + b_2\bar{Q} + C_2\frac{\bar{D}_0}{r_1} = a_1 \tag{30b}$$

$$a_3\bar{P} + b_3\bar{Q} + C_3\frac{\bar{D}_0}{r_1} = a_3 - \frac{M}{r_1^2}\,. \tag{30c}$$

Here,

$$a_1 = -\frac{1}{\lambda_1 + \lambda_2}$$

$$b_1 = \frac{r_1^2}{(\lambda_1 - \lambda_2)\,\rho_1^2}$$

$$c_1 = \frac{D_0 b_{11}}{\nu_{11} r_1 \rho_1}(-\lambda_2 - \lambda_1 + \lambda_2 D_1)/(\lambda_1^2 - \lambda_2^2)$$

$$b_2 = \frac{r_1^2}{(\lambda_1 - \lambda_2)\,\rho_2^2}$$

$$c_2 = \frac{D_0 b_{11}}{\nu_{11} r_1 \rho_2}\,(-\lambda_1 - \lambda_2 + \lambda_2 D_1)/(\lambda_1^2 - \lambda_2^2)$$

$$a_3 = -\frac{1}{2(\lambda_1 + \lambda_2)}\,(\rho_2^2 - \rho_1^2)$$

$$b_3 = -\frac{ln(\rho_2 / \rho_1)}{(\lambda_1 - \lambda_2)}$$

$$c_3 = \frac{b_{11}}{\nu_{11}}\,\frac{(\lambda_2 + \lambda_1 - \lambda_1 D_1)(\rho_2 - \rho_1)}{(\lambda_1^2 - \lambda_2^2)}\,.$$

Evaluating $\bar{P}$, $\bar{Q}$ and $\frac{\bar{D}_0}{r_1}$ from eqns (30a–c) and inserting them in eqns (29a–b), $\sigma_r(\rho)$ and $\sigma_\theta(\rho)$ are found. Following the same procedure the voltage generated between the upper and lower faces of the bar can be expressed as

$$\bar{V} = \frac{\bar{D}_0}{r_1 \epsilon_{11}}\, ln(\rho_2 / \rho_1)\left[1 - \frac{b_{11} d_{11}}{\nu_{11}}\,\frac{(D_1 - 2)}{(\lambda_1 - \lambda_2)}\right]$$

$$+ \frac{2d_{11}}{\epsilon_{11}}\,\frac{\bar{Q}}{(\lambda_1 - \lambda_2)}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right). \qquad (31)$$

5. Numerical results

Numerical computations have been carried out to obtain the radial, hoop stress components and voltage generated across the depth of the bar in both nonhomogeneous and homogeneous cases.

As a typical example, quartz has been chosen as an aggregate of concrete for which $b_{11} = 2{\cdot}2$ pc/N[11,16], $\nu_{11} = 4{\cdot}5 \times 8{\cdot}854 \times 10^{-12}$ F/m[11], $c_{11} = 13{\cdot}16 \times 10^{-12}$ m^2/N[11] and $c_{12} = -1{\cdot}53 \times 10^{-12}$ m^2/N[11].

The results have been computed choosing the outer radius $r_2 = 1{\cdot}25 r_1$ (r_1, the inner radius of the bent bar) and $M = r_1^2 \times 10^{12}$ (in magnitude).

The stress components are shown in Figs 2 and 3. Voltages generated for homogeneous and different types of nonhomogeneous cases are evident from Table I. The voltages are expressed in terms of (M/r_1) unit.

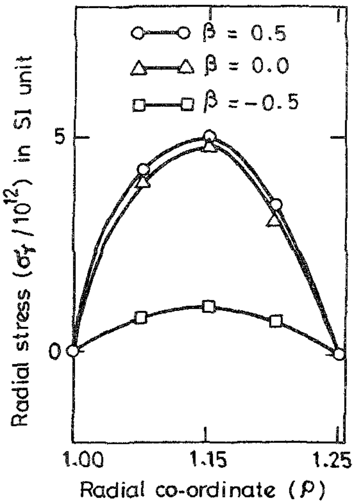

FIG. 2.

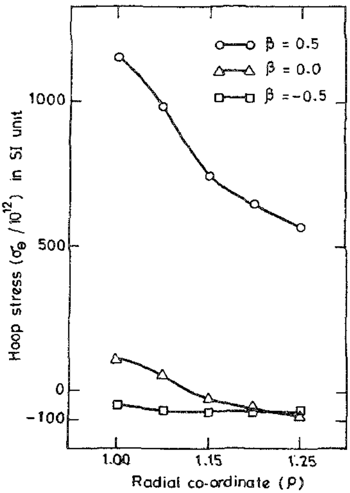

FIG. 3.

Table I

Nonhomogeneity parameter (β)	*Voltage generated* ($r_1 V/M$)
+ 4	− 1462·7
+ 2	− 1323·7
1	− 1310·1
+ ·5	− 1301·9
0	− 1228·4
− ·5	− 272·8
− 1	− 92·4
− 4	− 3·5

An interesting feature of this analysis is that the result shows a marked difference in the output voltage obtained across the depth of the bar for homogeneous and nonhomogeneous cases. Since the electrical properties of concretes are extremely variable and depend upon the proportion of the contents and gradation, such type of results are well expected, as the nature of nonhomogeneity also varied. The results obtained corresponding to $\beta > 0$ are of much significance. The magnitude of output voltages in these cases is greater than the voltage obtained in the case of a homogeneous piezoelectric bar ($\beta = 0$) having similar dimensions and under the action of the same bending moment M.

Another important aspect of the present discussion is that for a particular bar the voltage is found to be directly proportional to the applied bending moment whereas from theory of elasticity it is known that the tip displacement at one end of the bar varies directly as the bending moment. So it can be concluded that tip displacement should directly vary with voltage, which tallies with the experimental result[7].

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Nomenclature

ϕ	=	Electric potential.
σ_r	=	Normal stress in the radial direction at a point (r, Θ).
σ_Θ	=	Normal stress in the circumferential direction, *i.e.*, hoop-stress at a point (r, Θ).
$S_r, S_\Theta, S_{r\Theta}$	=	Strain components.
s_{11}, s_{12}	=	Elastic compliances at constant electric field.
d_{11}	=	Piezoelectric strain/charge parameter.
ϵ_{11}	=	Dielectric permittivity at constant stress.
r_1, r_2	=	Radii of lower and upper surfaces of the bar.
E_r	=	Electric field component.
D_r, D_Θ	=	Electric induction components.
M	=	Bending moment.
c_{ij}, b_{ij}, v_{ij}	=	Material parameters in relation to elastic, piezoelectric and dielectric properties, respectively, of the bar.

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