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Guest Editorial

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GUEST EDITORIAL

This special issue of the Journal of IISc is devoted to the interrelated areas of number theory and representation theory, two evergreen and central subfields of mathematics that continue to see groundbreaking progress with each passing decade. This volume contains a collection of articles by experts, each surveying an important and active domain in the two subfields. We now briefly describe each of these works, starting with three articles on distinct themes in number theory that have seen extensive activity in recent times.

The work of Wiles and Taylor on the Taniyama–Shimura conjecture catalysed a large amount of activity in the Langlands program. Roughly speaking, the Langlands program postulates a one-to-one correspondence between Galois representations on the one hand and certain automorphic forms on the other. Counterparts of some operations that are easily carried out for Galois representations turn out to be extremely difficult for automorphic forms (and vice versa). These form a part of Langlands's functoriality conjecture. One such operation is taking symmetric powers. The article "*Modularity of Galois representations and Langlands functoriality*" by James Newton gives a survey of recent results on modularity with a focus on his joint work with Thorne on proving Langlands functoriality of symmetric powers of a cuspidal Hecke eigenform. The article also provides an extensive list of references to expositions on various aspects and developments of modularity results that can be very helpful for someone to get initiated in the subject.

Another very important theme in algebraic number theory is the relationship between analytic and arithmetic invariants. The subject has its origin in the works of Dirichlet and Kummer in the nineteenth century. Although there is a very satisfying conjectural picture due to the insights of many eminent mathematicians including Stark, Tate, Deligne, Beilinson, Bloch, and Kato, very few cases of these conjectures are proved. Perhaps the most studied case is the Birch and Swinnerton-Dyer conjecture (BSD). It asserts that the rank of an elliptic curve (algebraic rank) is equal to the order of vanishing of its L-function at $s = 1$ (analytic rank). Furthermore, it gives a precise formula for the leading term of the L-function at $s = 1$ in terms of various arithmetic invariants attached to the elliptic curve. BSD remains elusive when the rank is bigger than 1. Nevertheless, there has been a tremendous amount of progress in the last 50 years when the rank is at most 1. Much of this progress consists of results that start with analytic rank equal to 0 or 1 and deduce equality of the analytic and algebraic ranks. The past decade or so has seen "p-converse" results due to Skinner, Zhang, and their collaborators. More precisely, starting with a bound on the rank of the Selmer group, one deduces the equality of the analytic and algebraic ranks. Ashay Burungale's article "*A p-adic Waldspurger formula and the conjecture of Birch and Swinnerton-Dyer*" gives a survey of the ideas surrounding such results.

Ritabrata Munshi's article "*Analytic number theory in the last decade*" gives an excellent survey of progress in analytic number theory in the second decade of this century. The number of breakthroughs in analytic number theory in this period is truly spectacular. The article gives a lucid survey of these. Let us mention two of these results that are easy to state. Perhaps the most striking development has been Zhang's proof of bounded gaps between consecutive primes and subsequent improvements by Maynard, Tao, and others. These results seem to take us very close to the twin prime conjecture: there are infinitely many prime numbers p such that $p + 2$ is also a prime. Another mystifying statement about prime numbers is the Goldbach conjecture—any even integer larger than 2 can be written as a sum of two prime numbers. While this remains out of reach, a weaker form, namely the ternary Goldbach conjecture—any odd integer larger than 5 is a sum of three prime numbers—was recently resolved by Helfgott. In 1937, Vinogradov proved that every odd integer larger than $e^{e^{16038}}$ is a sum of three prime numbers. This number $e^{e^{16038}}$ is simply too big to check the remaining cases of the

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ternary Goldbach conjecture, even with the most powerful computers available today. Helfgott improved Vinogradov's proof to show that every odd integer bigger than 10^{27} is a sum of three prime numbers. Helfgott and Platt then checked the remaining cases on a computer.

We now describe the articles in representation theory. Nicolas Libedinsky's "*Introsurvey of representation theory*" is an account of some parts of representation theory, which goes from the (affine) symmetric and general linear groups, to Iwahori–Hecke algebras and quantum groups, and ends by discussing Soergel bimodules and the p-canonical basis. The survey also includes historical references and is accessible to beginners; it ends with a list of open problems that will appeal to experts.

In "*The Frobenius characteristic of character polynomials*", Amritanshu Prasad begins with a crash course on representations of the symmetric group S_n and on symmetric function theory in characteristic zero. He then explains the notions of families $(V_n)_n$ of $(S_n)_n$-modules, and of (eventually) polynomial characters—e.g. the Specht character polynomials— and explains how to use this machinery and the Frobenius characteristic, using the well-known examples of symmetric and exterior powers of representations. These tools help study characters of S_n across all n at once. Prasad ends by explaining moment generating functions and vector partitions, illustrated for the above polynomial representations of GL_n, restricted to S_n.

The article "*The Duflo–Serganova functor, vingt ans après*" by Maria Gorelik, Crystal Hoyt, Vera Serganova, and Alexander Sherman has a dual purpose. Firstly, it includes the results in the well-known 2005 preprint on Lie superalgebras by Duflo–Serganova. This work was unpublished until now; it has led to tremendous subsequent activity in super-representation theory. Second, the authors have surveyed several subsequent applications and developments in the field, following the above preprint.

The article "*Quantum affine algebras, graded limits and flags*" by Matheus Brito, Vyjayanthi Chari, Deniz Kus, and R. Venkatesh surveys the representation theory of quantum affine algebras and of current algebras, and connections between the two via taking classical limits and graded limits. These lead to local Weyl modules, Kirillov–Reshetikhin modules, minimal affinizations, and prime representations. The prominent role played by Demazure modules is explained, as is the connection to Macdonald polynomials. The authors also discuss a category of finite dimensional representations, and how it is formally similar to Category $\mathcal{O}$ over semisimple Lie algebras, including BGG reciprocity/duality.

Gurbir Dhillon's account "*An informal introduction to categorical representation theory and the local geometric Langlands program*" lays out for the nonexpert reader an entryway into the local geometric Langlands program. This work collects together the results shown previously, e.g. Beilinson–Bernstein localization, as well as motivations for the program, moves on to categorical representations of algebraic groups, and then to the affine setting. The article provides several motivating connections and examples, and goes on to discuss the local geometric Langlands conjecture and recent works.

In conclusion, we hope that this special issue, with articles by experts in number theory and representation theory, will serve both as an entryway for interested mathematicians who are relatively inexperienced in these areas, as well as a useful reference for researchers in both fields.

Publisher's Note

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