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REVIEW ARTICLE

A *p*-adic Waldspurger Formula and the Conjecture of Birch and Swinnerton-Dyer

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Abstract | About a decade ago Bertolini–Darmon–Prasanna proved a *p*-adic Waldspurger formula, which expresses values of an anticyclotomic *p*-adic *L*-function associated to an elliptic curve $E_{/\mathbb{Q}}$ outside its defining range of interpolation in terms of the *p*-adic logarithm of Heegner points on *E*. In the ensuing decade an insight of Skinner based on the *p*-adic Waldspurger formula has initiated a progress towards the Birch and Swinnerton-Dyer conjecture for elliptic curves over $\mathbb{Q}$, especially rank one aspects. In this note we outline some of this recent progress.

1 Introduction

The Birch and Swinnerton-Dyer conjecture is a fundamental problem about the arithmetic of elliptic curves. It connects the structure of the rational points on an elliptic curve over $\mathbb{Q}$ to the analytic properties of its associated Hasse–Weil *L*-function.

A spectacular result towards the BSD conjecture is due to Gross–Zagier and Kolyvagin:

$$\mathrm{ord}_{s=1}L(s,E)\leq 1 \implies \mathrm{rank}_{\mathbb{Z}}E(\mathbb{Q}) = \mathrm{ord}_{s=1}L(s,E),\ \#\text{Ш}(E)<\infty. \quad (1)$$

Here $E_{/\mathbb{Q}}$ is an elliptic curve, $L(s,E)$ the associated Hasse–Weil *L*-function, $E(\mathbb{Q})$ the group of rational points and $\text{Ш}(E)$ the Tate–Shafarevich group. In the mid 1980's Gross and Zagier[38] proved a remarkable formula relating central derivative of the *L*-function $L(s,E_{/K})$ associated to *E* over an imaginary quadratic field *K* in terms of the Néron–Tate height of a Heegner point $y_K\in E(K)$. A few years later, Kolyvagin[52] invented the method of Euler systems, in particular showing the non-torsion-ness of y_K implies $\mathrm{rank}_{\mathbb{Z}}E(K)=1$ and $\#\text{Ш}(E_{/K})<\infty$.. An apt choice of *K* then yields (1).

The general principles of Iwasawa theory suggest that *p*-adic analogues of the Gross–Zagier formula may also shed some light on the BSD conjecture. In an inspired followup to[38] Perrin-Riou proved a *p*-adic analogue[55] for primes *p* of good ordinary reduction, which expresses derivative of a cyclotomic *p*-adic *L*-function $\mathcal{L}_p(E_{/K})$ in terms of the *p*-adic height of the Heegner point y_K. Unlike the Néron–Tate height, the *p*-adic height is not known to be non-zero on non-torsion points in general. However the non-vanishing is known for CM elliptic curves, which led Rubin to results towards the BSD conjecture for rank one CM curves, such as the finiteness of the *p*-part of the Tate–Shafarevich group[57] and the *p*-part of the BSD formula[60]. Rubin's approach relies on an Iwasawa theory of $\mathcal{L}_p(E_{/K})$, specifically the Iwasawa main conjecture and also transpires a converse to (1). Due to the lack of knowledge regarding non-vanishing of *p*-adic heights for non-CM curves, the general rank one case remained largely elusive.

In the early 90's Rubin discovered a formula[59] relating value of a Katz *p*-adic *L*-function outside its defining range of interpolation to the *p*-adic logarithm of a Heegner point on the underlying CM elliptic curve. This formula initiated Perrin-Riou's formulation of *p*-adic variant of the Beilinson conjecture, which concerns the arithmetic of motivic *p*-adic *L*-values outside the defining range of interpolation. In light of the *p*-adic Beilinson conjecture a natural problem: non-CM analogue of Rubin's formula. After almost two decades of search, Bertolini–Darmon–Prasanna found such an analogue[1], which links values of an anticyclotomic *p*-adic *L*-function $\mathcal{L}_v(E_{/K})$ outside the interpolation region to the *p*-adic logarithm of the Heegner point $y_K\in E(K)$. The formula echoes the Gross–Zagier formula as well as the

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Waldspurger formula. Subsequently, Liu–Zhang–Zhang developed an automorphic framework[53] which interprets the Bertolini–Darmon–Prasanna formula as a p-adic Waldspurger formula, thereby generalising it considerably.

It is intriguing that the p-adic world manifests distant incarnations of the Heegner point y_K in the guise of the p-adic Gross–Zagier formula and the p-adic Waldspurger formula.

Skinner sought the relevance of the p-adic Waldspurger formula to the BSD conjecture and observed that certain rank one aspects of the BSD conjecture may be amenable to Iwasawa theory. Notice the Heegner point y_K is non-torsion if and only if its p-adic logarithm is non-zero, that is, a special value of $\mathcal{L}_v(E_{/K})$ is non-zero. The Iwasawa–Greenberg main conjecture relates non-vanishing of such a p-adic L-value to finiteness of a Selmer group. Skinner[64] proved that a divisibility towards the main conjecture is closely related to a p-converse to the Gross–Zagier and Kolyvagin theorem:

Conjecture 1.1 *Let E be an elliptic curve over* $\mathbb{Q}$. *Let p be a prime and* $\mathrm{Sel}_{p^\infty}(E)$ *the associated* p^∞ *-Selmer group.* Then

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E) = 1 \implies \mathrm{ord}_{s=1}L(s,E) = 1. \qquad \text{(p−cv)}$$

A refinement of this approach due to Jetchev–Skinner–Wan[42] showed that the p-part of the BSD formula for rank one elliptic curves $E_{/\mathbb{Q}}$ is typically a consequence of the Iwasawa–Greenberg main conjecture for $\mathcal{L}_v(E_{/K})$. The insight of Skinner has initiated the exploration of Iwasawa theory of $\mathcal{L}_v(E_{/K})$, as well as the p-converse and the p-part of the BSD formula, often in conjunction with complementary tools. An important result towards the main conjecture is due to Wan[73]. About the same time as[64], Zhang[81] proved results towards (p-cv) via an independent approach, which has also been instrumental.

Since the work of Skinner and Zhang, the p-converse continues to be studied in various settings, leading to a key progress towards (p-cv). Iwasawa theory of elliptic curves underlies much of the progress. In combination with results of Bhargava and Shankar[2,3], this proves the BSD conjecture for a large proportion of elliptic curves over $\mathbb{Q}$[4,5]. As of now the resolution of (p-cv) is incomplete even for CM elliptic curves. For instance, a missing case relates to the congruent number problem.

This note is meant to be a brief introduction to some of the progress towards the BSD conjecture, especially rank one aspects.

After recalling the Birch and Swinnerton-Dyer conjecture for elliptic curves over $\mathbb{Q}$ in Sect. 2.1, we describe a few of the representative results toward the conjecture in Sects. 2.2 and 2.3. Some of the results are then viewed a little more closely in Sect. 3.

The papers of Bloch–Kato[7] and Kato[43–45,47] are excellent introductions to the arithmetic of special values of L-functions and Iwasawa theory, while[24,61,63,69,71,82] are some of the surveys of the BSD conjecture.

2 The Birch and Swinnerton-Dyer Conjecture

We describe the BSD conjecture for elliptic curves over $\mathbb{Q}$ and then survey some of the results.

2.1 The BSD Conjecture

2.1.1 The BSD Conjecture[77], As Stated by Birch[6] and Tate[68]:

Conjecture 2.1 (The Birch and Swinnerton-Dyer Conjecture) *Let* $E_{/\mathbb{Q}}$ *be an elliptic curve.*

(a) *The Hasse–Weil L-function* $L(s, E)$ *has an analytic continuation to the entire complex plane*[A] and

$$\mathrm{ord}_{s=1}L(s,E) = \mathrm{rank}_{\mathbb{Z}}E(\mathbb{Q}). \qquad \text{(BSD)}$$

(b) *The Tate–Shafarevich group* $Ш(E)$ *is finite and*

$$\frac{L^{(r)}(1,E)}{r!\cdot\Omega_E\cdot\mathrm{reg}(E)} = \frac{\#Ш(E)\cdot\prod_p c_p(E)}{\#E(\mathbb{Q})^2_{\mathrm{tor}}} \qquad \text{(BSD-f)}$$

for

- $r = \mathrm{ord}_{s=1}L(s,E)$,
- $c_p(E)$ *the Tamagawa number at p: the cardinality of the component group of the special fiber of the Néron model of E* over $\mathbb{Z}_p$,
- $\Omega_E \in \mathbb{C}^\times$ *the Néron period:*
- $$\Omega_E = \int_{E(\mathbb{R})}|\omega|,$$ *where* $\omega \in \Omega^1(\mathcal{E}_{/\mathbb{Z}})$ *is a* $\mathbb{Z}$*-basis of the differentials of the Néron model* $\mathcal{E}_{/\mathbb{Z}}$ of E,

[A] This conjecture of Hasse, Taniyama–Shimura and Weil is famously proved by Wiles[76] *etal.*

- $\text{reg}(E)$ *the regulator of the Néron–Tate height pairing on* $E(\mathbb{Q})$.
-

The order of vanishing $\text{ord}_{s=1}L(s,E)$ of $L(s, E)$ at $s=1$ is called the analytic rank of E. So the conjecture (BSD) is that the analytic rank of E equals the Mordell–Weil rank of E. The equality (BSD-f) is referred to as the BSD formula.

In the case $\text{ord}_{s=1}L(s,E) \leq 1$ the left hand side of (BSD-f) is known to be a rational number. In particular, we may ask whether the same power of a prime p appears in both sides of (BSD-f). This 'p-part of the BSD formula' is the focus of some of the results described in Sects. 2.2 and 2.3.

2.1.2 Selmer Groups and the BSD Conjecture

The Selmer group of E encodes $E(\mathbb{Q})$ and $\text{Ш}(E)$ at once, and is often more amenable to study.

Let p be a prime and $T = T_pE = \varprojlim_n E[p^n]$ the p-adic Tate module of E. Let $A = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p = E[p^\infty]$. The p^∞-Selmer group $\text{Sel}_{p^\infty}(E)$ is a subgroup of the Galois cohomology group $H^1(\mathbb{Q}, A)$ which appears as the middle term of the short exact sequence

$$0 \to E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \to \text{Sel}_{p^\infty}(E) \to \text{Ш}(E)[p^\infty] \to 0. \quad (2)$$

In view of the exact sequence (2) Conjecture 2.1 suggests the following.

Conjecture 2.2 *Let $E_{/\mathbb{Q}}$ be an elliptic curve. The following are equivalent:*

(a) $\text{rank}_{\mathbb{Z}}E(\mathbb{Q}) = r$ *and* $\text{Ш}(E)$ *is finite.*

(b) $\text{corank}_{\mathbb{Z}_p}\text{Sel}_{p^\infty}(E) = r$ *for p a prime.*

(c) $\text{ord}_{s=1}L(s,E) = r$.

Moreover, the BSD formula (BSD-f) *holds under any of (a), (b), and (c).*

Part (b) follows from part (a) just by (2). We refer to '$(b) \implies (c)$' as a p-converse: a p-adic criterion to have analytic rank r. In Sects. 2.2 and 2.3 we survey results towards this and other implications, including (BSD-f) when[B] $r \in \{0, 1\}$.

[B] For $r \in \{0,1\}$ a conjecture of Katz and Sarnak[48] posits that 50% of the elliptic curve over $\mathbb{Q}$ have analytic rank r.

2.2 Results I

We describe some of the principal results towards the BSD conjecture.

2.2.1 A Theorem of Coates–Wiles and Rubin

The first general results towards the BSD conjecture were proved for CM elliptic curves.

Theorem 2.3 *Let $E_{/\mathbb{Q}}$ be a CM elliptic curve. Then* $L(1,E) \neq 0 \implies \#E(\mathbb{Q})$ and $\#\text{Ш}(E) < \infty$.

The finiteness of $E(\mathbb{Q})$ is due to Coates–Wiles[31] and that of Ш due to Rubin[57].

The methods employed by Coates and Wiles made a surprising connection between Iwasawa theory and the BSD conjecture. Since then, Iwasawa-theoretic methods have been one of the main tools for studying the arithmetic of elliptic curves.

2.2.2 The Theorem of Gross–Zagier and Kolyvagin

After the work of Coates and Wiles in the 1970's, the next spectacular result towards the BSD conjecture came in the 1980's and was due to Gross–Zagier[38] and Kolyvagin[52].

Theorem 2.4 *Let $E_{/\mathbb{Q}}$ be an elliptic curve. Then*

$$\text{ord}_{s=1}L(s,E) \leq 1 \implies \text{rank}_{\mathbb{Z}}E(\mathbb{Q}) = \text{ord}_{s=1}L(s,E) \text{ and } \#\text{Ш}(E) < \infty.$$

In the case $\text{ord}_{s=1}L(s,E) = 1$ the method of proof yields a systematic construction of non-torsion points in $E(\mathbb{Q})$: Heegner points[39].

In the case $L(1,E) \neq 0$ the result was independently proved by Kato in the early 1990's[46]. In fact, Kato proved the upper bound for $\text{Sel}_{p^\infty}(E)$ predicted by (BSD-f) for all but finitely many explicit p. Inspired by[46], a cyclotomic approach to the the $r=1$ case of Theorem 2.4 is given in[25]. Unlike Kolyvagin, the approach leads to an upper for $\text{Sel}_{p^\infty}(E)$ predicted by (BSD-f).

2.2.3 A Rank Zero p-Converse

Advances on the Iwasawa theory of elliptic curves have led to p-converses to Theorem 2.4.

Theorem 2.5 *Let $E_{/\mathbb{Q}}$ be an elliptic curve with conductor N. Let $p \geq 3$ be a prime at which E has ordinary reduction. Suppose*:

($\text{irr}_{\mathbb{Q}}$) *The mod p Galois representation $E[p]$ is* *absolutely irreducible.*

(ram) *There exists a prime $\ell||N$, $\ell \neq p$, such that $E[p]$ is ramified at ℓ.*

Then

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E) = 0 \implies L(1,E) \neq 0.$$

This p-converse is largely due to Skinner and Urban around the mid 2000's[62,65]. More recently, Wan[75] established a p-converse for primes p of supersingular reduction. As for reducible primes, a progress is due to Greenberg and Vatsal[37].

2.2.4 The BSD Formula: Rank Zero Case

Theorem 2.6 *Let $E_{/\mathbb{Q}}$ be an elliptic curve with conductor N. Let $p \geq 3$ be a prime at which E has ordinary reduction. Suppose:*

($\mathrm{irr}_{\mathbb{Q}}$) *The mod p Galois representation $E[p]$ is absolutely irreducible.*

(ram) *There exists a prime $\ell||N$, $\ell \neq p$, such that $E[p]$ is ramified at ℓ.*

If $L(1,E) \neq 0$, then the p-part of the BSD formula (BSD-f) holds:

$$\left|\frac{L(1,E)}{\Omega_E}\right|_p^{-1} = \left|\#Ш(E)\cdot\prod_{\ell\nmid\infty}c_\ell(E)\right|_p^{-1}.$$

This p-part of the BSD formula is due to Kato and Skinner–Urban around the mid 2000's[46,62,65]. Indeed, the formula is a consequence of the cyclotomic main conjecture for E. Similarly, a supersingular case follows from[46,49,75].

If $p \nmid N$ and E is semistable, then the hypothesis (ram) of Theorem 2.6 is always satisfied. However, it is never satisfied by CM curves.

Theorem 2.7 *Let $E_{/\mathbb{Q}}$ be a CM elliptic curve. If $L(1,E) \neq 0$, then the BSD formula (BSD-f) holds:*

$$\frac{L(1,E)}{\Omega_E} = \frac{\#Ш(E)\cdot\prod_{\ell\nmid\infty}c_\ell(E)}{\#E(\mathbb{Q})_{\mathrm{tor}^2}}.$$

This is a main result of[15], which relies on the framework of the equivariant Tamagawa number conjecture[34,35]. The p-part of the CM BSD formula when $p \nmid \#\mathcal{O}_K^\times$ for K the CM field goes back to Rubin[58].

2.3 Results II

We describe more recent results towards the BSD conjecture.

2.3.1 p-Converse to a Theorem of Coates–Wiles and Rubin

Theorem 2.8 *Let $E_{/\mathbb{Q}}$ be a CM elliptic curve and p a prime. Then*

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E) = 0 \implies L(1,E) \neq 0.$$

In the early 1990's Rubin[58] proved this p-converse when $p \nmid \#\mathcal{O}_K^\times$ for K the CM field, the hypothesis being essential to employ the Euler system of elliptic units. In particular, the case $p = 2$ remained open. The unconditional p-converse is recent[13,18].

Remark 2.9 In combination with[67], the 2-converse leads to the first example of a quadratic twist family of elliptic curves for which the even parity case of Goldfeld's conjecture[33] holds. In[13,18] the even parity case of the conjecture is proved for the congruent number elliptic curve: Let $E^{(n)} : y^2 = x^3 - n^2x$ be a congruent number elliptic curve. Then,

$$L(1,E^{(n)}) \neq 0 \text{ for a density one set of integers } n \equiv 1,2,3 \mod 8.$$

2.3.2 p-Converse to the Gross–Zagier and Kolyvagin Theorem

Theorem 2.10 *Let $E_{/\mathbb{Q}}$ be an elliptic curve with conductor N and $p \nmid 6N$ a prime at which E has ordinary reduction. Suppose:*

($\mathrm{irr}_{\mathbb{Q}}$) *The mod p Galois representation $E[p]$ is absolutely irreducible.*

(ram) *There exists a prime $\ell||N$ such that $E[p]$ is ramified at ℓ.*

Then,

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E) = 1 \implies \mathrm{ord}_{s=1}L(s,E) = 1.$$

The first general results towards this p-converse were independently due to Skinner[64] and Zhang[81] a few years back. Other results in the same vein can be found in[25,28,30,74]. The version in this Theorem will appear in[26].

The hypothesis (ram) is never satisfied by CM curves. A rank one p-converse for CM curves:

Theorem 2.11 *Let $E_{/\mathbb{Q}}$ be a CM elliptic curve with conductor N and $p \nmid 6N$ a prime. Then,*

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E) = 1 \implies \mathrm{ord}_{s=1}L(s,E) = 1.$$

For p also a prime of ordinary reduction, this p-converse was proved in[13]. It is a rare instance where the non-CM case[64,81] preceded the CM case[C]. The hypothesis $p > 3$ is removed in[20,78]. The supersingular case will appear in[27] (see also[22,23]). Another approach which generalizes to CM curves over totally real fields is given in[20,21,24].

2.3.3 The BSD Formula: Rank One Case

Theorem 2.12 *Let $E_{/\mathbb{Q}}$ be an elliptic curve with conductor N. Let $p > 3$ be a prime at which E has ordinary reduction. Suppose:*

($\mathrm{irr}_{\mathbb{Q}}$) *The mod p Galois representation $E[p]$ is absolutely irreducible.*

(ram) *There exists a prime $\ell || N$, $\ell \neq p$, such that $E[p]$ is ramified at ℓ.*

If $\mathrm{ord}_{s=1}L(s,E) = 1$, then the p-part of the BSD formula (BSD-f) holds:

$$\left|\frac{L'(1,E)}{\mathrm{reg}(E)\cdot\Omega_E}\right|_p^{-1} = \left|\#\text{Ш}(E)\prod_{\ell\nmid\infty} c_\ell(E)\right|_p^{-1}.$$

The first general results towards the p-part but with additional conditions on p were independently due to Jetchev–Skinner–Wan[42] and Zhang[81] in the mid 2010's. Other results in the same vein were established in[4,30,66]. The result stated in Theorem 2.12 is proved in[25].

The p-part of the BSD formula for CM curves when $p \nmid 2N$ is a consequence of the main conjecture[58] in combination with the p-adic Gross–Zagier formulas[50,55].

3 A *p*-Adic Waldspurger Formula

We describe the p-adic Waldspurger formula and outline its key role in some of the recent results towards the p-converse and the p-part of the BSD formula. For a detailed introduction, one may refer to[63].

3.1 *p*-Adic Waldspurger Formula

3.1.1 Backdrop

Let $E_{/\mathbb{Q}}$ be an elliptic curve with conductor N and $f \in S_2(\Gamma_0(N))$ the associated elliptic newform. Let $p \nmid 2N$ be a prime and $a_p = p + 1 - \#E(\mathbb{F}_p)$.

Let K be an imaginary quadratic field. Fix an algebraic closure $\overline{\mathbb{Q}}$ and embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Let $\tau \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ denote the complex conjugation, which induces the non-trivial element in $\mathrm{Gal}(K/\mathbb{Q})$ via ι_∞.

Suppose the following Heegner hypothesis:

$$\text{Each prime dividing } N \text{ splits in } K. \qquad \text{(Heeg)}$$

Also suppose that p splits in K:

$$(p) = v\bar{v} \qquad \text{(spl)}$$

with v determined via ι_p and that

$$\text{The discriminant } D_K \text{ is odd and } D_K \neq -3. \qquad \text{(disc)}$$

In view of (Heeg) let $y_K \in E(K)$ be the Heegner point arising from the modular parametrisation $X_0(N) \twoheadrightarrow E$ and the CM points on $X_0(N)$ with endomorphism ring $\mathcal{O}_K$.

Let χ_K denote the quadratic character of $\mathbb{Q}$ associated to the extension $K/\mathbb{Q}$ and $\mathbb{1}$ the identity Hecke character of K. Let Γ be the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of K and $\Lambda := \mathbb{Z}_p[\![\Gamma]\!]$ the Iwasawa algebra. Let superscript ι denote the involution of Λ arising from the inversion on Γ. Let $\Lambda^{\mathrm{ur}} = \Lambda\hat{\otimes}_{\mathbb{Z}_p}W(\overline{\mathbb{F}}_p)$ and $\xi_{\Lambda^{\mathrm{ur}}}(\cdot)$ denote the Λ^{ur}-characteristic ideal.

3.1.2 The Formula

Let $\mathcal{L}_v \in \Lambda^{\mathrm{ur}}$ be the anticyclotomic p-adic L-function associated to f as in[1,8].

It interpolates the Rankin–Selberg central L-values associated to the self-dual pairs (f, ν) for $\nu \in \Xi$, where Ξ denotes the set of arithmetic Hecke characters of K with corresponding Galois character factoring through Γ and having Hodge–Tate weight at v at least 1. In particular, any finite order Hecke character of K—notably $\mathbb{1}$—lies outside the defining range of interpolation. An arithmetic interpretation of such p-adic L-values is given by the following[1].

Theorem 3.1 *Let $\mathcal{L}_v$ be the anticylotomic p-adic L-function as above. Then*

[C] In part the p-converse was sparked by the non-vanishing[10,14].

$$\hat{\mathbb{1}}(\mathcal{L}_\nu) = u\left(\frac{1-a_p+p}{p} \cdot \log_{E(K_\nu)}(y_K)\right)^2$$

for some $u \in (\mathbb{Z}_p^{\mathrm{ur}})^\times$. *In particular,*

$$\hat{\mathbb{1}}(\mathcal{L}_\nu) \neq 0 \iff \mathrm{ord}_{s=1} L(s, E_{/K}) = 1.$$

'In particular' part is a consequence of the Gross–Zagier formula[38,79,80]:

$$L'(1, E_{/K}) = (*) \cdot \langle y_K, y_K \rangle_{\mathrm{NT}} \quad (3)$$

for $(*)$ an explicit non-zero constant and $\langle -, - \rangle_{\mathrm{NT}}$ the Néron–Tate height pairing. Strikingly, it provides a criterion for the non-vanishing of central derivative of the Hasse–Weil L-function $L(s, E_{/K})$ in terms of a *value* of the p-adic L-function $\mathcal{L}_\nu$.

The construction of $\mathcal{L}_\nu$ is based on the Waldspurger formula[75] which expresses the Rankin–Selberg L-values in the interpolation region as the square of $K^\times$-toric periods of f. Coleman's theory of p-adic integration[32] lies at the heart of the proof of Theorem 3.1. This is a notable departure from backdrop of the Gross–Zagier formula and its p-adic analogue which is rooted in arithmetic intersection theory.

- *Remark 3.2* An analogous formula holds for any finite order characters of Γ.
- Liu–Zhang–Zhang[53] developed an automorphic framework which interprets Theorem 3.1 as a p-adic Waldspurger formula, thereby generalising Theorem 3.1 to modular elliptic curves over totally real fields, in particular allowing generalised Heegner hypothesis over $\mathbb{Q}$.
- The p-adic Waldspurger formula has influenced the arithmetic of elliptic curves over $\mathbb{Q}$ especially rank one aspects. For instance, an application to Mazur's conjecture on generic non-triviality of Heegner points is given in[11,12].

3.2 Main Conjectures

Iwasawa theory of the anticyclotomic p-adic L-function $\mathcal{L}_\nu$ has initiated a progress towards the BSD conjecture. We describe the underlying main conjectures, one of which involves $\mathcal{L}_\nu$ in the guise of Heegner points.

3.2.1 Greenberg Main Conjecture

The main conjecture for Iwasawa deformations satisfying the Panchishkin condition due to Greenberg[36] leads to a conjectural arithmetic interpretation of the p-adic L-function $\mathcal{L}_\nu$.

Let Σ be the finite set of places of K containing ∞ and the primes above Np. Let K^Σ be the maximal extension of K unramified outside Σ and set $G_\Sigma = \mathrm{Gal}(K^\Sigma/K)$. Consider a $\mathbb{Z}_p[G_\Sigma]$-module

$$M = T \otimes_{\mathbb{Z}_p} \Lambda^\vee,$$

where T denotes the p-adic Tate module of E and $(\cdot)^\vee$ the Pontryagin dual and G_Σ acts on Λ via $\Psi : G_\Sigma \twoheadrightarrow \Gamma \hookrightarrow \Lambda^\times$.

Define a discrete Selmer group

$$S_\nu = \ker\left\{H^1(G_\Sigma, M) \rightarrow \prod_{w \in \Sigma, w \nmid p} H^1(K_w, M) \times H^1(K_{\bar{v}}, M)\right\}$$

and let X_ν be its Pontryagin dual. These anticyclotomic Selmer groups interpolate the Bloch–Kato Selmer groups $H^1_{\mathrm{f}}(K, T \otimes \nu)$ for $\nu \in \Xi$.

Conjecture 3.3 *Let $E_{/\mathbb{Q}}$ be an elliptic curve with conductor N and $p \nmid 2N$ a prime. Let K be an imaginary quadratic field satisfying (Heeg), (spl) and (disc). Let S_ν and X_ν be the associated Selmer groups as above, and $\mathcal{L}_\nu$ the p-adic L-function. Then*

(a) $\mathrm{rank}_\Lambda S_\nu = \mathrm{rank}_\Lambda X_\nu = 0$,
(b) $\xi_{\Lambda^{\mathrm{ur}}}(X_\nu) = (\mathcal{L}_\nu)$.

Remark 3.4 The conjecture first appeared in[64]. It does not require any hypothesis on the image of the underlying Galois representation and is independent of the choice of a lattice (cf.[51, Prop. 2.9]).

3.2.2 Heegner Main Conjecture

The eponymous conjecture due to Perrin-Riou[56] concerns Iwasawa theory of Heegner points.

Let X be the Pontryagin dual of the discrete Selmer group

$$\mathcal{S} = \varinjlim_n \varinjlim_m \mathrm{Sel}_{p^m}(E_{/K_n})$$

and let $S = \varprojlim_n \varprojlim_m \mathrm{Sel}_{p^m}(E_{/K_n})$, where K_n is the nth layer of the extension K_∞/K.

Let $\kappa_0 \in \mathrm{Sel}_{p^\infty}(E_{/K})$ be the Kummer image of the Heegner point $y_K \in E(K)$. If E has good ordinary reduction at p, then a variant of the construction of y_K over the layers K_n leads to a norm-compatible system of generalized Heegner points, yielding a Heegner class $\kappa \in S$ which deforms κ_0.

Conjecture 3.5 *Let $E_{/\mathbb{Q}}$ be an elliptic curve with conductor N and $p \nmid N$ a prime of ordinary reduction. Let K be an imaginary quadratic field satisfying* (Heeg). *Then*

(a) *the Heegner class $\kappa \in S$ is not Λ-torsion,*
(b) $\mathrm{rank}_\Lambda S = \mathrm{rank}_\Lambda X = 1$,
(c) $\xi_{\Lambda_{\mathbb{Q}_p}}(S/\Lambda \cdot \kappa) \cdot \xi_{\Lambda_{\mathbb{Q}_p}}((S/\Lambda \cdot \kappa)^\iota) = \xi_{\Lambda_{\mathbb{Q}_p}}(X_{\mathrm{tor}})$,

for $\Lambda_{\mathbb{Q}_p} = \Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $(\cdot)_{\mathrm{tor}}$ the Λ-torsion submodule.

Moreover if $p > 2$ and $E(K)[p] = 0$, then

(c') $\xi_\Lambda(S/\Lambda \cdot \kappa) \cdot \xi_\Lambda((S/\Lambda \cdot \kappa)^\iota) = \xi_\Lambda(X_{\mathrm{tor}})$.

The p-adic Waldspurger formula intertwines the underlying main conjectures:

$$\text{Conjecture 3.3 and Conjecture 3.5 are equivalent} \quad (4)$$

(cf.[19, Thm. 5.2]). Indeed, the equivalence of the two main conjectures comes via the non-triviality of κ and Λ-adic analogue of the p-adic Waldspurger formula. The latter expresses the p-adic L-function $\mathcal{L}_v$ in terms of the Λ-adic logarithm[D] of κ.

3.3 Arithmetic Consequences

We outline an insight of Skinner: the main conjectures in Sect. 3.2 lead to the p-converse and the p-part of the BSD formula for rank one elliptic curves.

3.3.1 p-Converse

The subsection describes an approach to (p-cv) via Conjecture 3.5.

Let $E_{/\mathbb{Q}}$ be an elliptic curve with conductor N and $p \nmid N$ a prime of ordinary reduction. Suppose $\mathrm{corank}_{\mathbb{Z}_p} E(\mathbb{Q}) = 1$.

Let K be an imaginary quadratic field satisfying (Heeg) such that $L(1, E \otimes \chi_K) \neq 0$. The existence of K follows from the parity conjecture and[9]. By the Gross–Zagier and Kolyvagin theorem[E], notice $\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/K}) = 1$. On the other hand $L(s, E_{/K}) = L(s, E) L(s, E \otimes \chi_K)$ and so by the Gross–Zagier formula (3) the p-converse is equivalent to

$$\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/K}) = 1 \implies y_K \in E(K) \setminus E(K)_{\mathrm{tor}}.$$

[D] This interpolates the Bloch–Kato logarithm along the anti-cyclotomic tower.

[E] One may also resort to Kato[46].

To approach the p-converse, it suffices to show that the left-hand side of the conjectured equality in Conjecture 3.5 (c) divides the right-hand side:

$$\xi_{\Lambda_{\mathbb{Q}_p}}(S/\Lambda \cdot \kappa) \cdot \xi_{\Lambda_{\mathbb{Q}_p}}\left((S/\Lambda \cdot \kappa)^\iota\right) \mid \xi_{\Lambda_{\mathbb{Q}_p}}(X_{\mathrm{tor}}). \quad (5)$$

Indeed by Iwasawa-theoretic descent to K for X, the hypothesis $\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/K}) = 1$ implies $\xi_\Lambda(X_{\mathrm{tor}})$ is not divisible by the augmentation ideal of Λ. The same is then true of $\xi_\Lambda(S/\Lambda \cdot \kappa)$, which implies—again by descent to K—that κ_0 is non-torsion and hence that y_K is non-torsion.

For a large class of semistable elliptic curves the divisibility (5) is due to Wan[73]. It is based on the Eisenstein congruence approach for the unitary group $U(3, 1)$. The other divisibility may often be studied via the Kolyvagin system of Heegner points as in[30,40,41]. Some complementary results towards Conjecture 3.5 appear in[16,19,30,74].

- *Remark 3.6* The above approach to the p-converse generalises to primes p of good supersingular reduction[28], as well as the primes of multiplicative reduction. An important missing case is that of the primes of additive reduction, even a conjectural framework amiss.
- Though Conjecture 3.5 is equivalent to Conjecture 3.3, the p-converse is a little more attuned to Conjecture 3.5. Indeed Conjecture 3.3 yields a p-converse[64] under the additional assumption that $\#Ш(E)[p^\infty] < \infty$:: If

  $$\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E) = 1 \text{ and } Ш(E)[p^\infty] < \infty,,$$

  then observe $\hat{\mathbb{1}}(\xi_\Lambda(X_v)) \neq 0$. So Conjecture 3.1 (b) predicts $\mathbb{1}(\mathcal{L}_v) \neq 0$ and hence y_K is non-torsion by Theorem 3.1.
-
- The p-converse theorems in[64,74] inherit the hypothesis from[73], as does[81] from[65]. To treat the missing cases, additional ideas seem essential, shades of which appear in[16,30]. Theorem 2.10 is based on a new tool: the Euler system for E over K, the existence of which is the main result of[25,26].

3.3.2 p-Part of the BSD Formula

The subsection describes an approach to (BSD-f) via Conjecture 3.3.

Let $E_{/\mathbb{Q}}$ be an elliptic curve with conductor N such that $\mathrm{ord}_{s=1} L(s, E) = 1$. Let $p \nmid 2N$ be a prime with

$$E(\mathbb{Q})[p] = 0. \quad (6)$$

Let K be an imaginary quadratic field satisfying (Heeg), (spl) and (disc) so that $E(K)[p] = 0$ and

$$L(1, E \otimes \chi_K) \neq 0.$$

By the Gross–Zagier formula (3), the Heegner point y_K is non-torsion and so $\mathrm{rank}_{\mathbb{Z}} E(K) = 1$ and $\#Ш(E_{/K}) < \infty$by Kolyvagin. Thus, in light of the Gross–Zagier formula the p-part of the BSD formula for E over K is equivalent to

$$[E(K) \otimes_{\mathbb{Z}} \mathbb{Z}_p : \mathbb{Z}_p \cdot y_K]^2 = \left|\#Ш(E_{/K}) \cdot \prod_{\ell | N} c_\ell^2\right|_p^{-1} \quad (7)$$

The displayed formula for index of the Heegner point is a consequence of Conjecture 3.3: the specialisation of Conjecture 3.3 at the identity character $\mathbb{1}$ in conjunction with control theorem for the Selmer group X_v (cf. (6)) and the p-adic Waldspurger formula (cf. Theorem 3.1). This approach to the BSD formula first appeared in[42] to which we refer for details (see also[30, §5.3]).

As $L(s, E_{/K}) = L(s, E)L(s, E \otimes \chi_K)$, the p-part of the BSD formula for E over $\mathbb{Q}$ now reduces to the p-part of the BSD formula for $E \otimes \chi_K$ over $\mathbb{Q}$. At primes p of ordinary reduction this is the content of Theorem 2.6. The case of supersingular primes occasionally follows from[46,49,75]. We conclude that the p-part of the BSD formula for E over $\mathbb{Q}$ is a consequence of Conjecture 3.3 for E and the cyclotomic main conjecture for $E \otimes \chi$.

For a large class of semistable elliptic curves the $\mathcal{L}_v$-counterpart of the divisibility (5) is due[F] to Wan[73]. The desired upper bound for $\#Ш(E_{/K})$ in terms of the Heegner point $y_K \in E(K)$ may be deduced from the Kolyvagin system of Heegner points as in[30,40,41]. Some results towards Conjecture 3.3 can be found in[16,19,30,73].

- *Remark 3.7* This approach to the p-part of the BSD formula for E over K uniformly treats the primes of ordinary and good supersingular reduction. The above description of the strategy is simplistic: the implementation in[42] involves imaginary quadratic fields satisfying generalised Heegner hypothesis, not just (Heeg) .
- In contrast to the p-converse, the p-part of the BSD formula is more attuned to Conjecture 3.3.
- Theorem 2.12 is based on a cyclotomic method which relies on the zeta element associated to E over K[25].

[F] Actually Wan proves the divisibility in $\Lambda_{\mathbb{Q}_p}$ which in light of the vanishing of the μ-invariant of $\mathcal{L}_v$[11] can be refined integrally to Λ. While the $\Lambda_{\mathbb{Q}_p}$-divisibility suffices for the p-converse, the integral divisibility is crucial for the p-part of the BSD formula.

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