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On an inverse problem involving a second-order differential system

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Abstract

Given the spectral characteristics defined in the body of the paper, it is possible to construct a complete self-adjoint eigenvalue problem in the finite interval $(0, \pi)$ associated with a 2×2 matrix differential system. The solutions of the differential system so constructed satisfy certain prescribed boundary conditions at $x = 0$ and $x = \pi$.

Key words: Spectral functions, spectral characteristics, boundary characteristic vector, normalizing matrix, reciprocal kernel, orthonormal eigenvectors, P-identity, boundary condition vectors, bilinear concomitant, wronskian, Sturm-Liouville problem.

1. Introduction

In spectral theory, it becomes sometimes necessary to ascertain certain spectral data that determine (possibly uniquely) a differential operator and then to develop a method by which it may be possible to construct the operator from the data. The involved problem is an inverse problem associated with a differential operator. Inverse problems were first formulated and investigated way back 1929 by Ambarzumyan[1], and then since 1945 by Borg, Levinson, Marchenko, Krein, Gelfand, Levitan, Gasymov, and recently by Hochstadt and others. In 1951, Gelfand and Levitan[2] gave a method of reconstructing a second-order differential equation from its spectral function $\rho(\lambda)$ by reducing the problem to certain linear integral equations. Gasymov and Levitan[3], by adopting the same technique, solved *inter alia* an inverse problem for a finite interval from given spectral characteristics, *i.e.*, from the sequence of eigenvalues and the normalizing constants for the eigenfunctions associated with a second-order differential equation of the Sturm-Liouville type. For works on inverse Sturm-Liouville problems, reference may also be made to Levitan[4], and Marchenko[5]. In a recent paper, McLaughlin[6] presents a survey of the last forty years (till 1983) of researches on the methods and properties of these methods for recovering coefficients of differential equations from spectral data. Results are at first presented by invoking mathematical models for physical problems involving (i) a vibrating string with variable density but constant tension, (ii) propagation in the vocal tract, (iii) propagation in an isotropic elastic medium like the earth's crust, the curvature of the earth being ignored, and (iv) the Euler-Bernoulli model of transverse vibrations of a beam. Theoretical results concerning the inverse problems with special emphasis on the Sturm-Liouville operator then follow, the spectral data to be chosen being motivated by the physical examples considered earlier. Specially interesting is the consideration by the author of the inverse problem for the fourth-order equation

$$y^{(4)} + (Ay^{(1)})^{(1)} + By - \lambda y = 0, \quad 0 \leqslant x \leqslant 1,$$

$$y^{(1)} = y^{(1)}(1) = 0: \quad y^{(2)}(0) + a y^{(1)}(0) - b y(0) = 0,$$

$$y^{(3)}(0) + (b + A(0)) y^{(1)} + c y(0) = 0.$$

The problem is a self-adjoint eigenvalue problem when $A(x)$, $B(x)$ are real-valued and a, b, c are real constants. By adopting the Gelfand-Levitan technique[2] of integral equations and using a transformation operator which maps solutions of a known problem on to solutions of the to be derived problem, $A(x)$, $B(x)$ and a, b, c are determined from the spectral data comprising the sequence of eigenvalues, the spectral matrix for the fourth-order differential equations and the norming constants defined in the paper. Theoretical results obtained by the author are interspersed with pertinent remarks on a number of variations of the problem presented by some other authors. The paper ends with a rich and extensive bibliography on inverse problems on differential equations over a finite interval. The inverse problem involving fourth-order equations has not been considered as extensively as that involving the Sturm-Lieuville operator.

On eliminating v, say, from our system *i.e.* the system (1) with boundary conditions (2), (3) (section 2 below) we obtain a fourth-order differential equation in u with coefficients containing the parameter λ with boundary conditions at $x = 0$, $x = \pi$ also containing the parameter λ. Our problem is therefore entirely different from the type of fourth-order equations considered by McLaughlin. The method of solution of inverse problem involving fourth-order equations with coefficients as well as the boundary conditions containing the parameter λ does not appear to be quite well known; but if the same problem can be reduced to a second-order system as the one considered by us by some transformations, it may be possible to apply the Gelfand-Levitan technique and a transformation operator as we have done like McLaughlin, to solve such inverse problems.

The inverse problem associated with the Dirac system was initiated by Levitan and Gasymov in 1966, but very little work on the inverse problem appears to have been done for the system '$LY = \lambda MY$', consisting of m equations each of order n. Only in 1981, Ray Paladhi[7] dealt with an inverse problem associated with a special case of the system, *i.e.*, the system we are going to investigate. By defining a spectral matrix and adopting the Gasymov-Levitan method, he constructed the differential system from the spectral matrix. But his conclusion regarding non-unicity of the solution as made by him[7] in the last line of his theorem 5 (p. 191) appears wrong, the example cited in support on page 190 of the article being erroneous. We investigate the inverse problem for a finite interval from spectral characteristics to be defined for our system. The investigation appears important in view of the fact that a special case of our system, *i.e.*,

$$Y'' + \lambda^2 Y = (V(x) + 6x^{-2}P)\,Y, \quad 0 < x < \infty,$$

where $V(x)$ is a 2×2 Hermitian matrix and $P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is derived from the Schrödinger equation for a deuteron (in its ground state) when tensor interaction forces are taken into account.

It is interesting to note that the inverse problem was studied almost exclusively by mathematicians in the USSR, but elsewhere almost exclusively by physicists.

2. Preliminary results

The differential system under consideration is

$$L\phi = \lambda\phi, \quad L = \begin{pmatrix} -D^2 + p & r \\ r & -D^2 + q \end{pmatrix}, \quad D^2 = \mathrm{d}^2/\mathrm{d}x^2, \quad \phi = (u(x), v(x))^T \qquad (1)$$

where p, q, r are real-valued $C_{1-k}(0, \pi)$-class functions, $(k = 0, 1)$, summable on $(0, \pi)$. By $C_k(\alpha, \beta)$-class functions we mean sets of functions (real or complex) which are k-times differentiable on (α, β), the kth derivative being continuous in the interval. λ is the eigenvalue parameter.

The problem is one of the finite interval $(0, \pi)$ and the boundary conditions at $x = 0$ and at $x = \pi$ are, respectively,

$$a_{j1}u(0) + a_{j2}u'(0) + a_{j3}v(0) + a_{j4}v'(0) = 0 \qquad (2)$$

$$b_{j1}u(\pi) + b_{j2}u'(\pi) + b_{j3}v(\pi) + b_{j4}v'(\pi) = 0 \qquad (3)$$

where a_{ij}, b_{ij} are real-valued constants (independent of λ) satisfying the following conditions:

$$\text{(i)} \;\; \text{rank}\,(a_{ij}) = \text{rank}\,(b_{ij}) = 2, \quad i = 1, 2; \quad j = 1, 2, 3, 4 \qquad (4)$$

where at least one of

$$\begin{vmatrix} a_{j1} & a_{j3} \\ a_{k2} & a_{k4} \end{vmatrix}, \quad j, k = 1, 2; \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} \neq 0$$

$$\text{(ii)} \;\; a_{j1}a_{k2} + a_{j3}a_{k4} = 0, \quad j, k = 1, 2; \qquad (5)$$

$$\text{(iii)} \;\; b_{11}b_{22} - b_{12}b_{21} + b_{13}b_{24} - b_{14}b_{23} = 0; \qquad (6)$$

$$\text{(iv)} \;\; b_{j2}a_{k1} + b_{j4}a_{k3} = 0, \quad b_{j1}a_{k2} + b_{j3}a_{k4} = 0, \quad j, k = 1, 2; \qquad (7)$$

$$\text{(v)} \;\; b_{j2}a_{12} + b_{j4}a_{14} \neq 0. \qquad (8)$$

The system

$$\mathrm{d}^2u/\mathrm{d}x^2 + \lambda u = 0; \quad \mathrm{d}^2v/\mathrm{d}x^2 + \lambda v = 0 \qquad (9)$$

satisfying the same boundary conditions (2) and (3) at $x=0$, $x=\pi$, a_{ij}, b_{ij} satisfying conditions (4)–(8), is the Fourier system corresponding to (1).

Let ϕ_l, ϕ_j, $l=1,2$, $j=3,4$ be the boundary condition vectors at $x=0$, $x=\pi$ (*i.e.*, solutions of (1) which together with their first derivatives take prescribed constant values out of a_{ij}, b_{ij} in the boundary conditions (2), (3) at $x=0$, $x=\pi$). For example, we can choose $\phi_l'|_{x=0}=\phi_l'(0|0,\lambda)=-(a_{l1},a_{l3})^T$, $\phi_l|_{x=0}=\phi_l(0|x,\lambda)|_{x=0}=\phi_l(0|0,\lambda)=(a_{l2},a_{l4})^T$. Similarly, for $\phi_j|_{x=\pi}$, $\phi_j'|_{x=\pi}$ in terms of b_{ij}.

Then the boundary conditions (2) and (3) can be put in the form $[U,\phi_l]_0=[U,\phi_j]_\pi=0$, where $U=(u,v)^T$, $[\cdot]$ being the bilinear concomitant defined for two vectors $\alpha=(\alpha_1,\beta_1)^T$, $\beta=(\alpha_2,\beta_2)^T$ by.

$$\begin{vmatrix}\alpha_1 & \alpha_2\\ \alpha_1' & \alpha_2'\end{vmatrix}+\begin{vmatrix}\beta_1 & \beta_2\\ \beta_1' & \beta_2'\end{vmatrix}.$$

The relations (5) and (6) represented by $[\phi_1,\phi_2]_0=0$, $[\phi_3,\phi_4]_\pi=0$ are the self-adjointness conditions (see Chakravarty[8], p. 138). The wronskian is

$$W(\lambda)=[\phi_1,\phi_4][\phi_2,\phi_3]-[\phi_1,\phi_3][\phi_2,\phi_4]$$

and the eigenvalues are the simple or double zeros of $W(\lambda)$. In view of our observations[9] (p. 82), we can proceed with the assumption that the eigenvalues λ_n are the simple roots of $W(\lambda)$, which may be taken to be all positive. In particular, let the boundary conditions satisfied by the solution $(u,v)^T$ be the Dirichlet: $U(0)=U(\pi)=0$ or the Neumann: $U'(0)=U'(\pi)=0$ and $p>0$, $pq-r^2>0$, so that $Q=\begin{pmatrix}p\,r\\ r\,q\end{pmatrix}$ is positive definite (*i.e.*, the corresponding quadratic from is positive definite). Then it is easy to show that the eigenvalues λ_n are positive by considering the Dirichlet integral $D(f,g)=\int_0^\pi F(t)\,dt$, where $f=(f_1,f_2)^T$, $g=(g_1,g_2)^T$, $F(t)=f_1'g_1'+f_2'g_2'+pf_1g_1+r(f_1g_2+f_2g_1)+qf_2g_2$, and f', g' exist and are continuous in $(0,\pi)$ (see Chakravarty and Sen Gupta[10]).

Let $\{\lambda_n\}$ be the eigenvalues associated with the system (1) with (2) and (3); define $\lambda_n^{\frac{1}{2}}=\mu_n$, $\mu_n>0$, if $n\geqslant 0$ and $\lambda_n^{\frac{1}{2}}=-\mu_n$, $\mu_n>0$, if $n<0$.

Then

$$\lambda_n^{\frac{1}{2}}=n+\alpha_j/n+0(1/n),\quad j=1,2 \tag{10}$$

where α_j are constants depending on the coefficients in the boundary conditions at $x=0$ and $x=\pi$ and p, q, r which occur in the differential system. For derivation of the exact form of α_j we require the conditions (7) and (8); the vector $(\alpha_1,\alpha_2)^T$ is the boundary characteristic vector of the given problem[9] (pp 81–83).

Let

$$c_j(x,\lambda^{\frac{1}{2}})=\begin{pmatrix}c_{1j}(x,\lambda^{\frac{1}{2}})\\ c_{2j}(x,\lambda^{\frac{1}{2}})\end{pmatrix}=\begin{pmatrix}a_{j2}\cos\lambda^{\frac{1}{2}}x+a_{j1}\sin\lambda^{\frac{1}{2}}x\\ a_{j4}\cos\lambda^{\frac{1}{2}}x+a_{j3}\sin\lambda^{\frac{1}{2}}x\end{pmatrix},\quad j=1,2 \tag{11}$$

and

$$K(x,t)=\begin{pmatrix}K_{11}(x,t) & K_{12}(x,t)\\ K_{21}(x,t) & K_{22}(x,t)\end{pmatrix},$$

$K_{ij} = K_{ji}$, a 2×2 matrix (symmetric) having continuous partial derivatives up to the order two with respect to each of t and x, $K(x,t) = 0$ for $t > x$, $K(0,0) = 0$. Then putting $c_j(t) = c_j(t, \lambda^{\pm})$,

$$\phi_j(x, \lambda^{\pm}) = (\phi_{1j}(x, \lambda^{\pm}), \phi_{2j}(x, \lambda^{\pm}))^T$$
$$= c_j(x, \lambda^{\pm}) + \int_0^x K(x,t) c_j(t)\,\mathrm{d}t, \quad j = 1,2 \tag{12}$$

satisfy the system (1) with boundary conditions (2), if and only if the conditions (3.6)–(3.9) of Ray Paladhi[7] (Theorem 1, p. 175) are satisfied. Hence,

$$\phi_j(x, \lambda_n^{\pm})/\|\phi_j(x, \lambda_n^{\pm}),$$

where

$$\|\phi_j(x, \lambda_n^{\pm})\|^2 = \alpha_{jn} = \int_0^\pi |\phi_j|^2\,\mathrm{d}x$$

$j = 1, 2$, are two linearly independent sequences of normalized eigenvectors corresponding to the eigenvalue λ_n of the given differential system. ϕ_j, $j = 1, 2$, therefore, form a basis of the vector space of eigenvectors, any element of which is of the form $a_n \phi_2(x, \lambda_n^{\pm}) + b_n \phi_1(x, \lambda_n^{\pm})$, where a_n, b_n are any two constants independent of x. The linear combination

$$\psi_n(x, \lambda_n^{\pm}) \equiv (\psi_{1n}(x, \lambda_n^{\pm}), \psi_{2n}(x, \lambda_n^{\pm}))^T \equiv \alpha_{1n}\phi_2(x, \lambda_n^{\pm}) - \alpha_{2n}\phi_1(x, \lambda_n^{\pm})$$

is therefore an eigenvector chosen to correspond to the eigenvalue λ_n of the system under consideration[9] (pp 140–143).

Let

$$A_n = \alpha_{1n}\alpha_{2n}(\alpha_{1n} + \alpha_{2n} - 2\alpha_{3n}), \tag{13}$$

where

$$\alpha_{3n} = \int_0^\pi (\phi_1(x, \lambda_n^{\pm}), \phi_2(x, \lambda_n^{\pm}))\,\mathrm{d}x.$$

Then by the Schwarz inequality and the condition (4),

$$\begin{pmatrix} \alpha_{1n} & \alpha_{3n} \\ \alpha_{3n} & \alpha_{2n} \end{pmatrix},$$

to be called the normalizing matrix, is positive definite and therefore $A_n > 0$ and $\psi_n(x, \lambda_n^{\pm})/A_n^{\frac{1}{2}}$ is the normalized eigenvector corresponding to the eigenvalue λ_n of our system. $\{A_n\}$ are termed the normalizing constants. The sequence of eigenvalues $\{\lambda_n\}$ together with the sequences $\{\alpha_{jn}\}$ and $\{A_n\}$ may be called the spectral characteristics of our boundary-value problem (compare Levitan and Gasymov[3]).

The Parseval theorem corresponding to our system is

$$\int_0^\pi (f^T(x), g(x))\,\mathrm{d}x = \sum_{n=-\infty}^{\infty} 1/A_n \int_0^\pi (f^T(x), \psi_n(x, \lambda_n^{\pm}))\,\mathrm{d}x \int_0^\pi (g^T(t), \psi_n(t, \lambda_n^{\pm})\,\mathrm{d}t \tag{14}$$

for two vectors $f = (f_1(x), f_2(x))^T$ and $g = (g_1(x), g_2(x))^T$ each $\varepsilon L_2(0, \pi)$.

Let ε_j, $j = 1, 2$, be two unit vectors on $R \times R$ and let $x, t, x \neq t$ be given. In (14) put $f = \varepsilon_j$ for $0 \leqslant z \leqslant x$ and $= 0$, otherwise and $g = \varepsilon_k$ for $0 \leqslant z \leqslant t$, $= 0$, otherwise. Then

$$\min(x,t)\delta_{jk} = \sum_{n=-\infty}^{\infty} 1/A_n \int_0^x \psi_{jn}(u, \lambda_n^{\frac{1}{2}})\mathrm{d}u \int_0^t \psi_{kn}(v, \lambda_n^{\frac{1}{2}})\mathrm{d}v, \quad j,k = 1,2 \tag{15}$$

where $\delta_{j,k}$ is the kronecker delta.

3. Some asymptotic formulae

In (12) (with $j = 1$) put

$\begin{pmatrix} X_1(x,t) & Y_1(x,t) \\ X_2(x,t) & Y_2(x,t) \end{pmatrix} = \begin{pmatrix} a_{12} & a_{14} \\ a_{11} & a_{13} \end{pmatrix} K(x,t)$ and integrate by parts the integral on the right. Then in view of the relations[7] (p. 175) $X_2(x,0) = 0$, $Y_2(x,0) = 0$ and the asymptotic estimate (10), we obtain

$$\phi_1(x, \lambda_n^{\frac{1}{2}}) = \begin{pmatrix} a_{12}\cos nx + a_{11}\sin nx \\ a_{14}\cos nx + a_{13}\sin nx \end{pmatrix} - \alpha_j x/n \begin{pmatrix} a_{12}\sin nx - a_{11}\cos nx \\ a_{14}\sin nx - a_{13}\cos nx \end{pmatrix}$$
$$+ 1/n \begin{pmatrix} X_1(x,x)\sin nx - X_2(x,x)\cos nx \\ Y_1(x,x)\sin nx - Y_2(x,x)\cos nx \end{pmatrix} + O(1/n)$$

where α_j are the constants which occur in (10). Similarly, for $\phi_2(x, \lambda_n^{\frac{1}{2}})$.

Then substituting for $X_i(x,x)$, $Y_i(x,x)$ by the relations (3.6)–(3.9) of Ray Paladhi[7], we obtain after some reductions

$$\alpha_{1n} = \|\phi_1(x, \lambda_n^{\frac{1}{2}})\|^2 = \tfrac{1}{2}\pi|a_1|^2 + O(1/n) \tag{16}$$

where

$$|a_j|^2 = a_{j1}^2 + a_{j2}^2 + a_{j3}^2 + a_{j4}^2 \quad \text{for vectors } a_j = (a_{j1}, a_{j2}, a_{j3}, a_{j4}), j = 1,2 \tag{17}$$

$$\alpha_{2n} = \tfrac{1}{2}\pi|a_2|^2 + O(1/n) \tag{18}$$

$$\alpha_{3n} = \tfrac{1}{2}\pi(a_1, a_2) + O(1/n) \tag{19}$$

where (a_1, a_2) is the inner product $a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} + a_{14}a_{24}$. It may be noted that the restrictions on a_{ij} as imposed in (4) are needed in the asymptotic evaluation of α_{in}, $i = 1, 2, 3$.

Since rank $(a_{ij}) = 2$, it follows that

$$D_0 = \pi^3/8|a_1|^2|a_2|^2|a_1 - a_2|^2 > 0. \tag{20}$$

Then from (13), (16), (18), (19) it follows that

$$A_n = D_0 + O(1/n). \tag{21}$$

Put

$$\mathrm{d}_j = D_1 a_{2j} - D_2 a_{1j}, \quad j = 1,2 \tag{22}$$

where

$$D_k = \tfrac{1}{2}(\pi^{3/2}/D_0^{\frac{1}{2}})|a_k|^2, \quad k = 1,2$$

and

$$d_1^2 + d_2^2 = d_3^2 + d_4^2 = 1; \tag{23}$$

$$d_1 d_3 + d_2 d_4 = 0. \tag{24}$$

Let

$$A = 2\begin{pmatrix} d_2^2 & d_2 d_4 \\ d_2 d_4 & d_4^2 \end{pmatrix} \tag{25}$$

$$B = \begin{pmatrix} d_1^2 & d_1 d_3 \\ d_1 d_3 & d_3^2 \end{pmatrix} \tag{26}$$

$$c(x, \lambda_n^{\pm}) = \begin{pmatrix} c_{12}(x, \lambda_n^{\pm}) - c_{11}(x, \lambda_n^{\pm}) \\ c_{22}(x, \lambda_n^{\pm}) - c_{21}(x, \lambda_n^{\pm}) \end{pmatrix} \begin{pmatrix} \alpha_{1n} & \alpha_{1n} \\ \alpha_{2n} & \alpha_{2n} \end{pmatrix} \tag{27}$$

$$c^*(x, n) = \begin{pmatrix} c_{12}(x, n) - c_{11}(x, n) \\ c_{22}(x, n) - c_{21}(x, n) \end{pmatrix} \begin{pmatrix} D_1 & D_1 \\ D_2 & D_2 \end{pmatrix} \tag{28}$$

and

$$c(x, \lambda_n^{\pm}, A_n^{\pm}) = 1/A_n^{\pm} c(x, \lambda_n^{\pm}). \tag{29}$$

Then the series

$$2F(x, t) = \sum_{n=-\infty}^{\infty} c(x, \lambda_n^{\pm}, A_n^{\pm}) c^T(t, \lambda_n^{\pm}, A_n^{\pm}) - A/\pi - 1/\pi \sum_{n=-\infty}^{\infty}{}' c^*(x, n) c^{*T}(t, n) \tag{30}$$

(where the accent denotes that the term $n = 0$ is omitted) is uniformly convergent for $x, t \geqslant 0$, and either $x \leqslant \pi$, $t \leqslant \pi - \varepsilon$ or $t \leqslant \pi$, $x \leqslant \pi - \varepsilon$, $\varepsilon > 0$, $x \neq t$, as can be seen by using the asymptotic relations (10), (16)–(21) and the uniform convergence of the series $\Sigma_{\sin}^{\cos} kx/k^{\alpha}$ (alternatively written: $\Sigma \cos kx$ (resp. $\sin kx)/k^{\alpha}$) for $\alpha > 1$, if $-\infty < x < \infty$ and for $0 < \alpha \leqslant 1$, if $0 < \varepsilon \leqslant x \leqslant 2\pi - \varepsilon$. It can be further verified that under the stated conditions the series obtained by term-wise differentiation of (30) is also uniformly convergent in the intervals stated before.

4. Derivation of integral equations satisfied by $F(x, t)$

Let $M(x, t)$ be the matrix

$$M(x, t) = (M_{ij}(x, t)) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \text{ such that } M(x, t) = 0 \text{ for } t > x,$$

$M(x, 0) \neq 0$ and $M(x, t)$ satisfy the relations (4.6)–(4.9) as given by Ray Paladhi[7] (p. 177). Then

$$c_j(x, \lambda_n^{\pm}) = \phi_j(x, \lambda_n^{\pm}) - \int_0^x M(x, t) \phi_j(t, \lambda_n^{\pm}) \, dt \tag{31}$$

where ϕ_j and c_j are those given in (12) and (11) (see Ray Paladhi[7], p. 176). In view of (12) and (31) the kernels $K(x, t)$ and $M(x, t)$ are reciprocally related. We establish the following theorem.

Theorem 4.1. $F(x,t)$ defined in (30) satisfies the equation

$$F(x,t) = -M(x,s) + \int_0^t M(x,s)M^T(t,s)\,ds, \text{ if } 0 \leqslant t < x \leqslant \pi; \tag{32}$$

and

$$F(x,t) = -M^T(x,s) + \int_0^t M(x,s)M^T(t,s)\,ds, \text{ if } 0 \leqslant x < t \leqslant \pi. \tag{33}$$

$F(x,t)$ admits second-order partial derivatives, continuous with respect to x, t in $[0,\pi] \times [0,\pi]$ and $F(x,0) \neq 0$, $\partial/\partial t F(x,t)|_{t=0} = 0$ for $x \varepsilon [0,\pi]$.

Put

$$\psi(x, \lambda_n^{\pm}, A_n^{\pm}) = \begin{pmatrix} \psi_{1n}(x, \lambda_n^{\pm}, A_n^{\pm}) & \psi_{1n}(x, \lambda_n^{\pm}, A_n^{\pm}) \\ \psi_{2n}(x, \lambda_n^{\pm}, A_n^{\pm}) & \psi_{2n}(x, \lambda_n^{\pm}, A_n^{\pm}) \end{pmatrix} \tag{34}$$

where $\psi_{1n}(.)$, $\psi_{2n}(.)$ are the components of the normalized eigenvectors.

$$\psi_n(x, \lambda_n^{\pm}, A_n^{\pm}) \equiv 1/A_n^{\pm}\psi_n(x, \lambda_n^{\pm}) \text{ defined before (see section 2).}$$

Then from (27), (29) and (31), we have

$$c(x, \lambda_n^{\pm}, A_n^{\pm}) = \psi(x, \lambda_n^{\pm}, A_n^{\pm}) - \int_0^x M(x,t)\psi(t, \lambda_n^{\pm}, A_n^{\pm})\,dt. \tag{35}$$

Since term-wise integration is permissible in (30) on account of the uniform convergence of the series involved, we have

$$2\int_0^x \int_0^t F(u,v)\,du\,dv = \sum_{n=-\infty}^{\infty} \int_0^x \int_0^t c(u,v,\lambda_n^{\pm}, A_n^{\pm})\,du\,dv - Axt/\pi$$
$$- 1/\pi \sum_{n=-\infty}^{\infty}{}' \int_0^x \int_0^t c^*(u,v,n)\,du\,dv \tag{36}$$

where

$$c(u,v,\lambda_n^{\pm}, A_n^{\pm}) = c(u, \lambda_n^{\pm}, A_n^{\pm})c^T(v, \lambda_n^{\pm}, A_n^{\pm}),$$
$$c^*(u,v,n) = c^*(u,n)c^{*T}(v,n) \text{ and } 0 \leqslant u \leqslant x, 0 \leqslant v \leqslant t.$$

That is

$$2\int_0^x \int_0^t F(u,v)\,du\,dv = I - Axt/\pi - J, \text{ say.}$$

On substitution from (35), it follows that

$$I = \sum_{n=-\infty}^{\infty} \int_0^x \int_0^t \psi(u,v,\lambda_n^{\pm}, A_n^{\pm})\,du\,dv - \sum_{n=-\infty}^{\infty} \int_0^x \int_0^t \int_0^u M(u,s)\psi(s, \lambda_n^{\pm}, A_n^{\pm})\,ds$$
$$\times \psi^T(v, \lambda_n^{\pm}, A_n^{\pm})\,du\,dv - \sum_{n=-\infty}^{\infty} \int_0^x \int_0^t du\,dv\,\psi(u, \lambda_n^{\pm}, A_n^{\pm}) \int_0^v (M(v,s)$$
$$\times \psi(s, \lambda_n^{\pm}, A_n^{\pm})^T)\,ds + \sum_{n=-\infty}^{\infty} \int_0^x \int_0^t du\,dv \int_0^u M(u,s)\psi(s, \lambda_n^{\pm}, A_n^{\pm})\,ds$$
$$\times \int_0^v (M(v,s)\psi(s, \lambda_n^{\pm}, A_n^{\pm}))^T\,ds = I_1 + I_2 + I_3 + I_4, \text{ say,} \tag{37}$$

where

$$\psi(u, v, \lambda_n^{\pm}, A_n^{\pm}) = \psi(u, \lambda_n^{\pm}, A_n^{\pm})\psi(v, \lambda_n^{\pm}, A_n^{\pm}).$$

From (34) and the relation (15) it follows that $I_1 = 2\min(x, t)\, E$, where E is the 2×2 unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. By utilizing a modified version of the Parseval relation (14) with $f = (M_{1j}, M_{2j})^T$ $j = 1, 2$, the column vectors of $M(x, t)$, where $M(x, t) = 0$ for $t > x$ and $g = \varepsilon_j$ for $t \leqslant x$ and $= 0$, otherwise, it follows that

$$I_2 = -2\int_0^x \mathrm{d}u \int_0^t M(u, s)\,\mathrm{d}s.$$

Similarly,

$$I_3 = -2\int_0^t \mathrm{d}v \int_0^x M^T(v, s)\,\mathrm{d}s$$

and

$$I_4 = 2\int_0^x \mathrm{d}u \int_0^t \mathrm{d}v \int_0^u M(u, s) M^T(v, s)\,\mathrm{d}s.$$

Therefore, for all $x \neq t$ in $0 \leqslant t, x \leqslant \pi$,

$$I = 2\min(x, t)E - 2\int_0^x \mathrm{d}u \int_0^t M(u, s)\,\mathrm{d}s - 2\int_0^t \mathrm{d}v \int_0^x M^T(v, s)\,\mathrm{d}s$$
$$+ 2\int_0^x \mathrm{d}u \int_0^t \mathrm{d}v \int_0^u M(u, s) M^T(v, s)\,\mathrm{d}s. \tag{38}$$

Put

$$D = \begin{pmatrix} \mathrm{d}_2 & \mathrm{d}_1 \\ \mathrm{d}_4 & \mathrm{d}_3 \end{pmatrix}, \quad H(x, n) = \begin{pmatrix} \sin nx & \sin nx \\ 1 - \cos nx & 1 - \cos nx \end{pmatrix}$$

and

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{pmatrix} = DH(x, n)H^T(x, n)D^T.$$

Then, substituting for $c^*(x, n)$ by (28) and simplifying we obtain

$$J = 1/\pi \sum_{n=-\infty}^{\infty}{}' \; 1/n^2 \begin{pmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{pmatrix}. \tag{39}$$

If $x > t, \frac{1}{2}A_{11} = (\mathrm{d}_2 \sin nx + \mathrm{d}_1(1 - \cos nx)(\mathrm{d}_2 \sin nt + \mathrm{d}_1(1 - \cos nt))$. Similar results hold for A_{12}, A_{14} and A_{13} and also for $x < t$. Then

$$\sum_{n=-\infty}^{\infty}{}' A_{11}/n^2 = 2(\pi \min(t, x) - tx\,\mathrm{d}_2^2 + 1/3\pi^2\,\mathrm{d}_1^2) \tag{40}$$

by (23) and the formulae

$$\sum_{n=-\infty}^{\infty}{}' \cos n\theta/n^2 = 2\sum_{n=1}^{\infty} \cos n\theta/n^2 = 2(\pi^2/6 - \pi\theta/2 + \theta^2/4), \quad 0 \leqslant \theta < \pi$$

and

$$\sum_{n=-\infty}^{\infty}{}' \sin n\theta/n^2 = 0.$$

Similarly,

$$\sum_{n=-\infty}^{\infty}{}' A_{14}/n^2 = 2(\pi \min(t,x) - tx\,\mathrm{d}_4^2 + \pi^2 \mathrm{d}_3^2/3).$$

Also by using the condition (24)

$$\sum_{n=-\infty}^{\infty}{}' A_{12}/n^2 = \sum_{n=-\infty}^{\infty}{}' A_{13}^{*}/n^2 = -2tx\,\mathrm{d}_2\mathrm{d}_4 + 2\pi^2 \mathrm{d}_1\mathrm{d}_3/3.$$

Altogether, from (39)

$$\begin{aligned} J &= 2\min(t,x)E - 2tx/\pi \begin{pmatrix} \mathrm{d}_2^2 & \mathrm{d}_2\mathrm{d}_4 \\ \mathrm{d}_2\mathrm{d}_4 & \mathrm{d}_4^2 \end{pmatrix} + \frac{2}{3}\pi \begin{pmatrix} \mathrm{d}_1^2 & \mathrm{d}_1\mathrm{d}_3 \\ \mathrm{d}_1\mathrm{d}_3 & \mathrm{d}_3^2 \end{pmatrix} \\ &= 2\min(t,x)E - Axt/\pi + 2B\pi/3. \end{aligned} \tag{41}$$

Hence, from (36), (38) and (41), it follows that

$$\begin{aligned} \int_0^x \int_0^t F(u,v)\,\mathrm{d}u\,\mathrm{d}v &= \int_0^x \mathrm{d}u \int_0^t \mathrm{d}v \int_0^u M(u,s)M^T(v,s)\,\mathrm{d}s - \pi B/3 \\ &\quad - \int_0^x \mathrm{d}u \int_0^t M(u,s)\,\mathrm{d}s - \int_0^t \mathrm{d}v \int_0^x M^T(v,s)\,\mathrm{d}s. \end{aligned} \tag{42}$$

Now, operating with $\partial^2/\partial x \partial t$ both sides of (42) and then integrating by parts as and when necessary, we obtain after some reductions

$$F(x,t) = -M(x,t) - M^T(t,x) + \int_0^t M(x,s)M^T(t,s)\,\mathrm{d}s. \tag{43}$$

Since by definition $M(t,x)=0$ for $x>t$, the relation (32) follows from (43). Again since $M(x,t)=0$ for $t>x$, the relation (33) also follows from (43).

Since $M(x,t)$ has continuous second-order partial derivatives with respect to x, t, in $[0,\pi]$, $F(x,t)$ has also so for $0 \leqslant x,\ t \leqslant \pi,\ x \neq t$. Also since $M(x,0) \neq 0$, and $M_t(x,t)|_{t=0} = 0$, $F(x,0) \neq 0$ and $F_t(x,t)|_{t=0} = 0$ for all $x \in [0,\pi]$. The theorem therefore follows.

Theorem 4.2. The kernels $K(x,t)$ and $M(x,t)$ are connected by the equation

$$-M^T(x,t) = F(t,x) + \int_0^t K(t,u)F(u,x)\,\mathrm{d}u \tag{44}$$

where $F(x,t)$ is defined by (30).

The proof is a simple repetition of the arguments of Ray Paladhi[7] (pp 185–186) with our function $F(x,t)$ in place of $E(x,t)$ defined by him by the relation (I) on p. 184, and hence omitted (see also Gasymov and Levitan[3], pp 18–20, lemma 1.5.1.). The kernel $M(x,t)$ is the reciprocal of the kernel $K(x,t)$.

Theorem 4.3. If $0 \leqslant t < x \leqslant \pi$, the kernel $K(x,t)$ associated with (12) satisfies the integral equation

$$F(x,t) + K(x,t) + \int_0^x K(x,s)F(s,t)\,\mathrm{d}s = 0 \tag{45}$$

where $F(x,t)$ is given by (30). There is a similar relation for $0 \leqslant x < t \leqslant \pi$.

It follows from (12), (27), (29) and (34) that

$$\psi(u, \lambda_n^{\pm}, A_n^{\pm}) = c(u, \lambda_n^{\pm}, A_n^{\pm}) + \int_0^u K(u,t)c(t, \lambda_n^{\pm}, A_n^{\pm})\, dt. \tag{46}$$

Multiplying (46) by $c^T(v, \lambda_n^{\pm}, A_n^{\pm})$ and making use of (35) we obtain

$$\begin{aligned}&\int_0^x\int_0^t du\, dv \sum_{n=-\infty}^{\infty} \psi(u, v, \lambda_n^{\pm}, A_n^{\pm}) - \int_0^x\int_0^t du\, dv \sum_{n=-\infty}^{\infty} \psi(u, \lambda_n^{\pm}, A_n^{\pm}) \\ &\quad \times \int_0^v \psi^T(s, \lambda_n^{\pm}, A_n^{\pm}) M^T(v,s)\, ds \\ &= \int_0^x\int_0^t du\, dv \sum_{n=-\infty}^{\infty} c(u, v, \lambda_n^{\pm}, A_n^{\pm}) + \int_0^x du \int_0^t dv \sum_{n=-\infty}^{\infty} \int_0^u K(u,s)c(s, v, \lambda_n^{\pm}, A_n^{\pm})\, ds\end{aligned} \tag{47}$$

where $c(u, v, \ldots)$ and $\psi(u, v, \ldots)$ have the same meaning as in (36) and (37). By a modification of the Parseval theorem (14) as in the case of evaluation of I_2 and the relations (37), (38) and (42), we have from (47),

$$2/3\pi B + 2\int_0^x\int_0^t F(u,v)\, du\, dv + 2\int_0^t dv \int_0^x M^T(v,s)\, ds + \Omega = 0 \tag{48}$$

where

$$\Omega = \int_0^x du \int_0^t dv \sum_{n=-\infty}^{\infty} \int_0^u K(u,s)c(s, v, \lambda_n^{\pm}, A_n^{\pm})\, ds.$$

Substituting for $c(v, \lambda_n^{\pm}, A_n^{\pm})$ and $c(s, \lambda_n^{\pm}, A_n^{\pm})$ by (35), we have

$$\begin{aligned}\Omega &= \int_0^x du \int_0^v dv \sum_{n=-\infty}^{\infty} \int_0^u K(u,s)\psi(s, v, \lambda_n^{\pm}, A_n^{\pm})\, ds - \int_0^x du \int_0^t dv \\ &\quad \times \sum_{n=-\infty}^{\infty} \left(\int_0^u K(u,s)\, ds \int_0^s M(s,p)\psi(p, \lambda_n^{\pm}, A_n^{\pm})\, dp \right) \psi^T(v, \lambda_n^{\pm}, A_n^{\pm}) \\ &\quad - \int_0^x du \int_0^t dv \sum_{n=-\infty}^{\infty} \left(\int_0^u K(u,s)\psi(s, \lambda_n^{\pm}, A_n^{\pm})\, ds \right) \\ &\quad \times \left(\int_0^v \psi^T(p, \lambda_n^{\pm}, A_n^{\pm}) M^T(v,p)\, dp \right) + \int_0^x du \int_0^t dv \sum_{n=-\infty}^{\infty} \left(\int_0^u K(u,s) ds \right. \\ &\quad \times \left. \int_0^s M(s,p)\psi(p, \lambda_n^{\pm}, A_n^{\pm}) dp \right) \left(\int_0^v \psi^T(p, \lambda_n^{\pm}, A_n^{\pm}) M^T(v,p)\, dp \right) \\ &= J_1 + J_2 + J_3 + J_4, \text{ say.}\end{aligned} \tag{49}$$

By suitable modification of the Parseval relation (14), as before, we obtain

$$J_1 = 2\int_0^x du \int_0^t K(u,s)\, ds$$

$$J_2 = -2\int_0^x du \int_0^u K(u,s)\, ds \int_0^t M(s,p)\, dp$$

$$J_3 = -2\int_0^x \mathrm{d}u \int_0^t \mathrm{d}v \int_0^u K(u,s) M^T(v,s)\,\mathrm{d}s$$

$$J_4 = 2\int_0^x \mathrm{d}u \int_0^t \mathrm{d}v \int_0^u K(u,s)\,\mathrm{d}s \int_0^s M(s,p) M^T(v,p)\,\mathrm{d}p.$$

Hence, from (48) and (49), on substitution of the values of J_k, $k = 1,2,3,4$, and application of the operator $\partial^2/\partial x \partial t$ on both sides of the resulting relation, we obtain

$$F(x,t) + K(x,t) - \left(\int_0^x K(x,s)(M(s,t) + M^T(t,s) - \int_0^s M(s,p) M^T(t,p)\mathrm{d}p)\mathrm{d}s \right) = 0, \tag{50}$$

where we have used $M(t,x) = 0$ for $0 \leqslant t < x \leqslant \pi$, by definition. That is

$$F(x,t) + K(x,t) = R, \text{ say.} \tag{51}$$

In view of $M(s,t) = 0$, for $0 \leqslant s < t$, $M^T(t,s) = 0$, for $0 \leqslant t < s$ and the relations (32) and (33),

$$\begin{aligned} R &= \int_0^t K(x,s)\left(M^T(t,s) - \int_0^s M(s,p) M^T(t,p)\mathrm{d}p \right)\mathrm{d}s \\ &\quad + \int_t^x K(x,s)\left(M(s,t) - \int_0^s M(s,p) M^T(s,p)\,\mathrm{d}p \right)\mathrm{d}s \\ &= -\int_0^t K(x,s) F(s,t)\mathrm{d}s - \int_t^x K(x,s) F(s,t)\mathrm{d}s = -\int_0^x K(x,s) F(s,t)\mathrm{d}s. \end{aligned}$$

The required integral equation then follows from (51).

5. The inverse problem

Let (i) a_{ij}, $i = 1,2, j = 1,2,3,4$, be a sequence of real-valued constants satisfying the conditions (4) and (5);

(ii) $\{\alpha_{jn}\}$, $j = 1,2,3$, be a sequence of real numbers such that

(a) $\begin{pmatrix} \alpha_{1n} & \alpha_{3n} \\ \alpha_{3n} & \alpha_{2n} \end{pmatrix}$ is positive definite;

(b) α_{jn} have the asymptotic values as given in (16), (18) and (19);

(iii) $\{\lambda_n\}$ be a steadily increasing sequence of distinct positive real numbers with asymptotic estimates $\lambda_n^{\frac{1}{2}} = n + \alpha_j/n + \dot{O}(1/n)$, where α_j is a constant. $\lambda_n^{\frac{1}{2}}$ is interpreted in the same way as in (10).

Then, $A_n = \alpha_{1n}\alpha_{2n}(\alpha_{1n} + \alpha_{2n} - 2\alpha_{3n}) > 0$ and has the asymptotic estimates as in (21).

In the following we construct a boundary-value problem (1) with (2) and (3) when $\{\lambda_n\}$, $\{\alpha_{jn}\}$, $\{A_n\}$ are given. $\{\lambda_n\}$, $\{\alpha_{jn}\}$, $\{A_n\}$ together constitute the spectral characteristics of the eigenvalue problem to be constructed by us.

We begin by establishing the following lemma.

Lemma 5.1. The set of vectors $(\cos\lambda_n^{\pm}t, \sin\lambda_n^{\pm}t)^T$ is linearly independent and is complete in L on the interval $[0,\pi]$, if λ_n satisfies the asymptotic relation as assumed in (iii) above.

Since $\lambda_m \neq \lambda_n$, when $m \neq n$, the linear independence of the system follows from those of $\{\cos\lambda_n^{\pm}t\}$ and $\{\sin\lambda_n^{\pm}t\}$ in $[0,\pi]$. The completeness of the system in $L(0,\pi)$ is established by showing that

$$\int_0^{\pi} (f(x)\exp(i\lambda_n^{\pm}x) + g(x)\exp(-i\lambda_n^{\pm}x))\,\mathrm{d}x = 0 \tag{52}$$

for all large λ_n implies that $f = g = 0$ almost everywhere on $(0,\pi)$, if $(f,g)^T \varepsilon L(0,\pi)$.

If possible, let $(f,g)^T \neq 0$ in $(0,\pi)$ and (52) hold. Put

$$F(w) = \int_0^{\pi} f\exp(iw^2x)\,\mathrm{d}x + \int_0^{\pi} g\exp(-iw^2x)\,\mathrm{d}x.$$

Since each of $\{\cos nx\}$, $\{\sin nx\}$, $n = 0, \pm 1, \pm 2, \pm 3,\ldots$, is complete in $(0,\pi)$, $(\exp(inx), \exp(-inx))^T$ is also complete in $(0,\pi)$. Thus, $F(w)$ cannot vanish at all points $w^2 = n$, $n = 0, \pm 1, \pm 2,\ldots$. $F(w)$ is therefore an entire function which does not vanish identically. The validity of the lemma is now established in the same way as Levinson[11] (pp 3–5).

Let us construct the matrix $F(x,t)$ in the form (30) by making use of the conditions (i)–(iii) and the conditions (22)–(26). Then, $F(x,t)$ is uniformly convergent in the domain specified in (30). The following theorem is now established.

Theorem 5.1. The homogeneous integral equation

$$g^T(t) + \int_0^x g^T(s)F(s,t)\,\mathrm{d}s = 0 \tag{53}$$

where $g(t) = (g_1, g_2)^T$ is continuous in t and $g(t) = 0, t > x$, has only the null solution for $g(t)$ for every $x \in (0,\pi)$.

Taking the scalar product of (53) with $g(t)$ integrate with respect to t between the limits $(0,x)$ and replace $F(s,t)$ by its series expansion (30). Then, on reduction

$$\int_0^x g^T(t)g(t)\,\mathrm{d}t + \frac{1}{2}\sum_{n=-\infty}^{\infty}\left|\int_0^x g^T(t)c(t,\lambda,A_n^{\pm})\,\mathrm{d}t\right|^2 - 1/2\pi\int_0^x\int_0^x g^T(s)Ag(t)\,\mathrm{d}s\,\mathrm{d}t$$
$$- 1/2\pi\sum_{n=-\infty}^{\infty}\left|\int_0^x g^T(t)c^*(t,n)\,\mathrm{d}t\right|^2 = 0 \tag{54}$$

where

$$f^2 = f_1^2 + f_2^2, \quad \text{if } f = (f_1, f_2)^T.$$

In view of the conditions (4)–(7), it is easy to deduce that

$c_j(x,n) = \begin{pmatrix} c_{1j}(x,n) \\ c_{2j}(x,n) \end{pmatrix} = \begin{pmatrix} a_{j2}\cos nx + a_{j1}\sin nx \\ a_{j4}\cos nx + a_{j3}\sin nx \end{pmatrix}$, $j = 1,2$, are the eigenvectors of the Fourier system (9).

Then, $D_1c_2(x,n) - D_2c_1(x,n)$, where D_j are those defined in (22) is also an eigenvector with

$$\|D_1c_2(x,n) - D_2c_1(x,n)\|^2 = \tfrac{1}{2}\pi((D_1^2|a_2|^2 + D_2^2|a_1|^2 - 2D_1D_2(a_1,a_2))$$

where a_j are the vectors which occur in (16) and (17) and (a_1,a_2) is the inner product of a_1 and a_2. Substituting for $|a_j|^2$, (a_1,a_2) in terms of D_j, D_0 as obtained in (20) and (22) and simplifying, it follows that

$$\|D_1c_2(x,n) - D_2c_1(x,n)\|^2 = \pi.$$

Thus, $(D_1c_2(x,n) - D_2c_1(x,n))/\pi^{\frac{1}{2}}$ is a normalized eigenvector for the Fourier system (9) with boundary conditions (2) and (3).

Since $g(t) = 0$ for $t > x$, we have by the Parseval formula

$$\begin{aligned}\int_0^\pi g^T(t)g(t)\mathrm{d}t &= \int_0^x g^T(t)g(t)\mathrm{d}t = 1/\pi \sum_{n=-\infty}^{\infty}\left(\int_0^x g^T(t)(D_1c_2(t,n) - D_2c_1(t,n))\mathrm{d}t\right)^2 \\ &= 1/\pi \sum_{n=-\infty}^{\infty}{}' \left(\int_0^x g^T(t)(D_1c_2(t,n) - D_2c_1(t,n))\,\mathrm{d}t\right)^2 \\ &\quad + 1/\pi\left(\int_0^x g^T(t)(D_1c_2(t,0) - D_2c_1(t,0))\,\mathrm{d}t\right)^2 \\ &= 1/2\pi \sum_{n=-\infty}^{\infty}{}' \left|\int_0^x g^T(t)c^*(t,n)\,\mathrm{d}t\right|^2 + 1/2\pi\int_0^x\int_0^x g^T(s)Ag(t)\,\mathrm{d}s\,\mathrm{d}t \end{aligned} \tag{55}$$

where A is the matrix defined in (25) and $g = (g_1,g_2)^T \varepsilon L_2(0,\pi)$.

If further $f = (f_1,f_2)^T \varepsilon L_2(0,\pi)$, we have, more generally,

$$\begin{aligned}\int_0^\pi f^T(t)g(t)\,\mathrm{d}t &= 1/2\pi \sum_{n=-\infty}^{\infty}{}' \int_0^\pi\int_0^\pi f^T(s)c^*(s,t,n)g(t)\,\mathrm{d}t \\ &\quad + 1/2\pi\int_0^\pi\int_0^\pi f^T(s)Ag(t)\,\mathrm{d}s\,\mathrm{d}t\end{aligned} \tag{56}$$

where $c^*(s,t,n) = c^*(s,n)c^*(t,n)$ as before.

From (54) and (55), we, therefore, have

$$\left|\int_0^x g^T(t)c(t,\lambda_n^{\pm},A_n^{\pm})\,\mathrm{d}t\right|^2 = 0, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots.$$

From this, since $A_n > 0$, we have

$$\int_0^\pi g^T(t)c_j(t,\lambda_n^{\pm})\,\mathrm{d}t = 0, \quad j = 1,2;\ n = 0, \pm 1, \pm 2, \ldots.$$

Therefore, by lemma 5.1,

$$a_{j1}g_1(t) + a_{j3}g_2(t) = 0,$$

$$a_{k2}g_1(t) + a_{k4}g_2(t) = 0,$$

where $j=1,2$ and $k=1,2$; from these it follows that $g_1=g_2=0$ for $x\in(0,\pi)$, by (4). The theorem therefore follows.

The following lemma is only an extension to matrices of the Gasymov-Levitan lemma[3], p. 14, lemma 1.3.1.

Lemma 5.2. Let the matrices $g(t,a)$, $h(t,a)$ and $H(t,s,a)$ be connected by

$$g(t,a)=h(t,a)+\int_0^t H(t,s,a)h(s,a)\,\mathrm{d}s$$

where g and H are continuous in s and t. If for $a=a_0$, $g(t,a_0)=0$, and the resulting integral equation has only the null solution $h(t,a_0)$, then in some neighbourhood of a_0, $h(t,a)$ is continuous in t,a. Also $h(t,a)$ has continuous derivatives of order $m\geqslant 1$ with respect to a, if H and g have so with respect to a.

The following theorem is now proved.

Theorem 5.2. Let $F(x,t)$, $0\leqslant t<x\leqslant\pi$ be defined by (30) and satisfy the following conditions.

(a) $F(x,t)$ admits continuous second-order partial derivatives with respect to x, t in $[0,\pi]$;
(b) $F(x,0)\neq 0$, $F_t(x,t)|_{t=0}=0$ for $x\in[0,\pi]$.

Then,

(i) the integral equation (45) has a unique solution for the symmetric kernel $K(x,t)$, $0\leqslant t<x\leqslant\pi$ and $K(x,t)=0$ for $t>x$; $K(x,t)$ is a continuous function of x,t having continuous second-order partial derivatives with respect to x,t in x, $t\in[0,\pi]$, $x\neq t$;
(ii) $\phi_j(x,\lambda_n^{\pm})$, $j=1,2$, defined in terms of the matrix $K(x,t)$ and a_{ij} by the relation (12), satisfy the system (1) along with the boundary conditions (2) at $x=0$, where

$$2K'(x,x)=\begin{pmatrix} p & r\\ r & q\end{pmatrix}.$$

The first part of the theorem is an immediate consequence of theorem 5.1 and the iteration process. The continuity and differentiability of $K(x,t)$ follow from lemma 5.1 in the same way as in Gasymov and Levitan[3] (p. 15). The proof of the second part follows verbatim from the arguments of Ray Paladhi[7] (pp 183–185) with $F(x,t)$ in place of his $E(x,t)$ and hence is omitted. (See also Gasymov and Levitan[3], pp 15–18 and p. 26 for the classical Sturm-Liouville problem.)

6. Investigation of the eigenvectors for the inverse problem

Put

$$\psi_n(x,\lambda_n^{\pm})=(\psi_{1n}(x,\lambda_n^{\pm}),\psi_{2n}(x,\lambda_n^{\pm}))^T=\alpha_{1n}\phi_2(x,\lambda_n^{\pm})-\alpha_{2n}\,\phi_1(x,\lambda_n^{\pm}) \tag{57}$$

where ϕ_j are defined as in theorem 5.2 (ii) in terms of $K(x,t)$, the unique solution of (45) and the a_{ij}, the given sequence of real-valued constants. Obviously, $\psi_n(x)$ satisfy the differential system (1) with $\lambda = \lambda_n$ and the boundary condition (2) at $x = 0$. We prove the following theorem.

Theorem 6.1. $\{\psi_n(x, \lambda_n^{\frac{1}{2}})/A_n^{\frac{1}{2}}\}$ is a complete sequence of orthonormal eigenvectors associated with the system (1), (2) and a suitable boundary condition fixed at $x = \pi$, $\{\lambda_n\}$ being the eigenvalues for the system.

Put

$$G_1(\lambda_n) = 1/A_n^{\frac{1}{2}} \int_0^\pi f^T(x)\psi_n(x, \lambda_n^{\frac{1}{2}})\,\mathrm{d}x, \quad f = (f_1, f_2)^T \varepsilon L_2(0, \pi)$$

$$G_2(\lambda_n) = 1/A_n^{\frac{1}{2}} \int_0^\pi g^T(x)\psi_n(x, \lambda_n^{\frac{1}{2}})\,\mathrm{d}x, \quad g = (g_1, g_2)^T \varepsilon L_2(0, \pi)$$

and

$$G(\lambda_n) = G_1(\lambda_n)G_2(\lambda_n). \tag{58}$$

Then, from the definition of ϕ_j in terms of $K(x, t)$ by (12) we have

$$\psi_n(x, \lambda_n^{\frac{1}{2}}) = c_0(x, \lambda_n^{\frac{1}{2}}) + \int_0^x K(x, t)c_0(t, \lambda_n^{\frac{1}{2}})\,\mathrm{d}t \tag{59}$$

where

$$c_0(x, \lambda_n^{\frac{1}{2}}) = \alpha_{1n}c_2(x, \lambda_n^{\frac{1}{2}}) - \alpha_{2n}c_1(x, \lambda_n^{\frac{1}{2}}). \tag{60}$$

Let

$$h_1(t) = f^T(t) + \int_t^\pi f^T(s)K(s, t)\,\mathrm{d}s \tag{61}$$

$$h_2(t) = g^T(t) + \int_t^\pi g^T(s)K(s, t)\,\mathrm{d}s. \tag{62}$$

Then, since

$$\int_0^\pi f^T\phi_j\,\mathrm{d}x = \int_0^\pi f^T\left(c_j + \int_0^x K(x, t)c_j\,\mathrm{d}x\right)\mathrm{d}t = \int_0^\pi h_1^T c_j\,\mathrm{d}t,$$

by a change in the order of integration, with a similar relation involving g^T and h_2, it follows that

$$G_j(\lambda_n) = 1/A_n^{\frac{1}{2}} \int_0^\pi h_j(t)c_0(t, \lambda_n^{\frac{1}{2}})\,\mathrm{d}t, \quad j = 1, 2.$$

Therefore, by (30) and (56),

$$\begin{aligned}\sum_{n=-\infty}^{\infty} G(\lambda_n) &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \int_0^\pi \int_0^\pi h_1(s)c(s, \lambda_n^{\frac{1}{2}}, A_n^{\frac{1}{2}})c^T(t, \lambda_n^{\frac{1}{2}}, A_n^{\frac{1}{2}})h_2^T(t)\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_0^\pi \int_0^x h_1(x)F(x, t)\,\mathrm{d}x\,h_2^T(t)\,\mathrm{d}t + \int_0^\pi h_1(t)h_2^T(t)\,\mathrm{d}t.\end{aligned} \tag{63}$$

Here, $c(x, \lambda_n^{\frac{1}{2}}, A_n^{\frac{1}{2}})$ is the matrix $1/A_n^{\frac{1}{2}}c(x, \lambda_n^{\frac{1}{2}})$.

Substituting for $h_1(x)$ by (61), and changing the order of integration, we have

$$\begin{aligned}\int_0^\pi h_1(x)F(x,t)\,dx &= \int_0^\pi f^T(x)\left(F(x,t)+\int_0^x K(x,u)F(u,t)\,du\right)dx\\ &= \left(\int_0^t+\int_t^\pi\right)f^T(x)\left(F(x,t)+\int_0^x K(x,u)F(u,t)\,du\right)dx\\ &= -\left(\int_0^t f^T(x)M^T(t,x)\,dx+\int_t^\pi f^T(x)K(x,t)\,dx\right) \qquad (64)\end{aligned}$$

by (44) and (45).

From (62)

$$h_2^T(x)=g(x)+\int_x^\pi K^T(t,x)g(t)\,dt \qquad (65)$$

which is the Volterra integral system, giving

$$g(x)=h_2^T(x)-\int_x^\pi M^T(t,x)h_2^T(t)\,dt \qquad (66)$$

where the kernel $M^T(t,x)$ is the reciprocal of the kernel $K^T(t,x)$. (see Whittaker and Watson[12], p. 218). Hence, from (61), (63)–(66)

$$\begin{aligned}&\sum_{n=-\infty}^{\infty} 1/A_n \int_0^\pi f^T(x)\psi_n(x,\lambda_n^{\frac{1}{2}})\,dx \int_0^\pi g^T(t)\psi_n(t,\lambda_n^{\frac{1}{2}})\,dt\\ &= \int_0^\pi h_1(t)h_2^T(t)\,dt+\int_0^\pi f^T(x)(g(x)-h_2^T(x))\,dx-\int_0^\pi (h_1(t)-f^T(t))h_2^T(t)\,dt\\ &= \int_0^\pi f^T(x)g(x)\,dx. \qquad (67)\end{aligned}$$

(compare Gasymov and Levitan[3], pp 20–21).

Let

$$C_n=1/A_n\int_0^\pi f^T(x)\psi_n(x,\lambda_n^{\frac{1}{2}})\,dx, \quad f=(f_1,f_2)^T\varepsilon L_2(0,\pi).$$

Then, if

$$F(x)=\sum_{n=-\infty}^{\infty} C_n\psi_n(x,\lambda_n^{\frac{1}{2}})$$

is uniformly convergent is $[0,\pi]$, we have for any vector $g(x)\varepsilon L_2(0,\pi)$,

$$\int_0^\pi g^T(x)F(x)\,dx=\sum_{n=-\infty}^{\infty} C_n\int_0^\pi g^T(x)\psi_n(x,\lambda_n^{\frac{1}{2}})\,dx=\int_0^\pi g^T(x)f(x)\,dx, \text{ by (67).}$$

Thus, $F(x)=f(x)$ a.e. on $[0,\pi]$. Hence, for any vector $f\in L_2(0,\pi)$, we obtain formally the expansion formula

$$f(x)=\sum_{n=-\infty}^{\infty} C_n\psi_n(x,\lambda_n^{\frac{1}{2}}). \qquad (68)$$

The uniform convergence in $[0, \pi]$ of the series on the right of (68) is proved as follows. Let

$$\tilde{f}(x) = Lf = \begin{pmatrix} -f_1'' + pf_1 + rf_2 \\ -f_2'' + rf_1 + qf_2 \end{pmatrix} \varepsilon\, L_2(0, \pi), \quad \text{where} \quad f = (f_1, f_2)^T \varepsilon\, L_2(0, \pi). \text{ Also,}$$

let $f(x)$ satisfy the boundary condition (2) at $x = 0$. Then, since f and ψ_n satisfy the boundary conditions (same) at $x = 0$, $[f, \psi_n]_0 = 0$, $[.]$ being the bilinear concomitant of the vectors f and ψ_n. Since ψ_n satisfies the differential system (1) with $\lambda = \lambda_n$, we have by Green's theorem

$$\lambda_n C_n - \tilde{C}_n = 1/A_n [f, \psi_n]_\pi$$

where

$$\tilde{C}_n = 1/A_n \int_0^\pi \tilde{f}^T(x) \psi_n(x, \lambda_n^{\frac{1}{2}}) \mathrm{d}x. \tag{69}$$

Let a_{ij} satisfy the additional condition

$$|a_1|^2 [f, \phi_2]_\pi = |a_2|^2 [f, \phi_1]_\pi. \tag{70}$$

Then, it follows from the asymptotic relations (16)–(18) and the relation (69) that

$$\lambda_n C_n - \tilde{C}_n = O(1/n). \tag{71}$$

If $G(x, y, \lambda)$ be the Green's matrix for the differential system (1) and the boundary conditions (2) and (3), we have

$$\psi_n(x, \lambda_n^{\frac{1}{2}})/(\lambda_n - \lambda) = -\int_0^\pi G(x, y, \lambda) \psi_n(y, \lambda_n^{\frac{1}{2}}) \mathrm{d}y.$$

Hence, by the familiar Titchmarsh argument[13] (p. 27),

$$\sum_{n=-\infty}^{\infty} |\psi_{jn}(x)|^2/(A_n \lambda_n^2) \text{ converges boundedly with respect to } x \text{ in } [0, \pi].$$

Now

$$\sum_N^{N'} |C_n \psi_{jn}| = \sum_N^{N'} |\lambda_n C_n A_n^{\frac{1}{2}}| |\psi_{jn}| / |\lambda_n A_n^{\frac{1}{2}}| = \sum_N^{N'} (A_n^{\frac{1}{2}} \tilde{C}_n + O(1/n)) |\psi_{jn}| / |(\lambda_n A_n^{\frac{1}{2}})|$$

$$\leqslant \left(2 \left(\sum_N^{N'} A_n \tilde{C}_n^2 + O(1/n^2) \right) \sum_N^{N'} |\psi_{jn}|^2 / \lambda_n^2 A_n \right)^{\frac{1}{2}},$$

by the Cauchy inequality and the inequality $(a + b)^2 \leqslant 2(a^2 + b^2)$.

Since, by (67), $\Sigma_{n=-\infty}^{\infty} A_n \tilde{C}_n^2 < \infty$, the uniform convergence of $\Sigma_{n=-\infty}^{\infty} C_n \psi_n(x, \lambda_n^{\frac{1}{2}})$ in $[0, \pi]$ follows as in Titchmarsh[13] (p. 27). The series is also absolutely convergent. In particular, let $f(x) = \psi_k(x, \lambda_n^{\frac{1}{2}})$, so that

$$\psi_k(x, \lambda_n^{\frac{1}{2}}) = \sum_{n=-\infty}^{\infty} 1/A_n \left(\int_0^\pi \psi_k^T(x, \lambda_n^{\frac{1}{2}}) \psi_n(x, \lambda_n^{\frac{1}{2}}) \mathrm{d}x \right) \psi_n(x, \lambda_n^{\frac{1}{2}}). \tag{72}$$

Since $\left\{ \begin{matrix} \cos \\ \sin \end{matrix} \lambda_n^{\frac{1}{2}} x \right\}$ are two linearly independent sequences, and rank $(a_{ij}) = 2$, it follows from (72) that

$$\int_0^\pi \psi_n^T(x,\lambda_n^\pm)\psi_k(x,\lambda_n^\pm)\,\mathrm{d}x = A_n\delta_{k,n} \tag{73}$$

where $\delta_{k,n}$ is the kronecker delta.

Thus, $\{\psi_n(x,\lambda_n^\pm)\}$ is an orthogonal system satisfying (1) and (2) in $L_2(0,\pi)$. By Green's theorem applied to any two elements $\psi_j(x,\lambda_n^\pm)$, $j=m,n$, $m\neq n$, we have

$$[\psi_m(x,\lambda_m^\pm),\psi_n(x,\lambda_n^\pm)]_\pi = 0. \tag{74}$$

Define the boundary condition at $x=\pi$ by (74). Then $\{\psi_n(x,\lambda_n^\pm)/A_n^\pm\}$ represents a sequence of normalized eigenvectors for the system (1), (2) and (74). $\{\lambda_n\}$ is obviously a sequence of eigenvalues. The relation (67) is the Parseval relation for the system constructed. The system is therefore complete and the theorem is proved.

We next establish the following lemma which is the converse of a theorem of Chakravarty[8] (p. 138).

Lemma 6.1. If two linearly independent vectors U, V satisfy the condition $[U,V]_\pi=0$ with $[\phi_1,\phi_2]=0$, where ϕ_1, ϕ_2 are the boundary condition vectors at $x=\pi$, then $[U,\phi_j]_\pi = [V,\phi_j]_\pi = 0$, $j=1,2$ and hence U, V satisfy the boundary conditions at π given by (3).

Let θ_1, θ_2 be two linearly independent vectors connected with ϕ_1, ϕ_2 by the relations $[\theta_1,\theta_2]=0$ and $[\phi_r,\theta_s]=\delta_{r,s}$, $r,s=1,2$, where $\delta_{r,s}$ is the kronecker delta. Evidently, the choice of θ_1,θ_2 in the above way is not unique and in fact three more independent relations are necessary for complete determination of θ_1, θ_1', θ_2, θ_2'.

In the P-identity of Chakravarty[8] (identity 2.2, p. 135), let us identify $\phi_3=\theta_1$, $\phi_4=\theta_2$, $\phi_5=U$ and $\phi_6=V$, where the six vectors ϕ_j, $j=1,2,\ldots,6$, are linearly independent and for convenience we write $[\phi_i,\phi_j]_\pi = -[\phi_j,\phi_i]_\pi = P_{ij}$. It then follows from the P-identity, (since $P_{12}=0$, $P_{34}=0$, $P_{14}=0$, $P_{23}=0$, $P_{24}=1$, $P_{56}=0$, $P_{13}=1$), that

$$P_{25}P_{46} - P_{26}P_{45} + P_{15}P_{36} - P_{16}P_{35} = 0. \tag{75}$$

There can be three more relations of type (75) corresponding to three more independent choices of θ. Since θ_1, θ_2, U, V are linearly independent, the four vectors of type $(P_{35},P_{36},P_{45},P_{46})$ arising by considering four linearly independent choices of (θ_1,θ_2) are linearly independent. Giving these four vectors the values ε_j, $j=1,2,3,4$, where $\varepsilon_j\in R^4$ is the jth unit vector, we derive from (75), $P_{15}=P_{25}=P_{16}=P_{26}=0$. Hence the lemma follows.

We now have the following theorem.

Theorem 6.2. The sequence of constants $\{b_{ij}\}$, rank $(b_{ij})=2$, satisfying (6) having been given, the relation

$$[\psi_m(x,\lambda_m^\pm),\psi_n(x,\lambda_n^\pm)]_\pi = 0, \quad m\neq n,$$

implies that $\psi_n(x,\lambda_n^\pm)$ satisfies the boundary conditions (3) at $x=\pi$, by suitably choosing the boundary condition vectors at the point.

This is an immediate consequence of lemma 6.1.

Theorems 5.2, 6.1 and 6.2 taken together allow us to construct the eigenvalue problem (1) with (2) and (3) from the given spectral characteristics, *i.e.*, the sequence of eigenvalues $\{\lambda_n\}$, the sequence $\{A_n\}$ of normalizing constants, and the sequence $\{\alpha_{jn}\}$, $j = 1, 2, 3$, when certain asymptotic relations are known. The problem so derived is necessarily unique since $K(x, t)$, which plays a vital role in the construction, is unique by theorem 5.1.

We note that the condition (70) on a_{ij} can be waived, if f satisfies the relation (3) at $x = \pi$ for given b_{ij} along with (4), (6)–(8). In this case, (71) takes the simpler form $\tilde{C}_n = \lambda_n C_n$ and the rest of the analysis follows as before.

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