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BOOK REVIEWS

Lie groups beyond an introduction by Anthony W. Knapp; Birkhauser Verlag, AG, CH-4010, Basel, Switzerland, 1996, pp. 686, sFr. 78.

The main goal of this book is to unravel the structure of real semisimple (and reductive) Lie groups and their finite dimensional representations. In this direction, the books by Varadarajan, Chevalley and Helgason have achieved classic status. However, since each of these books was written long ago and with a particular slant, analytic, geometric of algebraic, there was clearly a need for a modern text which would, so to speak, unify and tidy up the presentation of the basics of this theory. Knapp himself wrote a book entitled 'Representation Theory of Semisimple Lie Groups: An Overview based on Examples' a few years ago, which was the first step in this direction, especially aimed at linear Lie groups. The volume under review should be seen as a sequel to that book.

The first chapter is a quick review of very basic material with which the reader should be already familiar before embarking on this book. It deals with essentials on Lie algebras, such as solvable and nilpotent lie algebras, the theorems of Lie, Engels, Cartan's criterion for semisimplicity. Representations of $\underline{sl}$ (2, **C**), which is again a key ingredient of all Lie theory, are analysed. The classical simple lie algebras are listed, and the chapter ends with a brief on Lie groups, analytic subgroups, semidirect products of Lie groups and a list of the Lie groups whose Lie algebras had been listed earlier.

The second chapter gets into the Cartan classification of complex semisimple Lie algebras. This is one of the cornerstones of Lie theory, and quite expectedly, very standard material which can be found in various places. The adjoint action of a Cartan subalgebra $\underline{h}$ (all Cartan subalgebras are conjugate in the complex semisimple case) splits the complex semisimple Lie algebra g into simultaneous $\underline{h}$-eigenspaces called root spaces, and this root-system completely determines g. The data of this root-system can be encoded in purely combinatorial terms, i.e., the Cartan matrix, and this in turn determines a graph known as the Dynkin diagram. Some elementary (though n on-trivial) combinatorial analysis allows the classification of these Dynkin diagrams, and hence all complex simple (and hence semisimple) Lie algebras. The Lie-algebraic and combinatorial aspects are carefully delineated in the book, and the pedagogical effort expended in discussing separately abstract root systems, Cartan matrices, Weyl groups, etc. is a good strategy, because it crops up with small modifications in real semisimple theory, and is useful to know even in the study of infinite dimensional Lie algebras. The last two sections prove the backand forth passage between isomorphism classes of complex simple Lie algebras and reduced root-systems/Cartan matrices.

To understand representations, and also to view the Lie algebra as an algebra of left-invariant first-order differential operators on the Lie group, it is necessary and fruitful to introduce the full algebra of all left invariant different operators, which is the universal enveloping algebra $U(g)$. By abstract nonsense, Lie algebra representations of g are in bijective correspondence with algebra representations of $U(g)$, and so it is crucial to understand this object. The fundamental key to this is the Poincare-Birkhoff-Witt theorem, proved in the second section. Using this, its associated graded algebra is seen to be the symmetric algebra of g, a fact that plays an important role later. The free lie algebra on a set is constructed in the last section.

With these tools in place, Chapter IV launches into the study of compact Lie groups, a beautiful and complete theory due to Cartan and Weyl in the earlier part of this century. The main ingredient here is integration with the invariant Haar measure, leading to a bi-invariant Reimannian structure on the group G. With just this weapon, one proves the fundamental results, e.g. the Schur orthogonality of characters of inequivalent finite-dimensional irreducible (complex) representations, the determination of equivalence classes of such representations by their characters, the Peter-Weyl theorem asserting density of matrix elements of irreducible complex finite dimensional representations in $L^2(G)$, etc. The fact that all compact lie groups are linear, the conjugacy of all maximal tori, and other nice global features that compact Lie groups are consequence of these few fundamental theorems, and are worked out in detail. It is also possible to give an analytical definition of the Weyl group of G in terms of a maximal tours, and it agrees with the Weyl group of the root system of the (complexified) Lie algebra g of G. (In fact, the invariant inner product shows that g is the direct sum of its center $Z(g)$ and its commutator $[g, g]$, and the latter algebra is (real) semisimple, and complexification yields a complex semisimple Lie algebra). Since the analytic subgroups corresponding to both these Lie algebra are closed, and the analytic subgroup of the former is just the identity component of the center of G, and hence a direct product factor isomorphic to a torus, one might as well assume that this center of G is discrete, i.e., that the first summand $Z(g)$ of g is trivial, i.e., that g is semisimple. (In fact it is more, its Killing form is negative definite, and such Lie algebras are called compact Lie algebras). Since the study of compact semisimple simply-connected Lie groups can be completely gleaned from their Lie algebras, the theory of (complex) semisimple lie algebras provides a 'complete' understanding of such groups.

Topological consequences follow from this study. For example, Weyl's theorem that the fundamental group of a compact semisimple Lie group is finite, and hence its universal cover is also compact semisimple, is proved in the last section. Of course, the material in this chapter is also quite standard and well represented in the literature, but Knapp's treatment is quite tidy and accessible, and good to have for the sake of completeness and in the context of his overall goals.

Chapter V launches into the study of finite dimensional complex representations of Lie groups, and clearly and first step is to nail it down for complex semisimple Lie algebras. There is again a complete answer, and the fundamental theorem is the famous Theorem of the Highest Weight, which sets up a bijective correspondence between equivalence classes of finite dimensional irreducible complex representations of complex semisimple g and dominant algebraic integral linear functionals on a (fixed at the outset) Cartan subalgebra $\underline{h}$. Verma modules, the objects which realize a given dominant integral character as the highest weight of a representation, are introduced, making use of universal enveloping algebras. In fact these modules are universal highest weight modules for the universal enveloping algebra, and thus a fundamental tool in representation theory. The approach to studying representations of g by investigating those of the universal enveloping algebra $U(g)$ is a fruitful (algebraic) way of going about it, and a Lemma due to Dixmier says that the only $U(g)$-linear maps of a unital irreducible left $U(g)$-module, where g is a complex Lie algebra, are scalars. Thus the center Z of $U(g)$ will act by scalar characters on such a module, and this is called the infinitesimal character of the representation, and is a fundamental invariant of the (equivalence class of) the irreducible representation. So to understand these, one must know what the center Z of $U(g)$ looks like, and the theorem of Harish-Chandra, proved in Section 5, answer it fully for a complex semisimple Lie algebra g. It is precisely the subalgebra of Weyl group invariants in $U(\underline{h})$, where $\underline{h}$ is a Cartan subalgebra of g, and thus a polynominal algebra due to a theorem of Chevalley. From this it is an easy matter to parametrise all infinitesimal characters. Facts such as the Weyl Character Formula and Weyl Dimension Formula are proved next, and allow the determination of the character and dimension of a finite dimensional irreducible complex representation from its highest weight. Incidentally, the complete reducibility of finite dimensional irreducible complex representation from its highest weight. Incidentally, the complete reducibility of finite dimensional complex representation of a complex semisimple Lie algebra is proved purely algebraically, and the neat "Weyl Unitarian trick" for proving it by finding a compact real form is deferred to a later chapter.

Chapters VI and VII are the real heart of this book. Chapter VI begins with the proof of the existence of compact real forms of complex semisimple Lie algebras. This is the assertion that every complex semisimple Lie algebra is the complexification of a compact real Lie algebra (the Weyl trick referred to earlier). Thus every real semisimple Lie algebra has a compact real form for its complexification. Further, any two such compact real forms are conjugate by an inner automorphism of this complexification. This leads to a Cartan Involution θ of the original real semisimple Lie algebra, and the +1 and −1 eigenspaces of this involution lead to its Cartan decomposition into the "compact part" $\underline{k}$ and "polar part" $\underline{p}$. If G is a semisimple Lie group with finite center and Lie algebra g, the analytic subgroup K corresponding to $\underline{k} \subset g$ is a closed compact subgroup of G, containing the center Z of G. Indeed, it turns out to be a maximal compact subgroup. The Lie algebra Cartan Decomposition lifts to a global Cartan Decomposition (called the KP-decomposition) of the group G.

The other global decomposition of a semisimple Lie group is the Iwasawa (or KAN) Decomposition. Here K is as before, and A is the analytic subgroup corresponding to a maximal abelian subspace $\underline{a}$ of $\underline{p}$. To describe N, one needs to do a root space decomposition of the real semisimple Lie algebra g, called the restricted root space decomposition. It turns out that $\underline{a}$ is an abelian subalgebra whose adjoint action on g is via self-adjoint endomorphisms, and hence g decomposes as a direct sum of simultaneous eigenspaces g_λ corresponding to real roots (real linear functionals) λ on $\underline{a}$, which is the restricted root space decomposition. Now, a lexicographic ordering of the roots defines positivity of roots, and one defines the sum of positive root spaces to be $\underline{n}$, which is a nilpotent subalgebra of g. N is the analytic subgroup corresponding to $\underline{n}$, and in the familiar case of $SL(\mathrm{n}, \mathbf{R})$ the KAN decomposition corresponds to the expression of a matrix as a product of an orthogonal, a real diagonal, and upper-triangular (with diagonal entries 1) matrix via the Gram-Schmidt process. A and N are simply connected, for a semisimple Lie group G. In Section 5, uniqueness of the Iwasawa Decomposition upto various conjugacies is established.

Cartan subalgebras of real semisimple Lie algebras are a trickier affair than the complex semisimple case, and one declares $\underline{h} \subset g$ to be a Cartan subalgebra if its complexification is a Cartan subalgebra of the complexification of g. However, unlike the complex case, not all Cartan subalgebras are conjugate. Not all is lost, because it turns out that every Cartan subalgebra is conjugate by an inner auto of g to a θ-stable Cartan subalgebra (where θ is the Cartan involution of g mentioned earlier). The Cartan decomposition of the θ-stable Cartan subalgebras brings about the notion of the maximally compact and maximally non-compact θ-stable Cartan subalgebras, and two maximally compact (resp. noncompact) θ-stable Cartan subalgebras are conjugate via K. It is ultimately proved that upto inner conjugation, there are only finitely many Cartan subalgebras in g. The last section introduces the Vogan Diagram, which is a Dynkin diagram of the complexification with some additional information introduced to keep track of the real Lie algebra. It retains the datum of a maximally compact θ-stable Cartan subalgebra and the Cartan Involution θ. Defining the Dynkin diagram (with respect to the positive root system Δ^+) determined by the complexification of this Cartan subalgebra, one labels the 2-element orbits under θ, and paints the 1-element orbits black if the corresponding imaginary simple root is a noncompact root (i.e., the root subspace of this root is contained in $\underline{p}$). The Vogan diagram determines the real semisimple algebra upto isomorphism, and the Section 11 carries out the classification of all simple (hence semisimple) real Lie algebras.

The last Chapter (number VII) concerns itself with further structure theory. It turns out to be convenient to find one unified notion which is somewhat easier to deal with than semisimplicity for a group, but carries many of its features and also covers most 'real life' Lie groups that one is likely to encounter. The notion is that of a reductive group.

In the reductive group situation, the Cartan Involution θ is part of the datum, and the group K corresponding to the Lie algebra $\underline{k}$ (the +1-eigenspace of θ) is a maximal compact subgroup. One further assumes that the analytic subgroup G_{ss} (called the semisimple part) corresponding to the semisimple part [$\underline{g}$, $\underline{g}$] of the Lie algebra $\underline{g}$ of G, is closed and has finite center.

Examples of reductive groups include semisimple groups with finite center, compact groups whose complexified lie algebra's automorphism via $Ad(g)$ are inner, closed connected Lie subgroups of the general linear (real or complex) defined by real-valued polynomials in the real and imaginary parts of matrix entries, closed under conjugate transpose inverse, and satisfying the automorphism condition above, among others.

In this reductive situation, two maximal abelian subspaces of $\underline{p}$ are conjugate via $Ad(k)$, $k \in K$, and as in the semisimple case, one can do a restricted root-decomposition with respect to such an $\underline{a}$. The Iwasawa decomposition carries through as in the semisimple case. The analytic Weyl group $W(G, A)$ defined as the quotient of the K-normaliser of $\underline{a}$ by its K-centraliser turns out to be the Weyl group defined by the restricted root system.

The subsequent sections discuss the KAK and Bruhat decompositions, in particular the structure of the subgroup M, the K-centraliser of $\underline{a}$. Then the Bruhat decomposition sets up a one to one correspondence between the double cosets of G under the subgroup MAN (the 'minimal parabolic' subgroup) and the Weyl group. Subsequently, the structure of M, and that of parabolic subgroups are investigated. The interesting geometric consequences of this theory, e.g., the Harish Chandra realization of G/K as hermitian bounded symmetric domains when G is reductive semisimple, are proved in Section 9.

The last chapter is a brief on Haar measure and integration, and decomposition formulas for the Haar measure corresponding to various decompositions of the group discussed earlier in the book, such as the Weyl integration formula. The Appendices include, for the sake of completeness, results on exterior algebra, the Levi decomposition, and detailed information on the classical irreducible root systems corresponding to the simple Lie algebras.

In sum, the book of Knapp is a compendious, carefully thought out and prodigiously detailed account of the subject. It is to be hoped that he will follow up with a final volume dealing with representation theory of semisimple groups, especially the pioneering work of Harish Chandra.

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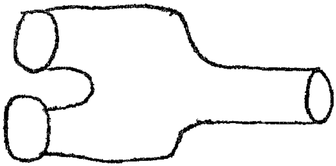
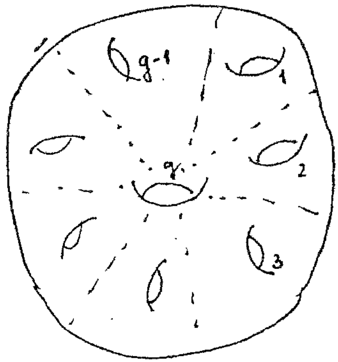
VISHWAMBHAR PATI

A first course in Geometrical Topology and Differential Geometry by Ethan D. Block., Birkhauser, Verlag AG, CH-4010, Basel, Switzerland, 1996, pp. 434, sFr. 98.

The book under review is an introductory text on geometric topology and differential topology via the study of surfaces. This book will serve as an introductory course on the topology and geometry of surfaces and culminating in the celebrated Gauss-Bonnet Theorem - a model theorem depicting the beautiful interplay between topology and geometry. The author leads the reader gently into the subject appealing more to his geometric intuition rather than taxing him with unnecessary flights of abstract rigour. At the end of this course, the student is well-prepared to embark on the study of the topology and geometry of higher dimensional manifolds.

After a review of point-set topology in the first chapter, the author takes up the study of surfaces in right earnest in the second chapter. Surfaces (like manifolds) come in three different avatars - topological,



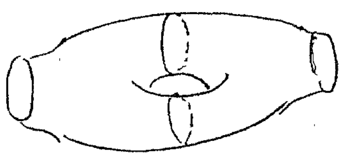


A surface of genus g.

piecewise linear and smooth. The novel feature of this book is the detailed study of the surfaces in all the three types. Rarely do we come across books treating the piecewise linear surface so well and in an intuitively appealing way. The second chapter mainly deals with the technique of gluing surfaces. The trinion composition of a surface has become very important in modern geometry especially symplectic geometry. A trinion (or a pair of pants) is a space homeomorphic to a sphere with three holes. (Figure)

These trinions are in some sense building blocks for surfaces and the bounding circles have geometric significance. It can be (pictorially) seen that (2g-2) - trinions are needed to build a surface of genus g. Cut the surface along the perforated lines in the figure below to get (g-1)-toris with two punctures each. Each two puntured toris is of two trinions.

Though the author hints at the trinion decomposition in Fig. 2.5.1, I wish he had given more details here. The second chapter ends with the statement of the theorem on the classification of compact surfaces. The third chapter is all about the proof of this classification theorem. After a leisurely discussion on triangulation. Euler characteristic, etc , the bare-hands convincing proof of the classification theorem using cut, paste and glue techniques is presented. The simplicial version of the Gauss-Bonnet Theorem developed in the exercises is a welcome addition. The fourth chapter is a small detour and studies the geometry of curves in R^3. The Serret-Frenet formulae are derived and used to prove that the curvature and torsion essentially characterise the curve in R^3 upto isometry.



A two punctured torus gotten by gluing two trinions.

The rest of the book is a liesurely course in Differential Geometry ending with the proof of the celebrated Gauss Bonnet Theorem. The author does well by introducing surfaces as sitting inside R^3 in Chapter V and works through several examples. The worked out examples with lots of pictures and the well selected graded exercises are pedagogically sound. The remaining chapters from the essentially standard material of basic differential geometry. This is indeed a well-written book and can be used for a one-year basic course in Geometric topology and differential geometry.

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Ellipsoidal calculus for estimation and control by Alexander Kurzhanski and Istvan Valyi, Birkhauser Verlag, AG, CH-4010, Basel, Switzerland, 1996, pp. 336, sFr. 128.

The volume under review is a timely-added book on ellipsoidal calculus approach to state estimation and control system synthesis. The fundamental problems in the modern control theory are: (a) to use the available data to specify or refine the mathematical model of controlled process which may or may not have uncertainty (dynamic modelling and estimation), (b) to analyse the model using suitable mathematical tools, (c) to predict or prescribe the future evolution of the system evolution of the system using the analysis (solvability and viability), and (d) to construct better control laws (feed-back control or regulation laws) so as to realize the desired goals. In this book, the authors selected an array of key problems in dynamic modelling, state estimation, viability and feed-back control under uncertainty. The main objective of the present work is to develop a unified framework for effectively solving these problems allowing parallel numerical computations and graphic animation.

The major mathematical tools used for the study are set-valued analysis and ellipsoidal calculus. The ellipsoidal calculus approach enables one to represent the set-valued solutions of a problem through ellipsoidal-valued functions. The solutions are thus constructed as elements that involve only ellipsoidal sets or ellipsoidal-valued functions and operations over such sets or functions. This enables one to carry out the actual computation using parallel algorithm and to animate the solution using graphic tools of computers.

The book is dived into four parts. The references and some historical comments are provided in the introduction of each part. A brief description of each part is as follows.

Part I: Evolution and Control - the Exact Theory

The system considered in the book is

$$\frac{dx}{dt} = u(t) + f(t) \text{ on } t_0 \le t \le t_1 \equiv T \qquad (1)$$

where state $x(t)$ is in R^n and control $u(t) \in P(t) : T \to \text{conv } R^n$, the disturbance $f(t) \in Q(t) : T \to \text{conv } R^n$, with the multi-functions $P(t)$ and $Q(t)$ are continuous. Also, the following closed-loop control system is also considered.

$$\frac{dx}{dt} \in U(t, x(t)) + f(t),\ t \in T, U(t, x(t)) \subseteq P(t) \qquad (1)$$

The attainability domain (reachability domain) for the system (1) at a time t from set X^0 is defined as

$$X[t] = \bigcup \{x(t, t_0, x^0) : x^0 \in X^0,\ u(t) \in P(t), t \in T\} \qquad (3)$$

where X^0 is a convex compact set in R^n. The multi-valued function $X[t]$, $t \in T$, $X[t_0] = X^0$ is defined to be the solution tube to system (1) from set X^0 for the interval T.

The oper-loop solvability set $W[\tau]$ is set of all states $x_\tau \in R^n$ such that there exists a control $u(t) \in P(t)$, that steers the system from $x_\tau \in R^n$ to a terminal set M due to a respective trajectory $x[t]$, $t \le t \le t_1$, so that $x[\tau] = x_\tau$, and $x[t_1] \in M$. The multi-valued function $W[t]$, $t \in T$, $W[t_1] = \mathrm{M}$ is defined to be the open-loop solvability tube to system (1) from 'set M for the interval T. A viability tube is also defined for the control system when there is a constraint on the state.

The problem of control synthesis under uncertainty consists in specifying a solvability set W and a set-valued feedback control strategy $u = U(t, x)$ such that all the solutions to differential inclusion

$$\frac{dx}{dt} \in U(t, x(t)) + Q(t)$$

that start from any given position $\{\tau, x_\tau\}$, $x_\tau = x[\tau] \in W$, $\tau \in [t_0, t_1]$ would reach the terminal set M at $t_1 : x[t_1] \in M$.

The ellipsoidal approach is applied to the following five cases:

1. System with no input uncertainty and no state constraints.
2. System with input uncertainty and no state constraints.
3. System with state constraints but no uncertainty.
4. System with uncertainty and with state constraints.
5. System with measurement output, with uncertainty in the inputs, initial states and measurement noise.

The first issue discussed is the calculation of attainability domains and attainability tubes. These were given through the solutions of the evolution funnel equations written in terms of Hausdorff distance with set-valued solutions. The attainability domain for the case 5 above is the information domain for the state estimation problem. The second issue considered is the problem of goal-oriented nonlinear control synthesis, that is to find nonlinear feedback control strategy $u(t, x)$ to steer the initial state to a preassigned terminal target set. The synthesized system is described by means of a nonlinear differential inclusion. It is possible to calculate the strategy $u(t, x)$ without introducing the tube $W[t]$, by solving the respective H-J-B equations.

Part II: The ellipsoidal calculus

This part is a separate text on ellipsoidal calculus, which is a technique of representing basic operations on ellipsoids. The geometrical sums and differences of two non-degenerate ellipsoids and their approximations both external and internal by a corresponding parametrized variety of ellipsoids are well presented. It is also shown how to construct varieties of internal ellipsoidal approximations of intersections of ellipsoids. The calculation of approximation error is also discussed in this part. One of the main advantages of this approach is that, if the upper limit of a set-valued integral varies, then the parameters of the ellipsoidal functions that approximate the integral can be described by ordinary differential equations. Therefore, the calculation of X[t] and W[t] would be parallelized into an array of identical problems each of which would consist in solving an ODE that describes an ellipsoidal-values function.

Part III: Ellipsoidal Dynamics –Evolution and control synthesis

The construction of external and internal ellipsoidal-valued approximations of attainability domains and tubes of system with out uncertainty is discussed. The approximating tubes are further described through ode's with rhs depending on parameters. The same done for systems with uncertainty later in this part. Some examples on solvability tubes and ellipsoidal control synthesis for 4-dimensional systems animated through computer windows is also presented towards the end of this part.

Part IV: Ellipsoidal Dynamics – State Estimation and Viability Problems

This last part of the book is concentrated around constructive techniques for state estimation and viability problems. Here, it is assumed that the initial states, system inputs and measurement noise are unknown in advance, but bounds on the unknows are specified in advance. The information set is the attainability domain which always includes the unknown actual state of the system and thus gives a set-valued guaranteed estimate of this state. The center of the unit ball that includes the information set (Chebyshev center) is taken as a single vector-values state estimator. The information sets are approximated by means of external ellipsoids, which are described as solutions of a system of ordinary differential equations.

The book is well-written and lucid in its presentation. I strongly believe that the authors have succeeded in presenting the new approach to stimulate readers to take up further investigation and to implementing this method to various real-life problems. This contains an exhaustive bibliography on the problems addressed in this book. I hope that control theorists and mathematicians would surely find it interesting.

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