J. Indian Inst. Sci., March–April 1998, **78**, 93–107.

Wavelets, PDEs and numerical analysis

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Received on February 15, 1998.

Abstract

The paper presents various drawbacks in the Fourier analysis of solutions of partial differential equations. Wavelets are gradually introduced and some of their seducing properties are presented. In particular, it is shown how some of the difficulties associated with Fourier analysis can be overcome. After highlighting the success of wavelets, this work speculates on the use of wavelets in making further progress in the analysis of solutions and their numerical approximation.

Keywords: Fourier analysis, partial differential equations, wavelets, numerical approximation.

1. Introduction

The objective of this paper is to motivate the need for wavelets in the study of partial differential equations (PDE) and the numerical analysis (NA) of their solutions. The use of Fourier analysis (FA) in this domain is classically well known in the form of Fourier transform (FT), Fourier series (FS) and spectral methods (SM): see for instance Hörmander[1], Gottlieb and Orszag[2], Voight *et al.*[3], Canuto *et al.*[4] and Bernardi and Maday.[5] Other techniques which are frequently used in numerical computations include finite-element method (FEM) and finite-difference method (FDM). We cite the works of Ciarlet[6], Girault and Raviart[7], Richtmyer and Morton[8], Fletcher[9], Peyret and Taylor[10], Strikwerda[11] and Davis[12] to convince the reader of the power of these methods and the class of problems they can solve. Thus, it seems natural to begin by recalling some of the virtues of these classical methods and the difficulties that we face in enlarging their field of applications. I think that this is the best way of motivating the definition of wavelets.

The discovery of wavelets was not sudden and it has been a slow evolution. Many scientists (mathematicians and engineers alike) were convinced of the need to modify the classical FA to tackle new classes of problems and they have been trying out various alternatives over a period of several decades. The idea of wavelets can be found in some vague form in several earlier works. One striking example is the so-called atomic decomposition used in the analysis of Hardy class functions: see Stein.[13] However, their final form and their applications are recent and are due to I. Daubechies, P.G. Lemarie, S. Mallat, Y. Meyer, J. Morlet and J.O. Stromberg. They can be found along with historical references in the monumental works of Meyer[14,15], Meyer and Coifman[16] and Daubechies[17].

National and international conferences and popular lectures organized in the last few years show the enormous interest of the scientific community on the subject. At the same time, we witness an explosion of articles and publications in journals presenting seducing properties of wavelets and their applications in various domains such as harmonic analysis, numerical analysis, computations, image processing, signal processing, fluid mechanics, etc. There exists even an up-to-date bibliography on wavelets available through e-mail at a nominal cost. Considering this situation, one feels the need to stimulate interest and develop the subject in India. (Of course, certain individuals in India realizing the importance of the subject are already pursuing research in this attractive field.) My aim here is to achieve this by probing the following questions: What are wavelets? Why wavelets? What are their properties? Why are they better suited than their predecessors to understand various classical phenomena in a different light? What new things can be achieved using them? As we shall see, they cannot replace FA; on the contrary, we need FA to understand and construct wavelets. We touch upon certain applications of wavelets, and conclude by discussing recent developments, modifications, improvements, various perspectives and an outlook into the future. For an exhaustive bibliographic material, we refer the interested reader to Meyer[14,15], Daubechies[17] and Ruskai *et al.*[18]

2. Fourier analysis

Since the initial ideas of Fourier, trigonometric series and FT have been the main tools to study the structure and regularity properties of functions. Because of their importance, their definitions have been extended to cover singular objects called tempered distributions: see Schwartz[19]. To see the impact of FA in PDEs, it is enough to cite the seminal works of Hörmander[1] not to speak of abstract harmonic analysis on groups and respresentation theory. The purpose here is not to discuss such advanced developments but merely to point out some motivating properties of FS and FT for which they were introduced.

The definition of FS and FT of a function f defined on $\mathbb{R}$ stems from our desire to represent f in terms of exponentials $\{e^{i\xi x}\}$. Since the latter functions are 'nice', we will be able to 'read off' and 'understand' the properties of f. Let us recall the definitions in one dimension:

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx, \xi \in \mathbb{R}. \tag{1}$$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, n \in \mathbb{Z}. \tag{2}$$

Needless to repeat, (2) is for functions which are 2π-periodic whereas (1) is for 'general' functions. Then we have the inversion formula which are the representations we seek:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{ix\xi}d\xi, \tag{3}$$

$$f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n)e^{inx}. \tag{4}$$

These formulae generally hold for tempered distributions as well. We observe that both direct and inverse formulae are non-local operations; for instance, the computation of $\hat{f}(\xi)$ for any fixed ξ requires the knowledge of f on the entire real line.

One crucial question is the following: do we lose any information in passing from f to $\hat{f}$? This is not easy to answer. It depends on the class to which f belongs. If $f \in L^2$ then we do not lose any information; more precisely, we have energy conservation in the form of Plancheral Identity:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 d\xi, \tag{5}$$

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} \left|\hat{f}(n)\right|^2, \tag{6}$$

i.e. one can read off whether a function $f \in L^2$ or not by merely looking at the magnitudes of its Fourier coefficients. The same is true of Sobolev spaces H^s which are based on L^2. More precisely, $f \in H^s$ if $\left(1+|\xi|^2\right)^{s/2} \hat{f}(\xi) \in L^2$. Further, the convergence in (3) and (4) takes place in the corresponding norm. The significance of these spaces is that their elements represent states of several mechanical systems with finite energy. To analyze finer properties of such systems, in particular to study nonlinear systems, we need to consider L^p, C^0 spaces and more generally $W^{s,p}, C^s$ spaces. We are then naturally led to ask the following questions:

– Can one characterize $f \in W^{s,p}\left(f \in C^s\right)$ in terms of the absolute values of $\hat{f}$?

– Does the convergence in (3) and (4) take place in the corresponding norm?

The answers to these types of questions are in general difficult and negative. The reason is that $\hat{f}$ has a phase even if f does not and these phases play a role which is too subtle to be completely unravelled by human beings.

On the other hand, let us recall the following striking property of FT with respect to differentiation which had enormous success in linear PDEs:

$$\widehat{\frac{df}{dx}}(\xi) = i\xi\hat{f}(\xi). \tag{7}$$

This signifies the fact that analytic operation 'derivation' goes into algebraic operation 'multiplication by a polynomial' under FT. This is because exponentials are eigenfunctions of constant coefficient operators:

$$P(D)e^{ix.\xi} = P(i\xi)e^{ix\xi} \tag{8}$$

where

$$P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha, a_\alpha \text{ being constants.} \tag{9}$$

These properties lie at the heart of the analysis of linear PDEs with constant coefficients. For instance, an initial value problem (IVP) involving PDE can be transformed to a parametrized family of IVPs for ODE. The latter can be solved 'explicitly' and the passage to PDE can be achieved using Inversion formulae (3) and (4) under a suitable hyperbolicity conditon; see for instance, Treves[20]. The case of operators with smooth variable coefficients is harder. However, the problem can be attacked by perturbation analysis and this requires sophisticated tools such as the calculus of pseudo-differential operators and Fourier integral operators[1]. When this works, we see that there is no major qualitative departure from the constant coefficient case.

However, these methods are not easily adaptable to cover nonlinear equations which are order of the day. This is because exponentials are no more left invariant as in (8) and this is a qualitative departure from the linear case. This simple reason is good enough to look for alternatives of FA, indeed nonlinearities tend to spread the support of $\hat{f}$. To see this, let us recall the formula which shows that ordinary product is converted into convolution product under FT.

$$\widehat{fg}(\xi) = \hat{f} * \hat{g}(\xi) = \int \hat{f}(\xi - \eta)\hat{g}(\eta)d\eta. \tag{10}$$

In particular, we have

$$\hat{f}^2(\xi) = \hat{f} * \hat{f}(\xi). \tag{11}$$

This implies, in particular, the spread of the support under FT:

$$\text{supp } \hat{f}^2 \subset \text{supp } \hat{f} + \text{supp } \hat{f}.$$

This indicates that new Fourier modes are generated by nonlinearities. To get it confirmed, let us consider the Hopf operator:

$$Nu \equiv u_t + u\, u_x.$$

If $u = e^{i(\xi x - \omega t)}$ has one Fourier mode, then $Nu = (-i\omega)e^{i(\xi x - \omega t)} + (i\xi)e^{2i(\xi x - \omega t)}$ has two Fourier modes. This is the chief mechanism behind the formation of shocks. This already shows that the solutions to nonlinear equations can be very rough and new ideas are needed to handle them.

In the theory of PDEs and also in NA, one is interested in the following aspects of solutions apart from their existence and uniqueness :

– regularity properties with respect to data,
– singularities, if any,
– position, size and nature of singularities given these data initially.

As an easy application of FA to prove regularity results, we show how $f \in L^2$, $\Delta f \in L^2$ imply $f \in H^2$. Indeed, our hypotheses are equivalent, via Plancheral Identity, to $(1 + |\xi|^2)\ \hat{f}(\xi) \in L^2$ which in turn is equivalent to

$$\hat{f}(\xi), |\xi|^2 \hat{f}(\xi) \in L^2.$$

Plancheral Identity once again shows that the above property is nothing but saying that all derivatives of f up to order 2 are all square integrable, i.e. $f \in H^2$.

To analyze the singularities of *f*, the so-called *singular support* of *f* is introduced. In order to keep track of the singularities, it is a discovery that one has to consider the corresponding wave numbers also which cause the singularities. This localization in (x, ξ) space leads us to one of the fundamental objects, $WF(f)$, called the *wave-front set* of *f*. Comparing the singularities with the energy of wave front of light, we conclude that $WF(f)$ obeys the laws of *linear geometrical optics* (LGO) in the case of linear PDEs. For the computation of this object in the case of linear PDEs and the subsequent beautiful analysis, see Hörmander[1] once again. For reasons already cited, the analysis of $WF(f)$ for nonlinear equations poses a great challenge to mathematicians. As we shall see, localization in the physical as well as Fourier space lies at the heart of wavelet analysis (WA). Indeed, $WF(f)$ should be compared with the set of points where the wavelet coefficients $W_f(b, a)$ do not 'vanish'.

If we examine the difficulties mentioned above a little more closely, we see that one of the principal reasons is that exponentials are localized to the maximum in ξ-space. They are very regular and have no decay at all. According to Heisenberg Uncertainty Principle, the more an object is localized in ξ-space the more it is inadequate to describe the local phenomena in x-space. This explains why we face serious difficulties in describing regularity properties of function f using its Fourier representation. Can one replace exponentials by other functions which do not concentrate in ξ-space and have nice decay properties in x-space? Can they be constructed by easy means? Do they form basis in the sense of (3) and (4)? Do we retain properties (7), (8) and (10)? The definition of wavelets is motivated through these questions.

Some of these questions were earlier asked in the context of generalizing LGO to nonlinear equations. This is the subject matter of nonlinear geometrical optics (NGO); cf. Whitham.[21] The crucial idea there is to superimpose exponentials over several wave numbers to obtain a suitable localization in x-space. In other words, replace exponential by a suitable function which will be determined in such a way that it has some desired properties. Some of these ideas are retained in the construction of wavelets also; however, the desired properties are not the same now.

3. Haar bases

If the difficulty with exponentials is what was described in §2 and the purpose is to describe local properties of functions then one obvious solution (as suggested in §2) is to look for basis which are localized in x-space rather than in ξ-space. The construction of the classical Haar bases is done with this in mind. Start with the following function which has compact support:

$$h(x) = \begin{cases} 1 \text{ if } 0 \le x < \frac{1}{2} \\ -1 \text{ if } \frac{1}{2} \le x < 1 \\ 0 \text{ if } x \notin [0,1]. \end{cases}$$

We then consider the following two collections of functions constructed by dilations and translations from h:

$$h_{jk}(x) = 2^{j/2} h\left(2^j x - k\right), k \in \mathbb{Z}, j \in \mathbb{Z}. \tag{H1}$$

$$\begin{cases} h_{jk}(x) = 2^{j/2} h\left(2^j x - k\right), k \in \mathbb{Z}, j \geq 0, \\ q(x-k), k \in \mathbb{Z} \text{ where } q(x) = \chi_{[0,1]}(x). \end{cases} \tag{H2}$$

These collections form individually orthonormal basis for $L^2(\mathbb{R})$. Compared with exponentials, the Haar bases have many advantages. For instance, the norm of an L^p function can be estimated by a function depending only on the absolute values of Haar coefficients of f. Nothing similar could happen with exponentials.

One of the drawbacks of Haar functions is that they fail with regard to property (8). They are not differentiable at all. Let us remember that (8) was an essential key point in the success story of FA in the theory of PDEs. This failure is due to the fact that Haar functions are on the other extreme; they are too much localized in x-space and poorly localized in ξ-space. This is reflected in their oscillations and lack of regularity. We measure the *oscillations* of a function f by looking at the averages. More precisely, we say f oscillates to the degree r if

$$\int_{-\infty}^{\infty} x^k f(x) dx = 0, \ 0 \leq k \leq r. \tag{12}$$

This can equivalently be phrased as

$$\left(\frac{d}{d\xi}\right)^k \hat{f}(0) = 0, \ 0 \leq k \leq r.$$

Thus, we observe that the regularity of a function signifies the decay of large Fourier modes whereas the oscillation property (12) signifies the decay of low Fourier modes. The stipulation that these two Fourier modes decay is a good measure of localization of the function in ξ-space.

The moral therefore is that we should not completely sacrifice the localization in ξ-space and the oscillation property available in FA even though there is a need to localize in x-space. So, the idea is to strike a middle ground between these two extremes without violating Heisenberg's Uncertainty Principle but touching the very limit set forth by it. Wavelets arise naturally in this way.

Another idea tried out in the past is to smoothen the Haar basis by taking their primitives, but then one loses the orthogonality property. By the classical Gram–Schmidt orthogonalization, one can recover it but then the functions obtained this way introduce enormous complexities in the computation. Recall that the computation of solutions is one of our principal aims. Complexity means a lot of operations in the computer and thereby increase in the cost and round-off error. Complexity is thus to be avoided.

4. Numerical analysis

Having seen some motivation for wavelets from the Fourier analysis of solutions of PDEs, we turn our attention to the computational aspects and point out some fundamental difficulties. What can be done to overcome them? As we shall see, this leads us to wavelets once again.

Usual procedures employed to discretize PDEs are FDM, FEM and SM. The basic idea is to approximate the spaces involved by finite dimensional spaces. To construct them, exponentials are used in SM whereas piece-wise polynomials are used in FEM. The question is to know how accurate the approximate solutions are. Apart from the order of the scheme, this is related to regularity of the solution and stability. In classical situations where the solution is regular enough, the error is of finite polynomial order in FEM and of infinite order in SM: see the works cited in §1. However, in situations where solution f is not regular, spectral approximations do not yield satisfactory results. One such example is the velocity field of a turbulent flow. Its principal characteristic is that Fourier representation is 'full', i.e. there exists N large such that $\left|\hat{f}(\xi)\right|$ for $|\xi| > N$ are all negligible and for $|\xi| \leq N$ are not negligible. Hence, it is intuitively clear that if we want a reasonable approximation of such functions, we must take into account all Fourier modes $\hat{f}(\xi), |\xi| \leq N$. The limitation of today's computers in terms of memory requirements and the speed of calculations prevent us from doing this. Rigorous mathematical analysis of these solutions is out of reach for the moment. The idea therefore is to look for alternative basis functions in which solutions will have 'controllable' number of terms which are significant. Once again, we see the need to superimpose exponentials over several wave numbers. Wavelet representation is motivated towards carrying out this idea.

The situation with FEM and FDM is not bad. On one hand, the FE basis of functions are easily constructed even on unstructured grid avoiding complexities. On the other, there are some adhoc procedures to handle singularities. From physical reasons, the location of singularities of solution is roughly estimated. For turbulent solutions, this is a hard problem and there are only conjectures: see Mandelbrot.[22] Once this is done, refinement of the mesh in those regions is performed. This amounts to a minimal increase in the dimension guaranteeing, at the same time, an enhanced accuracy of the approximate solution.

This practice has been in existence for quite sometime with the numerical analysts and it is found quite successful. In some cases, there has been mathematical justification. As we shall see later, the introduction of wavelets formalizes this adhoc procedure. The FE basis associated with such meshes are, of course, localized in x-space but nonuniformly distributed in space to take care of the variation of functions. Their main drawback is that they are not very smooth, and neither have the oscillation property mentioned earlier. Hence, it is necessary to combine this basis with that in SM in a suitable sense.

5. Wavelet transform

From our discussion in the previous sections, we feel the need to have a basis consisting of functions localized in (x, ξ) space. The notion of $WF(f)$ already incorporates such an idea. Another classical object which does the same job is the *windowed Fourier transform* (WFT) introduced by Gabor[23] (see also Daubechies[17]). The idea is to decompose the given function into small pieces (windows) and take FT of each piece. More precisely, WFT of a function $f(x)$ is defined by

$$Tf(\xi, x) = \int f(y) g(y - x) e^{-i\xi y} dy,$$

where g is the fixed window function. One of the drawbacks of this localization is that regardless of the frequency values (high or low) the windows have the same width defined by g. Intuitively, we feel the need for larger windows to see high frequencies and small windows to see low frequencies. The definition of wavelet transform (WT) can be seen to achieve this. A second reason to modify WFT is that is has been shown (see Daubechies[17]) that one can only generate 'frames' and not a basis via a lattice sampling in WFT. The main difference between frames and basis is that frames contain 'too many' vectors and so not ideally suited for NA. On the other hand, as we shall see in the sequel, it is a miracle that a suitable lattice sampling of WT will lead us to an orthonormal basis. For a good review about frames, see Heil and Walnut[24]. The formalization of the above ideas involve the following: since we wish to localize in x-space, we must have a variable to do this job. Since exponentials were localized in ξ-space, this was not possible in FT. Of course, as in FT, we must have a variable which measures the scale of variations of function. As agreed upon already, exponentials have to be grouped over several wave numbers and this gives to what is called a *mother wavelet* function ψ. Once ψ is chosen, the principle of WT is very simple. As in FT, given a function f, we test it against ψ. Let $b \in I\!R$ denote the position parameter which can be moved from one position to another by translation. This corresponds to localization in x-space. Let $a > 0$ be the scaling parameter which measures the scale of variations of functions. This corresponds therefore to localization in ξ-space. They form a group under multiplication. WT of f is defined as follows :

$$Wf(b,a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(x) \psi\left(\frac{x-b}{a}\right) dx. \tag{13}$$

It is quite clear that WT serves as a 'mathematical microscope' to analyze the structure of a function. Indeed, by fixing b, we can localize the behaviour of f around b and by decreasing the values of $a > 0$, we can see the structure of f to finer and finer details.

Of course, the mother wavelet ψ is not an arbitrary function. From our discussions in the earlier section, we wish ψ to have the following properties:

$$\begin{cases} \text{(a)} & \psi \text{ is regular} \\ \text{(b)} & \psi \text{ is localised in } x-\text{space} \\ \text{(c)} & \psi \text{ oscillates in the sence of (12)} \\ \text{(d)} & \text{There exists an inversion formula expressing } f \text{ in terms of } Wf. \end{cases}$$

A partial solution to the above question is found in the early 60s: see Calderón.[25] Indeed, let ψ satisfy

$$c(\psi) = \int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} = \int_0^\infty |\hat{\psi}(-\xi)|^2 \frac{d\xi}{\xi} < \infty. \tag{15}$$

Then it can be shown, using Parseval's relation in FA, that

$$W: L^2(I\!R) \to L^2\left(I\!R_+^2; \frac{da\,db}{c(\psi)a^2}\right)$$

is an isometry, i.e. $W^* Wf = f$. Moreover, the dual map W^* is given by

$$\begin{cases} W^*: L^2\left(\mathbb{R}^2_+; \frac{dadb}{c(\psi)a^2}\right) \to L^2(\mathbb{R}), \\ W^* g(x) = \int_{-\infty}^{\infty} \int_0^{\infty} g(a,b)\psi\left(\frac{x-b}{a}\right)\frac{dadb}{c(\psi)a^2\sqrt{a}}. \end{cases} \tag{16}$$

As a consequence, we obtain the following *inversion formula*:

$$f(x) = \frac{1}{c(\psi)}\int_{-\infty}^{\infty} \int_0^{\infty} Wf(b,a)\frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)\frac{dadb}{a^2}. \tag{17}$$

This can be generalized as follows : let ψ be such that

$$c(\phi,\psi) = \int_0^{\infty} \frac{\hat{\phi}(\xi)\hat{\psi}(\xi)}{\xi} d\xi < \infty.$$

Then, we have

$$f(x) = \frac{1}{c(\phi,\psi)}\int_{-\infty}^{\infty} \int_0^{\infty} Wf(b,a)\frac{1}{\sqrt{a}}\phi\left(\frac{x-b}{a}\right)\frac{dadb}{a^2}. \tag{18}$$

One important choice of ϕ in (18) is $\phi = \delta_0$ which is possible if ψ satisfies, in addition, that

$$c'(\psi) = \int_0^{\infty} |\hat{\psi}(\xi)| \frac{d\xi}{\xi} < \infty.$$

We then obtain the inversion formula:

$$f(x) = \frac{1}{c'(\psi)}\int_0^{\infty} \frac{Wf(x,a)}{m}\sqrt{a}\,\frac{da}{a}. \tag{19}$$

The single-most important property of WT which distinguishes it from FT is the following: f is regular at b if the wavelet coefficients $Wf(b, a)$ decay as $a \to 0$. To see this heuristically, let us observe that we can write $Wf(b, a)$ as follows using (12):

$$Wf(b,a) = \int \left(f(x) - f(b)\right)\frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)dx,$$

$$Wf(b,a) = \int \left[f(x) - f(b)) - f'(b)(x-b)\right]\frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)dx,$$

and so on. These expressions imply successively the decay of wavelet coefficients:

$$Wf(b,a) = O\left(a\sqrt{a}\right), O\left(a^2\sqrt{a}\right)\ldots$$

depending on the regularity of f. The converse part is a consequence of the inversion formulae (18) and (19). We remark that such local regularity analysis does not exist in FT.

Of course, properties (7) and (8) which are due to the maximum localization in ξ-space are not shared by WT. Better is the localization of ψ in x-space, the poorer is its localization in ξ-space and in this case, properties (7) and (8) are more violated by WT.

6. Wavelet series

Apriori the uncertainty principle in quantum mechanics seems to cast doubts of obtaining a basis consisting of localized functions in (x, ξ) space. However, this is not the case; indeed a refined version of the principle says the following (see Slepian[26]): there are exactly $2T\Omega/\pi$ independent functions that are essentially localized in $\{(x, \xi);\ |x| \le T,\ |\xi| \le \Omega\}$ as $T\Omega \to \infty$. Thus, there is a ray of hope producing a basis by letting $\Omega \to \infty$ or $T \to \infty$.

Though (17) is a formula analogous to (3), we would like to underline one important difference. As f varies over L^2, $\hat{f}$ fills up L^2 whereas $Wf(b, a)$ varies over a 'tiny' subset of $L^2\left(\mathbb{R}_+^2, \frac{da db}{a^2}\right)$. This suggests that from the family of wavelet coefficients in the inversion formula (17) representing a function, one can extract a countable number of them which are significant. The corresponding wavelets form a basis for L^2. From the numerical point of view, this is extremely important because it implies enormous reduction in storage.

To see this, let us start with the classical finite-element spaces which are defined on finer and finer meshes of $\mathbb{R}$:

$$V_j = \left\{f \in C^0(\mathbb{R}); f \text{ is linear on } \left[k2^{-j}, (k+1)2^{-j}\right] \forall k \in \mathbb{Z}\right\}, j \in \mathbb{Z}.$$

These spaces correspond to the so-called P_1-element in the finite-element literature: see Ciarlet.[6] The usual finite-element basis consisting of 'hat functions' has a particular structure which was not used in the theory of finite elements but becomes important now. It is the following: for V_0, there exists a basis of the form $\{g(x-k)\}_{k\in\mathbb{Z}}$ where $g \in V_0$. In fact, g is the unique hat function in V_0 such that $g(0) = 1$ and $g(k) = 0\ \forall\ k \in \mathbb{Z}\setminus(0)$.

The above example can be abstracted and put in the following form:

Definition: A *multiresolution analysis* (MRA) is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying

$$\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots$$

$$\cap V_j = (0) \text{ and } \overline{\cup V_j} = L^2$$

$$f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$$

$$f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0, k \in \mathbb{Z}.$$

There exists a function $g \in V_0$ such that $\{g(x-k)\}_{k\in\mathbb{Z}}$ forms a basis of V_0.

From g, we can produce a function $\phi \in V_0$ such that $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ forms an orthonormal (o.n.) basis for V_0. To get the required basis for L^2, we introduce W_j to be the orthogonal complement of V_j in V_{j+1} and write

$$L^2 = \oplus_{j\in\mathbb{Z}} W_j \text{ (or) } L^2 = V_0 \oplus \oplus_{j=0}^{\infty} W_j.$$

Since $V_0 \subset V_1$ and $\left\{\sqrt{2}\phi(2x-k)\right\}_{k\in\mathbb{Z}}$ forms on o.n. basis for V_1 we can express, for some $(h_k) \in l^2$

$$\phi(x) = \sum_k h_k \sqrt{2}\phi(2x-k).$$

Simply define $\psi(x) = \Sigma_k (-1)^{k+1} h_{1-k} \sqrt{2}\phi(2x-n)$. It is not hard to check that $\left\{\psi_{jk}\right\}_{k\in\mathbb{Z}}$ forms an o.n. basis for W_j for all $j \in \mathbb{Z}$. Here, ψ_{jk} stands for the function $\psi_{jk}(x) = 2^{j/2}\psi\left(2^j x - k\right)$ obtained from ψ by translation and dilation. We then have the following representations for L^2-function:

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{jk}\rangle \psi_{jk}, \tag{20}$$

$$f = \sum_{k\in\mathbb{Z}} \langle f, \phi_k\rangle + \sum_{j\geq 0, k\in\mathbb{Z}} \langle f, \psi_{jk}\rangle \psi_{jk}. \tag{21}$$

These are examples of expansions in the basis of L^2 formed by wavelets $\{\psi_{jk}\}$ which arise from MRA. The functions ψ and ϕ are referred to as *mother* and *father wavelets*, respectively.

Once their existence has been established, the next question concerns their choice which is more suitable for a local Fourier analysis, namely, can one choose ψ such that ψ is regular, ψ is localized in x-space and ψ has oscillation property? In this connection, the following results have been proved in literature: Stromberg[27] proved, for each r, s, the existence of $\psi \in C^s$ having exponential decay and satisfying (12). Daubechies[17] improved it by showing ψ can be chosen to have compact support. On the other hand, Lemaire and Meyer[28] showed that the choice of ψ is possible in the Schwartz class S and such that (12) is satisfied with $r = \infty$.

7. Wavelet and Fourier series

WS are destined to compete with FS. Thanks to double localization, WS permits analysis finer than FS. As far as NA is concerned, the most single property which distinguishes WS from FS is the following: WS is 'sparse' in the sense that wavelet coefficients with respect to scale parameter j is 'zero' for $|j|$ large where the function is regular. This property is responsible for enormous data compression. On the other hand, let us remark that Fourier series of important functions is 'full' whereas lacunary FS represent often pathological functions.

Moreover, wavelets provide a basis also for the classical standard space $L^p\, 1 < p < \infty, C^{0,\alpha}$ $0 < \alpha < 1$, etc. These spaces are characterized directly by conditions on the wavelet coefficients. Let us remember that such characterizations with Fourier coefficients are rather rare.

There are, of course, certain inconveniences in dealing with wavelets of which we mention two here. Recall that the derivation operator is transformed to multiplication operator under FA (see (7)). This property is no more true. However, some operators acquire special structure in the wavelet formulation depending upon the choice of the wavelet. For instance, the so-called Calderón–Zygmund operators are almost diagnosable in the wavelet basis.[15]

Next, turning our attention to nonlinear equations, let us recall that the usual multiplication in x-space is transformed to convolution product under FA (see (10)). In other words, the Fourier coefficients of f^2 are calculable entirely in terms of those of f. This is not the case with the wavelet coefficients. For the moment, this is done in x-space after computing f^2. Research is on as to how best the wavelet coefficients of nonlinear terms can be directly calculated without going to the physical space. To have a measure of difficulties in this context let us cite a recent paper.[29]

8. Wavelets in numerical computations

If there is one field where wavelets have enormous impact it is in the domain of numerics. Since wavelet basis lies between finite-element basis and spectral basis as explained already, it shares their properties: as efficient as FEM in localizing and capturing singularities of solution and at the same time providing good approximation in smooth regions. This latter phenomenon depends on the oscillation property (12) satisfied by the wavelets. This situation is to be compared with the difficulties one encounters with higher order schemes like Lax–Wendroff in the presence of singularities.

A major task is to exploit the presence of lacunarity in the wavelet series representing the solution. To this end, we must necessarily use non-uniform meshes. Indeed, a comparative study shows that on regular meshes, wavelet method and more traditional methods yield the same type of results. In practice, the mesh is rendered nonuniform in an iterative fashion by anticipating significant wavelet coefficients at the next iteration from the magnitude of the coefficients in the present iteration. Another technique is to use what are called *mobile wavelets*.[30] The idea here is to consider the wavelets as particles in the space (b, a) of position and scale and they move around as time evolves. The approximate solution is in the form

$$\sum_{i=1}^{n} c_i(t)\psi\left(\frac{x-b_i(t)}{a_i(t)}\right).$$

The aim here is to cook up suitable evolution equations for $a_i(t)$, $b_i(t)$ and $c_i(t)$ in such a way that there is strong concentration of wavelets in the region of singularities of solution.

Yet another theory on the horizon to achieve this is that of *wavelet packets* wherein the aim is to represent a function in a basis which is optimal, i.e. the number of elements of the basis representing the function is as small as possible. Each element of the basis is constructed starting from the mother-wavelet packet by the operations of dilation, translation and modulation. Thus, there are three parameters instead of the usual two in the classical construction of wavelets. This theory includes that of windowed FT and WT and seems to be full of promise in future applications. For details see Coifman *et al.*[31]

9. Conclusions

In this paper, we have tried to answer the following questions: what are wavelets? why wavelets? basis ideas behind their construction with examples and interpretations starting from the classically known objects, their immediate properties, limitations, comparison with trigonometric functions, etc. There are several issues which are not discussed and research is in full swing in these areas. To the set of several questions raised in earlier sections, we add the following ones: the choice of wavelets best suited to the problem at hand, construction of wavelets in the presence of boundaries, issues involved in the case of several variables, wavelets in nonhomogeneous media, etc. Wavelets are destined to compete with the more classical trigonometric functions. Consequently, some classical issues are viewed in a different light now. For instance, Calderón–Zygmund operators are almost diagnosable in wavelet basis and this explains aposteriori their success. Various algorithms using wavelets for these operators should show rapid convergence and this has to be confirmed. Some important observations in this regard have been made by Devore and Lucier.[32] Group-theoretical aspects of WT are discussed in Heil and Walnut.[33] For fast algorithms using WT, consult Beylkin *et al.*[34] The interaction of wavelets with geometrical surfaces (regular and fractal) is the subject matter in David[35], Arneodo *et al.*[36] and Holschneider.[37]

A basic goal of the subject is to analyze the singularities of solutions of nonlinear equations. One specific question in this context is the following: do Navier–Stokes equations and Euler equations in three dimensions exhibit non-smooth solutions with smooth initial data? Can one answer such questions using wavelets? FA did not have much success in this area. Argoul *et al.*[38] show numerical evidence of an affirmative answer to this question. However, rigorous mathematical analysis of this phenomenon is still elusive. The conjecture is that the set of singularities is concentrated on a small set[22,39]. If this is true then, by the very virtue of wavelet coefficients, we will be able to represent fluid flows by wavelet series where there exists only a 'controllable' number of significant terms. Fourier analysis enabled one to derive upper estimate on the dimension of the attractor which represents the fluid flow (Temam). The above arguments may imply that a significant improvement of this estimate can be achieved using wavelets. Can one then develop a nonlinear Galerkin method based on wavelets analogous to Marion and Temam[40]? Probably, these are some of the major issues with which the scientific community will be preoccupied in the future.

Acknowledgements

This paper was written by the author as a member of the Programme Advisory Committee in Mathematical Sciences (PAC-MS) of the Department of Science and Technology (DST), Government of India, New Delhi. A condensed version of it is appearing as a three-part serial in the *Vision Papers in Mathematical Sciences* in *Mathematics Newsletter* (MN) published by the Ramanujan Mathematical Society. Two parts have already appeared. Part 1: *MN*, 1997, **6**(4), 64–77; Part 2: *MN*, 1997, **7** (1), 8–21. The author gratefully acknowledges the permission granted by DST to publish the article in its present form.

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