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The Constantin–Lax–Majda model for the vorticity equation revisited*

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Abstract

We propose a new viscous term in the Constantin–Lax–Majda 1D model for the 3D vorticity equation. This overcomes the drawback associated with the canonical viscous term considered by Schochet.

1. Introduction

Physical arguments (e.g. Frisch[1], p. 115) and numerical computations (e.g. Grauer and Sideris[2]) strongly suggest that finite-time singularities develop in three-dimensional inviscid incompressible flow. The equations governing such a flow are the Euler equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = 0, (x,t) \in R^3 \times (0,\infty) \\ \nabla \cdot u = 0 \end{cases} \tag{1}$$

with the initial condition

$$u(x, 0) = u_0(x). \tag{2}$$

The main mathematical question regarding (1)–(2) are : Do smooth solutions exist for all time or do singularities develop in finite time? Regarding the first question, Beale *et al.*[3] have proved the following. Suppose u_0 is smooth, then there exists a global smooth solution if and only if the vorticity $\omega = \nabla \times u$ satisfies

$$\int_0^T \|\omega(.,t)\|_\infty \, dt < \infty$$

for every $T > 0$. Further, they have shown that if a solution which is initially smooth loses its regularity at some later time, then the maximum vorticity necessarily grows without bound as the critical time approaches. Thus, the formation of singularities in Euler equations depends on vorticity production or vortex stretching. Note that in two dimensions there is no vortex stretching. The interest in these possible singularities, as pointed out by Caflisch[4], are physical, numerical and mathematical: physical because singularity formation may signify the onset of turbulence and may be a primary mechanism of energy transfer from large to small scales, numerical because special methods to solve Euler equations would be required for tackling this

*Dedicated to Prof. R. Narasimha on his 65th birthday.

singularity formation, mathematical because singularities in Euler equations would prevent an establishment of global existence theorems for (1).

The need to understand the precise mechanism of formation of singularities in finite time has led to a consideration of some model problems that mimic the Euler equations. These models should not only be simpler than (1) but also possess some of the important features that are known about (1). Such models would be natural candidates as test problems in verification of numerical methods for (1).

In this direction, Constantin *et al.*[5], hereafter CLM, proposed a very simple model for the vorticity equation. We shall briefly explain the motivation for their proposal. The Euler equations (1)–(2) can be written as

$$\begin{cases} d\omega / dt = (\omega \cdot \nabla)u, \\ \omega(x,0) = \omega_0(x) \end{cases} \tag{3}$$

where

$$\frac{d}{dt} = \omega_t + (u \cdot \nabla)\omega,$$

and

$$\omega_0 = \nabla \times u_0.$$

Now u can be written in terms of ω as

$$u(x,t) = (K * \omega)(x,t) = \int_{R^3} K(x-y)\omega(y,t)dy \tag{4}$$

where

$$K(x) = \begin{bmatrix} 0, & -G_3(x), & G_2(x) \\ G_3(x), & 0, & -G_1(x) \\ -G_2(x), & G_1(x), & 0 \end{bmatrix},$$

$$G_i(x) = \frac{\partial G}{\partial x_i},$$

$$G(x) = \frac{1}{4\pi|x|}.$$

The matrix ∇u can be decomposed into its symmetric part

$$D(u) = \frac{1}{2}\left[\nabla u + (\nabla u)^t\right], \tag{5}$$

and its antisymmetric part

$$J(u) = \frac{1}{2}\left[\nabla u - (\nabla u)^t\right].$$

Since

$$J(u)\omega = 0$$

eqn (3) reduces to

$$\frac{dw}{dt} = D(\omega)\omega \tag{6}$$

where we have substituted (4) in (5). The explicit formula for D is not needed in the following but the following properties of D are worth noting. The matrix valued function D depends linearly on ω, the operator relating ω to $D\omega$ is a linear singular operator that commutes with translation and has a mean value on the unit sphere equal to zero. CLM made the remarkable observation that in one space dimension the only operator similar to D is the Hilbert transform

$$H(\omega) = \frac{1}{\pi} pv \int_{-\infty}^{\infty} \frac{\omega(y)}{x-y} dy.$$

CLM then went on to propose the one-dimensional analogue of (6)

$$\begin{cases} \omega_t = \omega H(\omega), (x,t) \in I\!R \times (0,\infty) \\ \omega(x,0) = \omega_0(x). \end{cases} \tag{7}$$

'Velocity' is defined as,

$$v(x,t) = \int_{-\infty}^{x} \omega(y,t) dy. \tag{8}$$

Surprisingly, (7) is explicity solvable and the solution is given by

$$\omega(x,t) = \frac{4\omega_0(x)}{[2 - t(H\omega_0)(x)]^2 + t^2\omega_0^2(x)}. \tag{9}$$

From the explicit formula it is clear that the solution ω blows up in a finite time T_0 if and only if there exists an x_0 such that $\omega_0(x_0) = 0$ and $(H\omega_0)(x_0) > 0$. CLM also showed that if x_0 is a simple zero of $\omega_0(x_0)$ then

$$\lim_{t\to T_0} \int_{-\infty}^{\infty} |\omega(x,t)|^p dx = \infty, \quad \text{for} \quad 1 \le p < \infty$$
$$\lim_{t\to T_0} \int_{-\infty}^{\infty} |v(x,t)|^p dx < M^p < \infty, \quad \text{for} \quad 1 \le p < \infty$$

Thus, the model vorticity equation (7) seemed to possess the most important feature of (6): finite-time blow up of vorticity but velocity remaining bounded. Now (7) with its explicit solution (9) is a challenging test problem for numerical methods designed to detect blow up. This has been demonstrated by Stewart and Geveci[6]. Extension of the model equation (7) to include viscous effects was taken up by Schochet[7]; who considered the equation

$$\begin{cases} \omega_t = \omega H\omega + \varepsilon\omega_{xx}, (x,t) \in I\!R \times (0,\infty) \\ \omega(x,0) = \omega_0(x) \end{cases} \tag{10}$$

Solution to (10) was explicitly written down by Schochet, who found that it blew up at time T_ε and

$$T_\varepsilon < T_0 \tag{11}$$

where T_0 is the blow up time for $\varepsilon \equiv 0$, i.e. for (9). In other words, adding diffusion makes the solution less regular! Clearly this is unsatisfactory, especially in view of the result by Constantian[8] which says that if the solution to the Euler equation is smooth then the solution to the slightly viscous Navier–Stokes equations with the same initial data is also smooth. Hence, the simple model (7) of CLM lost most of its interest. Some improvements of this model have been undertaken by De Gregorio[9,10]. The main aim of this work is to propose a nonlocal diffusion term which, when added to the CLM model, will reverse the inequility in (11) and thus remove the drawback mentioned above.

2. The proposed viscous model

In this section, we derive heuristically our proposal for including 'viscous effects' to (7). It is well known that solution to

$$\begin{cases} u_t = uu_x, & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x), \end{cases} \tag{12}$$

loses regularity in finite time no matter how smooth u_0 is. If we add viscosity to (12)

$$\begin{cases} u_t = uu_x + \nu u_{xx}, & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x), \end{cases} \tag{13}$$

then a global smooth solution exists for all time. Trying to propose a model for water wave phenomenon like sharp crests and breaking of waves, Whitham[11] asked the question: Is there a 'viscosity' which when added to (12) loses its regularity in finite time? Obviously, νu_{xx} is not the right candidate for reasons mentioned above. Whitham[11] conjectured that if we consider

$$\begin{cases} u_t = uu_x - K * u_x; & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x) \end{cases} \tag{14}$$

with the kernal K having the Fourier transform,

$$\hat{K}(\xi) = \sqrt{\xi \tanh \xi},$$

then solutions to (14) will lose its regularity in finite time. This conjecture has been completely settled by Naumkin and Shishmarëv[12]. In a similar vein, we ask the opposite question: which viscous will make the solution (9) blow up at a later time when added to (7)? In other words, inequality (11) holds from the results of Schochet[7] that it cannot be εu_{xx}. Now, CLM has shown that the blow up of (9) is different from the blow up of u_x where u is a solution to (12). Note that u_x satisfies, along the characterstics

$$\begin{cases} (u_x)_t = (u_x)^2 \\ u_x(x,0) = (u_0(x))_x, \end{cases} \tag{15}$$

and hence blows up in finite time. In other words, equation $u_t - uu_x$ is not a good model for the breakdown of smooth solutions to (1) but $\omega_t - \omega H(\omega)$ is a better model. Now $H(\omega)$ can be interpreted as differential operator via the Fourier transform

$$H(\omega) = (\mathrm{sgn}\frac{d}{dx})(\omega).$$

Arguing analogously one feels that $-\varepsilon Hu_x$ would be a better model to 3D 'viscosity' compared to εu_{xx}. So, we propose

$$\begin{cases} \omega_t = \omega H(\omega) - \varepsilon H\omega_{xt}; (x,t) \in I\!R \times (0,\infty) \\ \omega(x,0) = \omega_0(x) \end{cases} \tag{16}$$

as the 'viscous' analogue of (7). Note that $-\varepsilon H\omega_x$ is indeed a dissipative term as can be checked by solving the linear part of (16) using Fourier transform. Such a dissipative term has also been considered by Matsuno[13].

3. Explicit solutions

Solution to (16) can be explicitly obtained by the complexification

$$Z = H(\omega) + i\omega \tag{17}$$

It is easy to check that Z satisfies,

$$\begin{cases} Z_t + i\varepsilon Z_x = \frac{Z^2}{2}, \\ Z(x,0) = Z_0(x), \end{cases} \tag{17}$$

where,

$$Z_0 = H(\omega_0) + i\omega_0.$$

Making the transformations

$$Z = \frac{1}{\vartheta}, \bar{\vartheta} = \theta - \frac{ix}{2\varepsilon}, \tau = \varepsilon t,$$

(17) reduces to

$$\begin{cases} \bar{\vartheta}_\tau + i\bar{\vartheta}_x = 0, \\ \bar{\vartheta}(x,0) = \frac{1}{Z_0(x)} - \frac{ix}{2\varepsilon}. \end{cases} \tag{18}$$

This can be easily solved and we obtain the solution for (17)

$$Z^\varepsilon(x,t) = \left[\frac{1}{1+i}\left\{\frac{1}{Z_0^+(x,t)} + \frac{i}{Z_0^-(x,t)}\right\} - \frac{t}{2}\right]^{-1}, \tag{19}$$

where

$$Z_0^\pm(x,t) = Z_0(x \pm \varepsilon t).$$

Note that if $\varepsilon = 0$ then $Z_0^\pm = Z_0$ and we get

$$Z^0(x,t) = \left[\frac{1}{Z_0} - \frac{t}{2}\right]^{-1} = \frac{Z_0}{1 - tZ_0/2},$$

which is the solution to the inviscid case (7) obtained by CLM. To obtain the solution for (16) we need to take the imaginary part of (19). After a tedious algebra we obtain,

$$\omega^\varepsilon(x,t) = 2\frac{\omega_n^\varepsilon(x,t)}{\omega_d^\varepsilon(x,t)}, \tag{20}$$

where

$\omega_n^\varepsilon(x,t) = (h^+\omega^- + h^-\omega^+)(h^+ + h^- - \omega^+ + \omega^-) - (h^+h^- - \omega^+\omega^-)(h^+ - h^- + \omega^+),$

$\omega_d^\varepsilon(x,t) = \left[\omega_{d_1}^\varepsilon(x,t)\right]^2 + \left[\omega_{d_2}^\varepsilon(x,t)\right]^2,$

$\omega_{d1}^\varepsilon(x,t) = (h^+ + h^- - \omega^+ + \omega^-) - t(h^+h^- - \omega^+\omega^-),$

$\omega_{d2}^\varepsilon(x,t) = (h^+ - h^- + \omega^+ + \omega^-) - t(h^+\omega^- + h^-\omega^+),$

$h^\pm(x,t) = (H\omega_0)(x \pm t\varepsilon),$

$\omega^\pm(x,t) = \omega_0(x \pm t\varepsilon)$

Once again note that if $\varepsilon = 0$, then $h^\pm = h$ and $\omega^\pm = \omega$, and we obtain,

$$\omega_n^0 = 2\omega(h^2 + \omega^2),$$

$$\omega_d^0 = \left[4 - 4th + t^2(h^2 + \omega^2)\right](h^2 + \omega^2),$$

hence

$$\omega^0 = \frac{4\omega}{(2 - th)^2 + t^2\omega^2},$$

which is the same as (9). Now the solution to (20) will blow up in finite time T_ε if $\omega_{d_1}^\varepsilon$ and $\omega_{d_2}^\varepsilon$ vanish simultaneously at some point x_0. This is difficult to check unless we know precisely what $h^\pm$ and $\omega^\pm$ are. Let us consider the example considered in CLM

$$\omega_0 = \cos x$$

then

$$\omega^\varepsilon(x,t) = \frac{4\alpha_\varepsilon(t)\cos x}{\left[2\alpha_\varepsilon(t) - t\sin x\right]^2 + t^2\cos^2 x}$$

where

$$\alpha_\varepsilon(t) = \sin \varepsilon t + \cos \varepsilon t$$

Figure 1 shows the case for $\omega_0 = \cos \pi x$ for $\varepsilon = 0.1$ at $t = 2$ and for $\varepsilon = 0.1$ and 0.001 in Fig. 2. Clearly the solution is finite at $x = \frac{1}{2}$ but as $\varepsilon \to 0$, the solution is about to blow up at $x = \frac{1}{2}$.

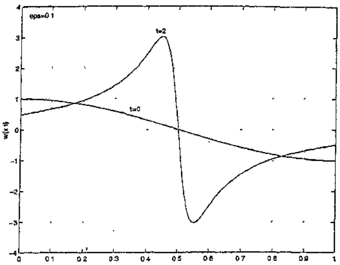

FIG. 1. Solution of the modified equation (16) at $t=0$ and 2 with $\omega_0 = \cos \pi x$ for $\varepsilon = 0.1$.

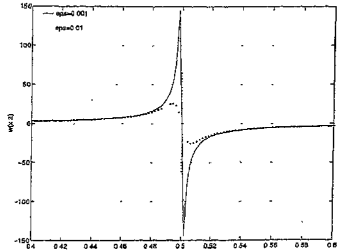

FIG. 2. Solution of the modified equation (16) at $t=2$ with $\omega_0 = \cos \pi x$ for $\varepsilon = 0.01$ and 0.001.

If $\varepsilon = 0$ the solution blows up at $x = \frac{1}{2}$ and $t = 2$. So for $\varepsilon > 0$ the solution blows up at T_ε and this is obtained by solving the nonlinear equation

$$2[\sin \varepsilon t + \cos \varepsilon t] - t = 0.$$

This function is shown in Fig. 3 and for $\varepsilon = 0.001$ T_ε is just above 2 which is the blow up point for $\varepsilon = 0$.

Since it is difficult to give an explicit value of T_ε where (20) blows up (as was shown above even in the explicit case of $\omega_0 = \cos \pi x$) we try to give an approximate answer by considering the Taylor expansion of (20) in powers of ε

$$\omega^\varepsilon_{d_i} = \omega^0_{di} + \varepsilon \frac{\partial \omega^\varepsilon_{d_i}}{\partial \varepsilon}|_{\varepsilon=0} + O(\varepsilon^2)$$

for $i = 1, 2$. This gives

$$\omega^\varepsilon_{d_1}(x,t) = 2h_0 - t(h_0^2 - \omega_0^2) - \varepsilon 2t\omega_0' + O(\varepsilon^2) \tag{21}$$

$$\omega^\varepsilon_{d_2}(x,t) = 2\omega_0\left[1 - th_0\right] - \varepsilon 2th_0' + O(\varepsilon^2). \tag{22}$$

We now assume the following on the initial data ε_0 and h_0. There exists an x_0 such that

$$\omega_0(x_0) = 0 \text{ and } h_0(x_0) > 0, \tag{23}$$

$$\omega_0'(x_0) < 0 \text{ and } h_0'(x_0) = 0 \tag{24}$$

Note that (23) is the same condition that is required for the inviscid solution (9) to blow up. Using (23)–(24) in (21)–(22) an approximate blow up time T_ε is obtained by requiring (21) to vanish up to $O(\varepsilon^2)$. This gives

$$T_\varepsilon = \frac{2h_0}{\left(h_0^2 + 2\varepsilon\omega_0'\right)}.$$

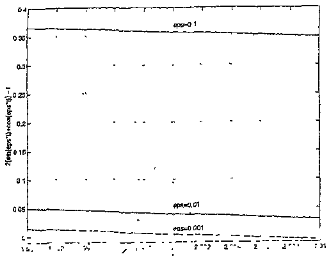

FIG. 3. The function $\alpha_\varepsilon(t) - t$ for various ε.

Since $\omega_0' < 0$ (cf. 24) we obtain

$$T_\varepsilon > T_0. \tag{25}$$

In the above example where $\omega_0 = \cos \pi x$ conditions (23)–(24) are satisfied; hence, $T_\varepsilon > T_0 = 2$ as is confirmed in Fig. 3.

To summarize, under conditions (23)–(24) on the initial data the viscous solution (20) blows up at time T_ε which satisfies (25). This argument is true up to $O(\varepsilon^2)$ and can be made rigorous by assuming that the solution is smooth in ε. A more systematic and complete study is required to characterize the precise conditions on ε_0 that is necessary for the blow up of the solution to (16) given by (20). This is being pursued and will be reported in Wegert and Vasudeva Murthy[14].

4. Conclusions

A new viscous version of the Constantin–Lax–Majda 1D model for the 3D vorticity has been proposed. The solution to this equation blows up at a later time than that of the inviscid case for the same initial data. This is in contrast to the canonical viscous version considered by Schochet.

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References

1. FRISCH, U. *Turbulence*, Camb. Univ. Press, 1995, p. 115.
2. GRAUER, R. AND SIDERIS, T. C. Finite time singularities in ideal fluids with swirl, *Physica D*, 1995, **88**, 116–132.

3. BEALE, J. T., KATO, T. AND MAJDA, A — Remarks on the breakdown of smooth solutions for the 3D Euler equations, *Commun. Math. Phys.*, 1984, **94**, 61–66.

4 CAFLISCH, R. E. — Singularity formation for complex solutions of the 3D incompressible Euler equations, *Physica D*, 1993, **67**, 1–18.

5. CONSTATIN, P , LAX, P. D. AND MAJDA, A. — A simple one-dimensional model for the three dimensional vorticity equation, *Commun. Pure Appl. Math.*, 1985, **38**, 715–724.

6. STEWART, K AND GEVECI, T — Numerical experiments with a nonlinear evolution equation which exhibits blow up, *Appl Num. Math.*, 1992, **10**, 139–147.

7. SCHOCHET, S. — Explicit solutions of the viscous model vorticity equation, *Comm. Pure Appl. Math.*, 1986, **39**, 531–537.

8. CONSTANTIN, P. — Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations, *Commun. Math. Phys.*, 1986, **104**, 311–326.

9. DE GREGORIO, S. — On a one-dimensional model for the three-dimensional vorticity equation, *J. Stat. Phys.*, 1990, **59**, 1251–1263.

10. DE GREGORIO, S — A partial differential equation arising in a 1D model for the 3D vorticity equation, *Math. Meth. Appl. Sci.*, 1996, **19**, 1233–1255.

11. WHITHAM, G. B — Variational methods and applications to water waves, *Proc. R. Soc. Lond. A*, 1967, **299**, 6–25.

12. NAUMKIN, P. I. AND SHISHMAREV, I. A. — *Nonlinear nonlocal equations in the theory of waves*, Translations of Math. Monographs 133. Am. Math. Soc., 1994.

13. MATSUNO, Y. — Pulse formation in a dissipative nonlinear system, *J Math. Phys.*, 1992, **33**, 3039–3045.

14. WEGERT, E. AND VASUDEVA MURTHY, A. S. — Blow-up in a modified Constantin-Lax-Majda model for the vorticity equation (under preparation).