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Some issues in dynamical systems

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Abstract

We consider some basic issues in the theory of dynamical systems. We assess the progress achieved by citing some landmark results with corresponding references. We briefly speculate on what could probably be unaccomplished major tasks for the future.

1. Introduction

The subject of dynamical systems (DS) is so vast and varied that it is impossible to address all the problems and questions treated in it. We thus propose to confine ourselves to a certain important corner of the subject and discuss some significant issues. The discussion centres around questions arising in various applied fields such as classical mechanics (CM), fluid mechanics (FM), economics, biology, electrical circuits, etc.

We start with a smooth manifold M which represents the state space of the system. In this paper, by a DS, we mean either a smooth diffeomorphism $f : M \to M$ or a vector field (vf) F on M. The evolution is then modelled by the iterates $\left\{f^n(x); n \in \mathbb{Z}\right\}$ or the flow $t \to X(t, x)$ generated by F on M, i.e. $dX/dt = F(X)$, $X(0, x) = x$. In the sequel, we will alternate between these two depending on the simplicity of the presentation. As an immediate example, one can take $M = \overline{D}$ where D is a smooth, bounded open set in $\mathbb{R}^d$ and F is a vf on $\overline{D}$ which is tangent to the boundary of D. The central problem of DS can be easily phrased as follows: what is the behaviour of flows as $t \to \infty$ or $n \to \infty$? To bring out the difference in qualitative properties, one must distinguish between the following cases: M finite or infinite dimensional, f invertible or not, dissipative or conservative systems. In the case of PDEs, we will have infinite dimensional manifold M.

2. Well-definedness of DS

To enable one to discuss the properties of dynamics, it is necessary that they be uniquely defined for all times. Existence proof is usually carried out in two steps: first one obtains local solution which can blow up in finite time. In order to prevent this, we try to establish a priori estimates. If we succeed, then local solution becomes global. For example, unique local solutions are assured if F is of class C^1. They become global if F is Lipschitz or M is compact. There are many examples of nonlinear PDEs where existence of suitable global solution is not known. One also comes across another kind of difficulty. In the important case of Navier–

Stokes equation (NSE) in three dimensions one can prove the existence of a suitable global weak solution, but it is not known whether it is unique or not. This question is related to the regularity of weak solutions, an issue which is not satisfactorily resolved in the case of many important nonlinear PDEs.

Let us point out another important direction which has received less attention; it is the case of singular vf. A classical example is the n-body problem. Another one arises if we wish to define the dynamics of dye carried by fluid flow. One idea[1] is to exploit the variational characterization of the trajectories. Another is to approximate singular vf by smooth vfs. In DiPerna and Lions[2], this has been successfully carried out with the notion of *almost everywhere* (ae) *flows*, i.e. the flow is defined omitting a null set of initial conditions (IC). As far as I know, this is the only work which follows the point of view of Poincaré and throws away naturally an exceptional set of 'non-representative' trajectories. Many qualitative aspects discussed below are yet to be investigated in the case of ae flows.

Nonlinear hyperbolic equations give birth to another type of singular vfs. Because of the appearance of shocks and other singularities in these equations, the corresponding vfs are not well understood. Nevertheless, let us mention that there is another concept of solution, named after Fillipov, associated with singular vfs. It is defined pointwise and is useful in hyperbolic equations.[3]

Because of these difficulties, in the sequel, we work with a compact manifold M without boundary and smooth vfs F. There are not many works which incorporate the effects of boundary and the behaviour at infinity.[4]

3. The main problem

The goal is to draw the *phase portrait* of the flow; especially, we are interested in its behaviour as $t \to \infty$. This fascinating subject has attracted the attention of several great mathematicians like Poincaré who investigated the stability of the solar system. Fluid turbulence is another model phenomenon. This problem remains essentially unsolved because of the rich and varied behaviour of DS. In olden days, resolution of ODEs was achieved by obtaining smooth *invariants*. This yielded positive results in cases now known as *completely integrable systems*. A major result in this area is Liouville's Theorem.[5] When it was realized that nonintegrable systems are the order of the day, other approaches were sought. There are five major methods.

(A) Geometric and topological point of view initiated by Poincaré,
(B) Statistical approach originating in the works of Boltzmann and Maxwell,
(C) Algebraic formulation of Koopman for which we refer to Arnold and Avez[6],
(D) Numerical integration, and
(E) Nonlinear functional analytical approach.

(A), though old, is dominant even today. After the invention of computers, method (D) has become powerful and enabled scientists to get insight into the behaviour of DS. In the context of fluid turbulence, it has been felt that (A) is helpful in understanding its onset while moderately excited regimes require new tools provided by (B). In fully developed turbulence, no method is found suitable except perhaps (D). Method (E) is relatively new and is proving very powerful in the analysis at large. It is especially very effective in the case of Hamiltonian systems when combined with other methods. The method invokes variational properties of the trajectories. In the sequel, we do not discuss this fast-growing approach except to give some important references.[1, 7–11] In the important field of infinite-dimensional DS, the techniques of nonlinear functional analysis are proving to be very useful whereas the other methods have their limitations.

Part A: Geometrical approach in finite-dimensional case

The classification problem: Whatever be the method of attack, the canonical programme towards the understanding of the dynamics seems to be the following:

STEP (1) Identify two dynamics with the same behaviour, i.e. define an equivalence relation in V, the set of vfs on M. Present the phase diagram inside each equivalence class. Also obtain a canonical form of the vf in each equivalence class. Define a vf to be *stable* if it has a neighbourhood of equivalents in V.

STEP (2) Prove that stable vfs are dense in V.

STEP (3) Characterize the stable classes in simple terms. Classify them in terms of invariants (algebraic, numerical, etc.).

STEP (4) Classify the unstable classes of codimension 1, 2, ...

STEP (5) Study the bifurcation at unstable classes and the nature of the unfolding.

Let me explain the above programme. Among the many equivalence relations used depending on the context, we single out *topological conjugacy* which is defined in terms of homeomorphism mapping the orbits onto themselves. Stability w.r.t. this equivalence is known as *structural stability*. Obviously, this is an important concept if our model is to represent the reality. If our model is not stable then we must be able to choose a perturbation that is. This is Step (2) which has remained a dream in the theory of DS. Our equivalence relation should be fine enough to distinguish things that are qualitatively different but sufficiently coarse to prove Step (2). Secondly, weaker the topology on V the more chance we have to establish Step (2). It is a common practice to use C^k-type topology on V but now there is a weaker topology which is found more natural in view of Diperna and Lions.[2]

The unstable elements, hopefully, will form submanifolds of finite codimension which are to be classified. The significance is that any r-parameter family of vfs inside V will generically cross only those submanifolds of codimension $\leq r$. At this crossing, one can expect bifurcation and new qualitative behaviour.

Physical and numerical experiments exhibit various 'stable' phenomena which are not stable structurally. In practice, one may thus run into difficulties with the concept of structural stability because we often have to deal with a restricted class of vfs in which case we will not accept arbitrary small perturbations inside V. Stability is thus to be understood in a broad sense.

In the classification problem it is always advocated to ignore vfs and orbits which are not representative. The genericity of a phenomenon is therefore usually evaluated from its validity on a large set in a topological sense (e.g. Baire set) or in a measure sense. It is true that there is no canonical measure on V, but, if we restrict our attention to vfs depending on a set of parameters we can take measure induced from the space of parameters.

In these questions of classification, smoothness of vfs plays an essential role. In particular, one can see in the literature different behaviours depending on whether a vf is C^1, C^2... C^∞, C^ω. Another related question of importance is the following: if a vf is smooth and it is equivalent to another, can one choose the equivalence map to be smooth? These questions are not answered in a general way because of the presence of *resonances.*[12]

Local classification: The classification problem is solved at each point of M in a satisfactory manner.[13, 14] We are, of course, interested in a global description which depends on the interaction between dynamics and the geometry of the manifold. The interest in the local description is that it brings out certain fundamental concepts which are then generalized to attack the global problem. One such notion is that of *equilibrium points*, i.e. points at which vf F vanishes. If it does not then the flow is equivalent to a *tubular flow*. In the local study it is natural to linearize the flow at an equilibrium point x^*. Stability of orbits then depends on the distribution of eigenvalues $\{\lambda_i\}$ of the linearized operator. One realizes the importance of the notion of *hyperbolicity* (i.e. Re $\lambda_i \neq 0\ \forall i$). Flow around x^* is then governed by *stable* and *unstable manifolds*: $W^s(x^*)$ and $W^u(x^*)$. x^* is a *saddle* if dim $W^s \neq 0 \neq \dim W^u$. Hyperbolicity is a generic property. A flow near its hyperbolic equilibrium point is equivalent to its linearization. Linear hyperbolic flows are essentially characterized by their indices ($= \dim W^u(x^*)$). At non-hyperbolic points, one has also to deal with the *centre manifold.*

Gradient vector fields: Here we refer to those systems where the vf $F = -\nabla g$ for some function g called potential. In this category, it is natural to consider perturbations of g rather than those of F. The resulting stability concept is different from structural stability. A function is stable if it has a finite number of critical points, each nondegenerate and having distinct critical values. Thom[15] has characterized the unstable function classes of codimension ≤ 4 in terms of singularities. They are just the elementary catastrophes! He also assumes that one can pass from the bifurcation of gradient dynamical systems to the unfolding of their potential functions in studying catastrophes. This is not entirely correct. The unfoldings of gradient dynamical systems can be of higher dimension than the unfoldings of their potentials.[16]

Two-dimensional flows: The landmark results characterizing two-dimensional flows are theorems of Andranov–Pontryagin, Poincaré–Bendixson and Peixoto. One of the new generic phenomena exhibited is that of (hyperbolic) *periodic orbits*, i.e. the ω and α limit sets can be only equilibrium or periodic orbits. Further, saddle connections are not allowed. For these results, see Palis and de Melo.[13] Though there are vfs which are not stable (e.g. a quasi-periodic motion on a 2-torus where the frequencies are independent over Q. The orbit is then dense.), the stable ones are dense. What about the characterization of stable vfs in terms of invariants and the classification of unstable ones? It is not clear whether these questions have been completely answered. See, however, Hale and Kocak.[17]

If, instead of two-dimensional flows, we consider diffeomorphisms on one-dimensional manifolds, say S^1, then *rotation number* allows us to classify the maps.[12]

Stable systems: In trying to generalize the above to general flows, one runs into enormous difficulties. This can be vaguely explained as follows: $2d$ flows correspond to maps in $1d$ and there is a natural ordering in $\mathbb{R}$ which can be exploited. In dimensions ≥ 3, there are other stable phenomena which will be described in this paragraph. Even after adding these, the density (Step (2)) is not true. The classification programme is a real challenge posed by nature to mathematicians. Efforts are now on to collect various possible behaviours and it is hoped that the density will be proved some day in future.

Smale[18] was one of the first to group various known examples and generalize them to higher dimensions. It is known for a long time that recurrence properties play an essential role in the study of asymptotic behaviour of a DS. Thus, *non-wandering* set Ω which includes equilibria and periodic orbits was introduced. Next, Morse–Smale systems were introduced where Ω consists of finitely many equilibria and periodic points (all hyperbolic). Next, the condition of 'no-saddle connection' is replaced by the *transversality* of stable and unstable manifolds of elements of Ω. We emphasize that the intersections of the stable and the unstable manifolds have to be preserved by any topological equivalence. It is therefore natural to require that these intersections be transversal since this will guarantee that they persist under small perturbations.

Even though M-S systems are stable,[13] they are far from being dense. World is not as simple as M-S systems. Indeed, a much richer structure was noticed by Poincaré himself at a *transverse homoclinic orbit.* In particular, the system was sensitive to initial conditions (SIC) near it. Such systems are called *chaotic*. Smale noticed three main mechanisms responsible for this effect: contraction, expansion and folding of state space volume by trajectories. Using these, he constructed his *horseshoe* where, in contrast to M-S systems, there are infinitely many periodic saddle points coexisting. Moreover, horseshoe is stable. It is worth remarking that the set of IC attracted in a horseshoe has measure zero if the system is of class C^2. However, there are C^1 horseshoe examples of positive measure.

Of course, there are other types of stable systems, e.g. *Anasov systems* where Ω is the entire manifold. Examples include the geodesic flows on a manifold with negative curvature.

Generalizing these objects, Smale introduced the notion of (uniformly) *hyperbolic sets* associated with flows/maps. They are compact invariant sets at every point of which there are contracting and expanding directions. Such sets are stable under perturbations of the map. *Hyperbolic systems* are the ones for which Ω is a hyperbolic set. A system is said to satisfy *Axiom-A* if it is hyperbolic and the set of periodic points is dense in Ω (which is true generically[19]). For such systems, Smale obtained the following satisfying picture[18]: there are finite number of *attractors* (compact invariant sets whose basin of attraction contains a neighbourhood of it). Basins put together cover a dense open subset of the manifold. Each attractor is transitive (it has a dense orbit) and is contained in Ω. Further, attractors which are not just fixed or periodic sink exhibit SIC. They are called *strange attractors*. Thus, Axiom-A systems can be decomposed into Anasov pieces assembled together somewhat like M-S case! This result can be viewed as a nonlinear analogue of the decomposition of the space in terms of generalized eigenvectors of a matrix.

The culmination of this circle of ideas is the following remarkable result of Mañé[20]: a diffeomorphism is C^1 structurally stable if it satisfies Axiom-A and all stable and unstable manifolds are transversal. Thus, we have a grand picture of structurally stable diffeomorphisms and their dynamics. The role of hyperbolicity in this cannot be overemphasized. The corresponding question for flows remains essentially unsolved.

From this analysis, it is clear that one must have efficient algorithms to find equilibria, periodic points, their stable and unstable manifolds, homo- and heteroclinic orbits and criteria to test their transversal intersection. Some tools are dynamical zeta function[18, 21] and Melnikov technique.[22] Many more are required.

Unstable systems: Having obtained a nice picture of stable systems, we might ask whether they are dense. It is known for a long time that they are not. Structural stability is thus of more limited significance than anticipated. The world of dynamics is very rich and fascinating. The classification of unstable systems and the resulting bifurcation is a problem that remains essentially unsolved. Attempts are being made to understand them by looking through the boundaries of stable ones. For instance, one may consider one parameter family of systems $\{F_\mu\}_{\mu\in\mathbb{R}}$ such that F_μ is stable for $\mu < 0$ and $\mu > 0$ and F_0 is not stable. One expects a different qualitative behaviour as $\mu \to 0$. To understand the situation, the concept of *attractors* is useful. At the bifurcation point $\mu = 0$ there is a change in the topology of the attractor. Multiparameter bifurcations are poorly understood.

In literature[23, 24] one can observe a long list of unstable situations. On one hand, one considers the cases where stable and unstable manifolds are not transversal or Ω loses hyperbolicity. On the other hand, there are scenarios obtained by Ruelle–Takens, Feigenbaum, Manneville–Pomeau, Hénon, etc. In each of these scenarios, not only a description of the attractors involved is presented but also unfoldings of them (i.e. the route which yields them) are also given. In Ruelle–Takens scenario, it is shown how a stationary point becomes unstable and gives rise to a periodic orbit via Hopf bifurcation. A 2-torus then appears through another bifurcation. If another instability occurs then typically a strange attractor appears instead of 3-torus. This is in sharp contrast to the picture projected by Landau and Hopf in the context of the onset of turbulence. Period doubling cascades occur as unfoldings of Feigenbaum attractor. What is surprising is that all these bifurcations are often really seen to follow each other and to converge asymptotically on a geometric sequence. In other words, in the space of maps of the interval there seems to exist a 'Feigenbaum manifold' of codimension 1 which is geometric limit of bifurcation manifolds corresponding to period doubling. In the intermittency route proposed by Manneville–Pomeau, the system oscillates in a regular fashion and is stable under small perturbations up to a critical value of the parameter appearing in the system. Beyond this critical value, the system exhibits abnormal fluctuations from time to time.

These systems are not structurally stable but are stable in some restricted sense. The big question is whether the union of these along with Axiom-A systems forms a dense subset in V. Are more phenomena to be included? There are several conjectures.

Nowadays, attention is focused on non-hyperbolic systems. Homoclinic bifurcation then becomes important and this can be obtained through homoclinic tangencies, for instance. The work of Newhouse[25] is pioneering in this context. Another important breakthrough is achieved to understand the Hénon map.[26] There is also a progress towards mathematical basis to explain Feigenbaum cascades and universality.[23] However, a lot remains to be done. Lorenz attractor is poorly understood,[27] and there are many conjectures.[24] Intensive research is on to prove them. Only time will tell if they are a success or a failure.

Even though several mechanisms producing instabilities are known, it is not clear whether a given system undergoes bifurcations when the parameters cross through critical values. It is an open problem whether a given model exhibits SIC. Bifurcations in the presence of symmetry is another vast area which we have not touched.[28, 29]

In dynamical problems, the following questions are usually raised and the answers are hard to obtain: how the trajectories are attracted towards the attractors, the rate of attraction, the nature of motion on the attractor, topology and geometry of attractors and their basins, etc. These things keep changing as parameters are varied and at the bifurcation point, one expects drastic changes. Mandelbrot[30] has been advocating *fractal geometry* to study attractors.

Another question that may be posed is the following: what happens to the dynamics under stochastic perturbations? In other words, we replace ODEs by stochastic differential equations and ask similar questions. There is also intense activity to generalize the above to delay differential equations.[31]

Part B: Statistical approach

As in Part A, we concentrate here on finite-dimensional DS. The geometric approach presented in Part A has enabled one to attack problems with a few degrees of freedom and thereby explain the onset of turbulence. There are difficulties with large degrees of freedom. For instance, fully developed turbulence is out of reach for the moment. However, there are physical models where only a moderate number of modes are excited, e.g. flame propagation and combustion problems. To understand such chaotic systems we require new tools provided by *ergodic theory* such as *dimensions, entropy* and *Lyapunov characteristic exponents*. Dimension represents the number of excited modes. The inverse of entropy quantifies the time up to which the state can be predicted with precision $0(\varepsilon)$ if IC is specified with tolerance ε. Characteristic exponents describe sensitivity to IC (SIC). In this approach, one deals with a measure μ invariant under the dynamics which replaces invariant sets of Part A. It is then natural to generalize hyperbolicity as follows: μ is *hyperbolic* if μ-almost all points are hyperbolic, i.e. characteristic exponents are non-zero μ ae. The goal of this approach is to prove that these quantities exist, discover the relations between them and use them to extract qualitative behaviour of DS. The theory is quite developed[5, 32–34] especially w.r.t. nonuniform hyperbolic attractors. A spectacular application of these tools will be pointed out in Part C. One of the major problems for the future is to know how the descriptions given in Parts A and B change when one takes, say thermodynamic limit, i.e. when the number of degrees of freeedom goes to infinity in a certain sense. Does it give a reasonable picture of continuous systems? What properties are lost in this passage? Reversibility? Another major difficulty is that there are too many measures invariant under the dynamics (e.g. Hénon map). Which one is the most relevant? In this context, SRB measures were introduced but proving their existence is a hard mathematical problem. Roughly, these measures represent the time spent by the orbits near the attractor. For Axiom-A systems, such measures exist and this is the content of the Bowen–Ruelle Theorem.

Most of the mathematical work in this approach has been restricted to either completely integrable or completely chaotic (ergodic) systems. Little work has been done in the case of intermediate systems which form the bulk of what is encountered in practice.

Pact C : Infinite-dimensional systems

There is a large and growing industry to extend whatever we have said about finite-dimensional systems in Parts A and B to the infinite-dimensional case, in particular, to the systems of nonlinear PDEs.[35–38] One of the impressive results is that the NS equation in two dimensions has a finite-dimensional attractor. The same is also true in three dimensions provided we assume the existence of a unique solution. The estimate on the number of degrees of freedom predicted by Kolmogorov theory of turbulence is thus recovered. We are not going to dwell on how this result is proved. We merely point out two radically new aspects in infinite dimensions of which little is known.

(i) We have been discussing about what are known as *temporal chaos*. In PDEs one can also have *spatial chaos*. Examples include flow past a sphere where chaos develops in the wake region. Similar situations arise in turbulent jets and plumes.

(ii) There is a possible occurrence of singularities in space. For instance, it has been conjectured that curl of the fluid velocity (obeying incompressible Euler equation) can become infinite in some parts of $I\!R^3$ at finite time. It has been proved that this set has to be small.[39] But one does not know whether this is empty or not. It is also conjectured that such a set is fractal. There is some numerical evidence supporting this. Another example is the appearance of shock waves. In these cases, the space in which dynamics takes place is to be so chosen as to include these singularities. Unfortunately, one then risks to lose the uniqueness of solution if the nature of the singularities is not properly understood. In many practical problems, this difficulty exists.

Part D: Numerical approach

The biggest question is how to do stable numerical computations in nonlinear systems which exhibit instabilities, bifurcation and SIC, and even if we can do, is there a basis to rely on them? In the hyperbolic case, there is *shadowing lemma*.[19] Non-hyperbolic situations should be looked into. If the system is governed by PDE and the dimension of the attractor is large (which is usually the case) then the power of present-day computers does not allow us to integrate the equations for long times. That is where the insight gained out of the theories developed in Parts A, B and C is going to be very useful. Long-time integration demands a good approximation of the attractor. When it has a complicated structure, this is not going to be easy. Thus was born the concept of *inertial manifold* which contains the attractor, is reasonably smooth and attracts orbits in an exponential way.[36] If N is the dimension of the attractor, it does not mean that the first N Fourier modes are sufficient to describe the motion. Because of the complicated geometry, the choice of the modes is subtle. Let us briefly indicate the ideas: split the unknown u into large and small 'eddies': $u = y + z$. Inertial manifolds are sought in the form $z = \Phi(y)$. Next, the idea is to project our equations onto this manifold. These projections can be done in various set-ups: finite-difference method (FDM), finite-element method (FEM), spectral method (SM) and wavelet method. In literature, one sees at least two ways of achieving *inertial projections*: nonlinear Galerkin method[40] using SM and incremental unknown method[41] using FDM. These are worked out and tested in only some examples. Much more remains to be done; for instance, one can employ wavelets here.

Numerical computations also offer some first-hand clues on the possible behaviour of dynamical systems. Lanford[42] has given computer-assisted proof of Feigenbaum's conjectures.

Part E: Dynamical system and some applications

The theory of dynamical systems plays an important role in computation and allied subjects, most notably in the area of 'analog' computation. There have been several interesting developments in this interface in recent years and this remains one of the most active areas of 'applications' of dynamical systems theory. Some of the notable topics are:

(1) *'Analog' algorithms*: Traditionally, these are continuous time, i.e. differential equation analogs of the classical algorithms for numerical analysis and optimization[43] because of the advances in analog device technology and the hope of embedding hard discrete algorithms into more tractable analog 'relaxations'. This, in turn, has spawned much mathematical activity of independent interest. Two 'high points' of this trend are:

(a) *Global Newton methods*: Originally studied as schemes for computing market equilibria in mathematical economics,[44] these have attracted much attention since. A related topic is the 'homotopy' method for optimization where one tracks the global minimum of a convex function to a 'good' local minimum of the function to be minimized as the former gets homotopically distorted into the latter.[45] This trajectory satisfies a differential equation similar to the global Newton method.

(b) *Brokett's double brackets*: These equations are of the type $\dot{x} = [x,[s,h]]$ on a Lie group and originally arose out of efforts to embed discrete optimization problems into continuous flows. They are also related to Karmarkar's interior point method.[46–48] These have led to much sophisticated mathematics of independent interest.[49, 50]

(2) *Complexity theory for analog computation*: Computational complexity theory for discrete computation based on the Turing machine formalism is a mature subject. Efforts are on to develop a continuous counterpart.[51, 52]

(3) *Neural networks*: Analog neural networks for classification problems provide interesting inverse problems.[53, 54] A related activity is a study of cooperative/competitive phenomena leading to self-organization or otherwise in interesting systems of differential equations. These are of interest to evolutionary biologists, economists and engineers in addition to mathematicians.[55–57]

(4) *Control theory*: The long-standing relationship between control theory and dynamical systems theory continues unabated, with some of the more exciting developments being the use of differential geometric techniques[58] and nonsmooth analysis.[59, 60]

(5) *Inverse problems*: In engineering sciences, one is not interested in chaos as such but in ways to control it. Indeed, by choosing properly the control parameters present in the system, we wish to have a prescribed behaviour. In other words, the attractor is given and one is required to produce a suitable and meaningful system whose behaviour is described by the given attractor.[61] Another related question is the compression of data which is represented by the attractor. The attractor is, the general, difficult to describe and store. If we can get hold of the

corresponding map/vf then life becomes easy. Research activities in this direction are in full swing.

(6) *Connection with nonlinear hyperbolic conservation laws*: A characteristic feature of these systems of equations is the appearance of shocks. Physically, these are limits of suitable viscous profiles as viscosity goes to zero. Finding these viscous profiles leads one to finding a heteroclinic orbit connecting the two states of a shock.[62]

Part F: Hamiltonian systems

Hamiltonian systems are special DS in which the vf is given in terms of a function H (called Hamiltonian) defined on the manifold. Celestial mechanics provides the first examples. Since H is a constant of motion, it is natural to restrict our attention to $M = \{H = \text{constant}\}$. The natural measure on this is preserved by the flow. Under suitable hypotheses, *Poincaré recurrence theorem* then shows that almost all points on this constant energy surface are non-wandering points. Of course, the big classical questions are to know statistical properties of the system (Part B): for example, whether a given system is ergodic on M; if not, can one obtain it at the thermodynamic limit? If so, it will justify the traditional apparatus of Gibbs ensemble in statistical physics of many particles.

In the case of Hamiltonian DS, it is customary to perturb the Hamiltonian and look for stable properties. Note that this is a restricted perturbation. Hence, we may expect new stable pheomena. Indeed, the new concept emerging is that of *elliptic equilibria* and *periodic orbits* whereas hyperbolicity is crucial in Part A. The celebrated KAM theory[63, 64] studies the effect of perturbations on a completely integrable system near an elliptic point.

Quasi-periodic motions are shown to be stable depending on how irrational their frequencies are. This is a surprising result establishing some unexpected connections with number theory. The fate of rational quasi-periodic motion is described by *Poincaré–Birkhoff theorem.* They break into 'island chains' with elliptic and hyperbolic points alternately placed. As in Part A, one can expect SIC near hyperbolic points. As the perturbation increases, Aubry and Mather have shown that even the irrational quasi-periodic motions disintegrate.[14] In higher dimensions, there is an additional phenomena called *Arnold diffusion.* There are many numerical experiments[8] which give a picture of possible instabilities and bifurcations.

In the context of numerical integration of Hamiltonian systems, let us mention that the usual algorithms do not work as they do not preserve the Hamiltonian nature of the system. Hence, special efficient algorithms are needed for long-term numerical studies. In this context, let us mention the Lie algebraic perturbation theory of Dragt–Finn.[65] See also Yoshida[66] and Sanz–Serna and Calvo.[67]

4. Conclusion

We have presented very rapidly some important phenomena occurring in dynamical systems. Various approaches to analyse them are outlined. Apart from highlighting the progress made so far, we have also pointed out the limitations of various approaches. Through this description, it is hoped to make clear the major remaining tasks to achieve further progress in the field. Undoubtedly infinite-dimensional dynamical systems constitute a major challenge of the future. To handle them, on one hand, various existing approaches will have to be generalized and strengthened and on the other, new approaches have to be discovered.

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References

1. AMBROSETTI, A. AND COTI ZELATI, V. *Periodic solutions of singular Lagrangian system*, Birkhäuser, 1993.
2. DIPERNA, R. AND LIONS, P. L. Ordinary differential equations, Sobolev spaces and transport theory, *Invent. Math.*, 1989, **98**, 511–547.
3. DAFERMOS, C. Characteristics in hyperbolic conservation laws, in *Nonlinear analysis and mechanics*, Vol. I, R. J. Knops, (ed.), Pitman, 1977.
4. LABARCA, R. AND PACIFICO, M. J. Stability of singular horseshoe, *Topology*, 1986, **25**, 337–352.
5. CORNFELD, I. P., FOMIN, S. V. AND SINAI, YA. G. *Ergodic theory*, Springer–Verlag, 1982.
6. ARNOLD, V. I. AND AVEZ, A. *Ergodic problems of classical mechanics*, W.A. Benjamin Inc., 1968.
7. EKELAND, I. *Convexity methods in Hamiltonian mechanics*, Springer–Verlag, 1990.
8. HÉNON, M. *Numerical exploration of Hamiltonian systems in chaotic behaviour of deterministic systems*, G. Iooss, R. H. G. Helleman, and R. Stora, (eds), North-Holland, 1983.
9. STRUWE, M. *Variational methods*, Springer–Verlag, 1990.
10. MAWHIN, J. AND WILLEM, M. *Critical point theory and Hamiltonian systems*, Springer–Verlag, 1989.
11. *ICTP Autumn School on Variational and Local Methods in the Study of Hamiltonian Systems*, 1994.
12. IRWIN, M. C. *Smooth dynamical systems*, Academic Press, 1987.
13. PALIS, J. AND DE MELO, W. *Geometric theory of dynamical systems*, Springer–Verlag, 1982.
14. ARROWSMITH, D. K. AND PLACE, C. M. *Dynamical systems*, Cambridge University Press, 1990.
15. THOM, R. *Structural stability and morphogenesis*, Benjamin-Addison Wesley, 1975.

16. GUCKENHEIMER, J. — Bifurcations and catastrophes, in *Dynamical systems*, M.M. Peixoto (ed.), Academic Press, 1973.

17. HALE, J. AND KOCAK, H. — *Dynamics and bifurcations*, Springer–Verlag, 1991.

18. SMALE, S. — Differential dynamical systems, *Bull. Am. Math. Soc.*, 1967, **73**, 747–817.

19. SHUB, M. — *Global stability of dynamical systems*, Springer–Verlag, 1987.

20. MAÑÉ, R. — A proof of the C^1 stability conjecture, *Publ. Math. Inst. Hautes Etud. Sci.*, 1988, **66**, 161–210.

21. RUELLE, D. — *Dynamical zeta function for piecewise monotone maps of the interval*, American Mathematical Society, 1994.

22. GUCKENHEIMER, J. AND HOLMES, P. — *Nonlinear oscillations in dynamical systems and bifurcation of vector fields*, Springer–Verlag, 1983.

23. RUELLE, D. — *Elements of differentiable dynamics and bifurcation theory*, Academic Press, 1989.

24. PALIS, J. AND TAKENS, F. — *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge University Press, 1993.

25. NEWHOUSE, S. — *Lectures on dynamical systems*, CIME Lectures, Italy, Birkhäuser, 1980.

26. BENEDICKS, M. AND CARLESON, L. — The dynamics of Hénon map, *Ann. Math.*, 1991, **133**, 73–169.

27. SPARROW, C. — *The Lorenz equations: Bifurcations, chaos and strange attractors*, Springer–Verlag, 1982.

28. GOLUBITSKY, M. AND SCHAFFER, D. G. — *Singularities and groups in bifurcation theory*, Vol. I, Springer–Verlag, 1985.

29. GOLUBITSKY, M., STEWART, I. AND SCHAFFER, D. G. — *Singularities and groups in bifurcation theory*, Vol. II, Springer–Verlag, 1988.

30. MANDELBROT, B. — *The fractal geometry of nature*, Freeman, 1982.

31. HALE, J., MAGALHÃES, L. T AND OLIVA, W. M. — *An introduction to infinite dimensional dynamical systems–geometric theory*, Springer–Verlag, 1984.

32. ECKMANN, J. P. AND RUELLE, D. — Ergodic theory of chaos and strange attractors, *Rev. Med. Phys.*, 1985, **57**, 617–656.

33. RUELLE, D. — *Chaotic evolution and strange attractors*, Cambridge University Press, 1989.

34. MAÑÉ, R. — *Ergodic theory and differentiable dynamics*, Springer–Verlag, 1987.

35. TEMAM, R. — *Infinite dimensional dynamical systems in mechanics and physics*, Springer–Verlag, 1988.

36. CONSTANTIN, P., FOIAS, C., NICOLAENKO, B. AND TEMAM, R. — *Integral manifolds and inertial manifolds for dissipative PDEs*, Springer–Verlag, 1989.

37. SELL, G., FOIAS, C. AND TEMAM, R. (EDS) — *Turbulence in fluid flows*, IMA, Vol. 55, Springer–Verlag, 1993.

38. HENRY, D. — *Geometric theory of semilinear parabolic equations*, *Lecture Notes in Mathematics* 840, Springer–Verlag, 1981.

39. CAFFARELLI, L., KOHN, R. AND NIRENBERG, L. — Partial regularity of suitable weak solutions of the N-S equations, *Commun. Pure Appl. Math.*, 1982, **35**, 771–831.

40. MARION, M. AND TEMAM, R. Nonlinear Galerkin Methods, *SIAM J Numerical Anal.*, 1989, **26**, 1139–1157.

41. TEMAM, R. Inertial manifolds and multigrid methods, *SIAM J. Math. Anal.*, 1990, **21**, 154–178.

42. LANFORD, O. E. A computer-assisted proof of the Feigenbaum conjectures, *Bull. Am. Math. Soc.*, 1982, **6**, 427–434.

43. CHU, T. MOODY On the continuous realization of iterative processes, *SIAM Rev.*, 1988, **30**, 375–387.

44. SMALE, S. A convergent process of price adjustment and global Newton methods, *J. Math Econ.*, 1976, **3**, 107–120.

45. GARCIA, C. B. AND ZANGWILL, W. I. *Pathways to solutions, fixed points and equilibria*, Prentice-Hall, 1981.

46. BROCKETT, R W. Dynamical systems that sort lists and solve linear programming problems, *Proc. 27th IEEE Conf. on Decision and Control*, 1988.

47. BLOCH, A.M. Steepest descent, linear programming and Hamiltonian flows, *Contemporary Math* , 1990, **114**, 77–88.

48. BROCKETT, R. W. Differential geometry and the design of gradient algorithms, *Proc. Symp. in Pure Math.*, Vol. 54, American Mathematical Society, 1993.

49. BLOCH, A. M., BROCKETT, R. W. AND RATIU, T. Completely integrable gradient flows, *Commun. Math. Phys.*, 1992, **147**, 57–74.

50. BLOCH, A. M., BROCKETT, R. W. AND RATIU, T. A new formulation of the generalized Toda lattice equations and their fixed point analysis via the moment map, *Bull. Am. Math. Soc.* (New Series), 1990, **23**, 477–485.

51. SIEGELMAN, H. AND SONTAG, E. *Neural networks with real weights: Analog computational complexity*, preprint, Rutgers Center for Systems and Control, 1992.

52. BLUM, L., SHUB, M. AND SMALE, S. On a theory of computation and complexity over the real numbers: NP completeness, recursive functions and universal machines, *Bull. Am. Math. Soc* , 1989, **21**, 1–46.

53 HIRSCH, M. Convergent activation dynamics in continuous time networks, *Neural Networks*, 1989, **2**, 331–349.

54. COHEN, M. A. The construction of arbitrary stable dynamics in nonlinear neural networks, *Neural Networks*, 1992, **5**, 83–103.

55. HOFBAUER, J. AND SIGMUND, K. *The theory of evolution and dynamical systems*, Cambridge University Press, 1984.

56. HIRSCH, M. Systems of differential equations that are competitive or cooperative I. Limit sets, *SIAM J. Math. Anal.*, 1982, **13**, 167–179.

57. HIRSCH, M. Systems of differential equations that are competitive or cooperative II. Convergence almost everywhere, *SIAM J. Math. Anal.*, 1985, **16**, 423–439.

58. ISIDORI, A. *Nonlinear control systems: an introduction, Lecture Notes in Control and Information Sciences*, No. 72, Springer–Verlag, 1985.

59. CLARKE, F. *Optimization and nonsmooth analysis*, Wiley, 1983.

60. AUBIN, J. P. *Viability theory*, Birkhauser, 1993.

61. LIONS, J. L. *Controllabilité exacte, perturbations et stabilisation de systèmes distribués*, Recherches en Mathematiques Appliquées (RMA) Masson, Paris, Vols 8, 9, 1988.

62. MAJDA, A. AND PEGO, R. L. Stable viscosity matrices for systems of conservation laws, *J. Differential Equations*, 1985, **56**, 229–262.

63. MOSER, J. *Stable and random motions in dynamical systems*, Princeton University Press, 1973.

64. ARNOLD, V. I. *Mathematical methods of classical mechanics*, Springer–Verlag, 1978.

65. DRAGT, A. J. AND FINN, J. M. Lie theories and invariant functions for analytic symplectic maps, *J. Math. Phys.*, 1976, **17**, 2215–2227.

66. YOSHIDA, H. Construction of higher order symplectic integrators, *Phys. Lett. A*, 1990, **150**, 262.

67. SANZ-SERNA, J. M. AND CALVO, M. P. *Numerical Hamiltonian problems*, Chapman-Hall, 1994.