Fractals and differential equations of fractional order

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Abstract

In this review, we consider the applications of fractional calculus to fractals. Some occurrences and applications of fractals are listed. We also give selected definitions of fractional derivatives and integrals. The review concentrates on works specifically applying fractional differential equations to fractals. These include diffusion on fractals, Lévy processes, continuous time random walks, etc. We also consider the recent notion of local fractional derivative and local fractional analog of Fokker–Planck equation.

1. Introduction

Fractal structures and processes[1] are omnipresent in the field of nonlinear and nonequilibrium phenomena. Fractals are used to model many objects in nature such as clouds, coastlines, rivers, etc. Fractals arise in the phase space of Hamiltonian systems. Percolating clusters are fractals. In turbulent fluid the energy is dissipated on the fractal set. Growing surfaces in nonequilibrium growth phenomena are modeled by fractals. Owing to frequent occurrences of fractals, various tools—numerical and analytical—are being developed to study such structures and processes. It is important to note that fractals, which are sets or objects having fractional dimension, are very irregular and the usual calculus fails to apply to such objects.

On the other hand, fractional calculus is an area of classical mathematics which deals with generalization of derivatives and integrals to arbitrary orders. Since ordinary calculus has been so useful in handling objects with integer dimension it is natural to ask if the formalism of fractional calculus can be used succesfully to deal with fractals. One of the early applications of fractional calculus to a fractal process can be traced down to the work of Mandelbrot and Van Ness.[2] Recently, there has been a surge of activity which probes this connection further. There have been many reports where fractional derivatives and integrals have been used to study fractals either directly or using equations involving them. In this review, we'll restrict our attention to those works where some kind of fractional differential equations (FDEs) have been used in connection with fractal structures or processes. Fractional calculus has also been found to be useful in many other areas which are not directly related to fractals. These include electrochemistry,[3,4] electrostatics,[5–7] acoustics[8], Navier–Stokes equation,[9,10] phase transitions[11, 12] and other topics.[13–16]

The plan of the review is as follows. In the next section we study various disciplines where fractals occur in science. It is planned to deal separately with two trends in fractional calculus, viz. the conventional fractional calculus (which is nonlocal) and the local fractional calculus. Accordingly, we introduce the conventional fractional calculus in Section 3 and the recent local

fractional calculus in Section 4. In this review, we focus on the applications of differential equations to fractals. So the next section, Section 5, deals with FDEs involving nonlocal fractional derivatives. Since fractional differential equations involving local fractional derivatives are fundamenatlly different from those involving nonlocal fractional derivatives, their discussion is postponed to Section 6. In Section 7, we make some concluding remarks.

2. Fractals

The concept of fractals has found innumerable applications in pure as well as in applied sciences. We do not go into various definitions of fractal dimension apart from saying that they are sets having noninteger dimension (see Mandebrot[1] and Falconer[17] for definitions). In the following we discuss some applications of fractals. More examples can be found in other references.[18-21]

2.1. *Fractal functions*

The graph of continuous but nowhere differentiable functions is known to have fractal dimension. Such functions occur naturally and abundantly in formulations of physical theories. The work on Brownian motion[17] showed that the graphs of projections of Brownian paths are nowhere differentiable and have dimension 3/2. A generalization of Brownian motion, called fractional Brownian motion,[1,17] is known to give rise to graphs having dimension between 1 and 2. It is also observed that typical Feynmann paths,[22, 23] like the Brownian paths, are continuous but nowhere differentiable.

2.2. *Turbulence*

In fluid systems, passive scalars advected by a turbulent fluid have been shown[24, 25] to have isoscalar surfaces which are highly irregular, in the limit of diffusion constant going to zero. Also points at which the energy is dissipated in the turbulent fluid forms a fractal set.

2.3. *Kinetic aggregation*

The diffusion-limited aggregation (DLA) model[21, 26] is a simple model of a fractal, generated by diffusion of particles in the following manner. One starts with a 'seed' particle which is fixed at a point. A second particle starts on the border of a circle around this point and performs a random walk. When it comes near the seed it sticks to it and a cluster of two particles is formed. Then a third particle performs a similar random walk and forms a cluster of three particles. This procedure is repeated many times, thereby giving rise to a large cluster having the shape of a fractal. DLA models many phenomena in nature, such as viscous fingering, dielectric breakdown, snowflake growth, etc.

2.4. *Biological systems*

There is abundance of fractal structures and processes in biological systems. Human lungs, branching of trees, root system in plants, etc. have a self-similar branching structure which is typical of fractals.[27] The above-mentioned DLA model has also been used to understand the shape of a neuron, growth of bacterial colonies, etc. Long-range correlations, which are believed to give rise to fractal geometry, are found in DNA sequences, human heart beats, etc.

2.5. *Percolation*

Consider a large square lattice where each site is occupied randomly with probability p and empty with probability $1 - p$. At low values of p, the occupied sites form small clusters. As one increases p there exists a threshold p_c at which a large (macroscopic) cluster appears and which connects opposite edges of the lattice. This is called the percolation cluster. This percolation cluster is self-similar in nature and hence has fractal structure.

3. Conventional fractional calculus

There are various ways in which an operator D^n can be constructed which gives derivative of order n for positive integer n and/or an integral of order n for negative integer n. Once one has such an operator it raises an interesting question if this operator can be generalized to all real values (or even complex) of the order. The branch of mathematics which deals with such a generalization is called fractional calculus.

Though the idea of such a calculus dates back to 1695, as evidenced in a letter by Leibniz to L'Hospital, there were confusions about the definition. It was only at the end of the last century that this confusion was cleared and an understanding of the connection among different definitions was obtained (see Miller and Ross[28] for a historical survey). In this section, we'll review some of the definitions of fractional derivatives and integrals and study some of their properties. Details can be found in Miller and Ross,[28] Oldham and Spanier[29] and Samko *et al.*[30]

3.1. *Definitions*

There are various definitions of derivatives and integrals of fractional order not necessarily equivalent to each other. All these definitions have different origin. The definitions introduced in this section have one thing in common, viz. they are nonlocal.

3.1.1. *Grunwald's definition*

The simplest definition can be given by generalizing the first-principle definition of derivative and integral, viz. difference quotients and Riemann sums, respectively. According to this definition, first given by Grunwald, the qth derivative of a real-valued function f of real-variable is given by

$$\frac{d^q f}{[d(x-a)]^q} = \lim_{N \to \infty} \left\{ \frac{\left[\frac{x-a}{N}\right]^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x - j\left[\frac{x-a}{N}\right]\right) \right\}, \tag{1}$$

where q, $a \in \mathbb{R}$ and $\Gamma(p)$ is the usual gamma function defined by the integral

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx, \quad p > 0, \tag{2}$$

and by the relation $\Gamma(p + 1) = p\Gamma(p)$. For $q > 0$ the above formula yields derivatives of order q, and for $q < 0$ it gives integrals of order q. It can be easily checked, using the properties of the gamma functions, that when $q = 1$ one gets back the first-principle definition of the derivative

of first order and when $q = -1$, eqn (1) reduces to the Riemann sum. The above definition involves the fewest restrictions on the functions to which it applies and, unlike some other definitions introduced below, does not use ordinary derivatives and integrals of the function. However, in practice it is somewhat difficult to use this definition, except in cases of simple functions.

3.1.2. *Riemann–Liouville definition*

The most frequently used definition of a fractional integral is that of Riemann–Liouville according to which a fractional integral of order q of a function f is given by

$$\frac{d^q f(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(-q)} \int_a^x \frac{f(y)}{(x-y)^{q+1}} dy \text{ for } q < 0, \tag{3}$$

where the lower limit a is some real number. The fractional derivative of order q is

$$\frac{d^q f(x)}{[d(x-a)]^q} = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_a^x \frac{f(y)}{(x-y)^{q-n+1}} dy \text{ for } n-1 < q < n \tag{4}$$

$$= \sum_{k=0}^{n-1} \frac{(x-a)^{-q+k} f^{(k)}(a)}{\Gamma(-q+k+1)} + \frac{d^{q-n} f^{(n)}}{[d(x-a)]^{q-n}} \text{ for } n-1 < q < n. \tag{5}$$

As is clear, this definition uses the concepts of ordinary derivatives and integration. Since it amounts to evaluating an integral, it is more convenient to use. It can be shown[29] that the Grunwald and Riemann–Liouville definitions are equivalent to each other.

3.1.3. *Weyl's definition*

Hermann Weyl defined fractional integral of order q of a function f by

$$\frac{d^q f(x)}{[d(x+\infty)]^q} = \frac{1}{\Gamma(-q)} \int_{-\infty}^x \frac{f(y)}{(x-y)^{q+1}} dy \text{ for } q < 0. \tag{6}$$

This definition is suitable for periodic functions as it leaves the periodicity of the function unaffected, unlike in the cases of previous two definitions. Because of this property, Zygmund[31] has used this definition extensively in working with trigonometric series.

3.1.4. *Other variants*

Generalization of Cauchy's integral: The Cauchy's integral formula for nth order derivative of a complex-valued function is given by

$$\frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \int_C \frac{f(\xi) d\xi}{(\xi - z)^{n+1}}, \tag{7}$$

where C is a closed contour surrounding point z and enclosing a region of analyticity of f. This formula can be generalized to give derivative of fractional order by replacing n by a real number q. Note that at the point $\xi = z$ the integrand no longer has a pole but a branch point and contour C cannot be deformed freely. The integral will depend on the point where C crosses the branch cut.

Definition arising from Fourier transforms: If we deonote by $\tilde{f}$ the Fourier transform of the function f then it is well known that $\tilde{f}^n = (ik)^n \tilde{f}$ where f^n is the nth derivative of f. This can be generalized to give a definition of the fractional order derivative by replacing integer n by a real number q. This definition is equivalent to the Weyl's definition introduced above.

There are many other variants introduced by various authors with a specific application in mind. It is not possible to list all the definitions over here but may be mentioned whenever needed in the following. We refer the reader to some books[28–30] for more definitions.

3.2. *Some properties and examples*

Many properties of the fractional integrals and derivatives, like chain rule, Liebniz rule, composition law, etc. have been studied.[29] Here we note one property which is useful in the context of scaling functions. When the argument of the function is scaled by a factor β, the differintegral (differentiation and integration of arbitrary order) satisfies

$$\frac{d^q f(\beta x)}{[dx]^q} = \beta^q \frac{d^q f(\beta x)}{[d(\beta x)]^q}.$$ (8)

For a more general formula, with nonzero lower limit a, see Oldham and Spanier.[29]

In general, it is difficult to evaluate a derivative or integral of fractional order except in a few cases. We consider one example below which may be needed later in the review. If we choose $f(x) = x^p$ then using Riemann–Liouville definition it can be shown that

$$\frac{d^q x^p}{dx^q} = \frac{\Gamma(p+1)}{\Gamma(p-q+1)} x^{p-q}, \, p > -1.$$ (9)

4. Local fractional calculus

Recently, a new notion called local fractional derivative (LFD) was introduced with the motivation of studying the local properties of fractal structures and processes. An interesting feature of the LFD is that it naturally appears in fractional Taylor expansion suitable for local approximations of scaling functions.

4.1. *Definitions*

The definitions of the fractional derivative were discussed in the last section. These derivatives differ in some aspects from integer-order derivatives. In order to see this, one may note, from eqn (1) or (4), that except when q is a positive integer, the qth derivative is nonlocal as it depends on the lower limit a. The same feature is also shown by other definitions. However, if

one wants to study local scaling properties then those definitions are not suitable and one has to modify them accordingly. Secondly, from eqn (9) it is clear that the fractional derivative of a constant function is not zero. Therefore, adding a constant to a function alters the value of the fractional derivative. Such a dependence on origin is again undesirable. While constructing the LFD operator, we have to correct for these two features. This forces one to choose the lower limit as well as the additive constant before hand. The most natural choices are as follows: (i) We subtract, from the function, the value of the function at the point where we want to study the local scaling property. This makes the value of the function zero at that point, canceling the effect of any constant term, and (ii) The natural choice of a lower limit will again be that point itself where we intend to examine the local scaling.

Definition 1. *If, for a function* $f:[0,1] \to \mathbb{R}$, *the limit*

$$\mathbb{D}^q f(y) = \lim_{x \to y} \frac{d^q(f(x)-f(y))}{d(x-y)^q}, 0 < q \le 1 \tag{10}$$

exists and is finite, then we say that the LFD of order q (denoted by $\mathbb{D}^q f(y)$), at y, exists.

This defines the LFD for $0 < q \le 1$. It was first introduced in Kolwankar and Gangal[32] and later generalized[33] to include all positive values of q as follows.

Definition 2. *If, for a function* $f: [0, 1] \to \mathbb{R}$, *the limit*

$$\mathbb{D}^q f(y) = \lim_{x \to y} \frac{d^q(f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(y)}{\Gamma(n+1)}(x-y)^n)}{[d(x-y)]^q} \tag{11}$$

exists and is finite, where N is the largest integer for which Nth derivative of f(x) at y exists and is finite, then we say that the LFD of order q ($N < q \le N + 1$), at x = y, exists.

We subtract the Taylor series term in the above definition for the same reason as one subtracts $f(y)$ in Definition 1. We do this to suppress any regular behavior that may mask the local singularity.

4.2. *Fractional Taylor expansion*

Following the usual procedure to derive Taylor expansion with a remainder[34] one arrives at the fractional Taylor expansion for $N < q \le N + 1$ (provided $\mathbb{D}^q$ exists), given by,

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(y)}{\Gamma(n+1)}(x-y)^n + \frac{\mathbb{D}^q f(y)}{\Gamma(q+1)}(x-y)^q + R_q(x,y) \tag{12}$$

where

$$R_q(x,y) = \frac{1}{\Gamma(q+1)} \int_0^{x-y} \frac{dF(y,t;q,N)}{dt}(x-y-t)^q \, dt \tag{13}$$

where

$$F(y, x - y; q, N) = \frac{d^q(f(x) - \sum_{n=0}^N \frac{f^{(n)}(y)}{\Gamma(n+1)}(x - y)^n)}{[d(x - y)]^q}. \tag{14}$$

We note that the LFD (not just fractional derivative) as defined above provides the coefficient A in the approximation of $f(x)$ by the function $f(y) + A(x - y)^q / \Gamma(q + 1)$, for $0 < q < 1$, in the vicinity of y. We further note that the terms on the RHS of eqn (12) are nontrivial and finite only in the case $q = \alpha$.

The existence of such Taylor expansion assigns a geometrical interpretation to the LFD. In order to see this note that when q is set equal to unity in eqn (12) one gets the equation of the tangent. It may be recalled that all the curves passing through a point y and having the same tangent form an equivalence class (which is modeled by a linear behavior). Analogously, all the functions (curves) with the same critical order α and the same $I\!D^\alpha$ will form an equivalence class modeled by x^α. This is how one may generalize the geometric interpretation of derivatives in terms of 'tangents'.

A fractional Taylor expansion was also given by Osler[35] which involved Riemann–Liouville definition. But this Taylor series is valid only for analytic functions. Further, it also contains terms with negative powers of $(x - y)$. Hence, this Taylor series is not useful for local approximations.

5. Fractional differential equations

Once we have a definition of fractional derivatives and integrals it is natural to ask a question if we can write equations in terms of such quantities. And will such equations have applications? In the last decade or so, many fractional differential equations have been proposed. Many of them are generalizations of the differential equations of mathematical physics. In these generalizations one replaces the usual integer order derivative by a fractional one. This replacement is either ad hoc or involves some plausible arguments. In this section, we review some of the works involving the applications of FDEs to fractals. The instances of FDEs not directly related to fractals include Wyss[36], Schneider and Wyss[37] and Jumarie.[38] There is also a prominent activity trying to generalize various relaxation equations to fractional orders.[39-43] Stability and controllability properties of fractional differential equations have been studied.[44-46] It should be pointed out that since fractional derivative can be written in terms of ordinary integral, the FDEs are nothing but integral equations.

5.1. Fractional diffusion equations

In recent years, studies of diffusion on fractals has attracted attention owing to possible applications in various phenomena such as diffusion in porous media, adsorption kinetics across interfaces, etc. The probability distribution $P(r, t)$, which gives the probability to find the random walker at time t at distance r from its starting point at $t = 0$, is given by

$$P(r, t) \sim \frac{1}{R^{d_f}} \exp\left[-\text{const.} (r / R)^u\right] \tag{15}$$

when $t \to \infty$ and $r/R \to \infty$, where d_f is a fractal dimension. In this equation, $u = d_\omega/(d_\omega - 1)$ where $d_\omega > 2$ is a diffusion exponent, i.e. the root-mean-square displacement of the random walk $R(t) \sim t^{1/d_\omega}$. The form of $P(r, t)$ above is known as stretched gaussian.

It is of natural interest to formulate a suitable diffusion equation to describe diffusion on fractals. Earlier attempts to modify the diffusion constant in the usual diffusion equation did not yield expected results. Another way is to make orders of derivatives in the diffusion equation noninteger. First such attempt was perhaps by Le Mehaute.[47] Giona and Roman[48–50] also considered the modification of the diffusion equation by replacing the integer order derivatives by those of fractional order. Thus, they postulated a fractional diffusion equation

$$\frac{\partial^{1/d_w} P(r,t)}{\partial t^{1/d_\omega}} = -G\left(\frac{\partial P(r,t)}{\partial r} + \frac{\kappa}{r} P(r,t)\right), \tag{16}$$

where $G > 0$ and κ is argued to be equal to $(d_f - 1)/2$. The solution of this equation results in expected behavior asymptotically. There have been other works which relate the fractional diffusion equations to continuous time random walk.[51–53]

5.2. *Fractional Fokker–Planck–Kolmogorov equation*

The purpose of this section is to review the work of Zaslavsky and coworkers.[54–57] Zaslavsky argued that the trajectory of the weakly chaotic Hamiltonian system in the phase space can be described by a fractional generalization of the Fokker–Planck–Kolmogorov (also known as Fokker–Planck or forward Kolmogorov) equation. In the case of strongly chaotic Hamiltonian system the transport in the phase space can be described by usual FPK equation; this description fails in the case of incomplete chaos owing to the presence of cantori. Cantori are fractal structures of invariant tori which, due to their stickiness, change the kinetics to something called strange kinetics.[58] Therefore, a new kinetic equation is needed to describe the transport properties in such a phase space.

He starts with a Chapman–Kolmogorov consistency condition for Markov processes given by

$$W(x, t+\tau) = \int P(x, t+\tau | x', t) W(x', t) dx' \tag{17}$$

where $W(x, t)$ is a probability density at time t given that $W(x_0, 0) = \delta(x_0)$, $P(x_1, t_1 | x_2, t_2)$ is a transition probability from x_1 at time t_1 to x_2 at time t_2 and $\tau \geq 0$. Different notations, P and W, have been used to emphasize the fact that the former is the transition probability for small times compared to the latter. Then, assuming the expansions

$$\lim_{\tau \to 0} \frac{1}{\tau^\beta} \{W(x, t+\tau) - W(x,t)\} = \frac{\partial^\beta W(x,t)}{\partial t^\beta}, \tag{18}$$

$$P(x, t+\tau | x', t) = \delta(x-x') + A(x', \tau)\delta^{(\alpha)}(x-x') + \frac{1}{2} B(x, \tau)\delta^{(2\alpha)}(x-x'), \tag{19}$$

where $0 < \alpha, \beta \leq 1$, and following the usual procedure[59] to derive the Fokker–Planck equation one arrives at the equation

$$\frac{\partial^{\beta} W(x,t)}{\partial t^{\beta}} = \frac{\partial^{\alpha}(\mathcal{A}W(x,t))}{\partial(-x)^{\alpha}} + \frac{1}{2}\frac{\partial^{2\alpha}(\mathcal{B}W(x,t))}{\partial(-x)^{2\alpha}}, \tag{20}$$

where

$$\mathcal{A} = \lim_{\tau \to 0} \frac{A(x,\tau)}{\tau^{\beta}} = \lim_{\tau \to 0} \frac{<|x|^{\alpha}>}{\tau^{\beta}} \tag{21}$$

and

$$\mathcal{B} = \lim_{\tau \to 0} \frac{B(x,\tau)}{\tau^{\beta}} = \lim_{\tau \to 0} \frac{<|x|^{2\alpha}>}{\tau^{\beta}}. \tag{22}$$

The $<>$ in the above equation denotes average.

5.3. Lévy processes and FDEs

Lévy processes have found many applications[58,60] ranging from fluid dynamics[61,62] to polymers.[63] Since Lévy processes do not possess second moment the derivation of the Fokker–Planck equation fails and one has to look for alternative formulations of kinetic equations for their description. In this section, we first give a brief introduction to the Lévy processes and then review various attempts to derive fractional differential equations to describe Lévy processes.

5.3.1. Introduction to Lévy processes

For translationally invariant stationary Markov process the transition probability $P(x, t + \tau|x', t)$ in eqn (17) can be replaced by $P(x - x', \tau)$ since it depends only on the difference between the initial and final positions and time only. If the characterstic function $\phi(k, t)$ is defined as the Fourier transform of the probability density,

$$\phi(k,t) = \int_{-\infty}^{\infty} dx e^{ikx} P(x,t), \tag{23}$$

then eqn (17) becomes

$$\phi(k, t + \tau) = \phi(k, \tau)\phi(k, t). \tag{24}$$

The most general form of the solution of the above equation is given by (see Seshadri and West[64] and references therein)

$$\phi(k,t) = \exp\left[-bt|k|^{\mu}\left(1 + ic\omega(k,\mu)\frac{k}{|k|}\right)\right] \tag{25}$$

where $0 \leq \mu \leq 2$, $b \geq 0$ and $-1 \leq c \leq 1$. The function $\omega(k, \mu)$ is defined by

$$\omega(k, \mu) = \tan(\pi\mu/2) \text{ if } \mu \neq 1 \tag{26}$$

$$= \frac{2}{\pi} \ln| \, kbt| \ \text{ if } \ \mu = 1. \tag{27}$$

The processes whose characteristic functions satisfy eqn (25) are called Lévy processes.

The important feature of these processes is that except for $\mu = 2$ they do not possess finite moments of all orders. All the α moments defined by

$$<|x|^{\alpha}> = \int_{-\infty}^{\infty} |x|^{\alpha} P(x,t) dx \tag{28}$$

are finite for $\alpha < \mu$ and are infinite for $\alpha \geq \mu$. The asymptotic behavior of the Lévy distribution is given by

$$P(x,t) \sim x^{-1-\mu} \ x \text{ large.} \tag{29}$$

If we consider a random walk in which the step size is drawn from a Lévy distribution then the trace of sites visited by the walker is a fractal with dimension μ.

As mentioned above, since the second moment of the Lévy distribution does not exist one cannot write a simple evolution equation of the diffusion type for the probability densities. However, it is known[64] that the Lévy distribution satisfies an integral equation of the form

$$\frac{\partial P(x,t)}{\partial t} = \frac{b}{\pi} \sin(\pi\mu/2)\Gamma(\mu+1)\int_{-\infty}^{\infty} dy \{1 + c\,\text{sign}(y-x)\} \frac{P(y,t)}{|x-y|^{\mu+1}}. \tag{30}$$

It is also known[64] that when μ is rational, differential evolution equations can be obtained, with some specific values of c, for the probability density involving higher time derivatives. Thus, when $c = 0$ and $\mu = m/n$, $P(x,t)$ satisfies

$$\frac{\partial^n P(x,t)}{\partial t^n} = (-1)^{n+m/2} b^n \frac{\partial^m P(x,t)}{\partial x^m}, m \text{ even} \tag{31}$$

and

$$\frac{\partial^{2n} P(x,t)}{\partial t^{2n}} = (-1)^m b^{2n} \frac{\partial^{2m} P(x,t)}{\partial x^{2m}}, m \text{ odd.} \tag{32}$$

These observations give a hint for developing fractional differential equations for Lévy distributions. Now that many applications of Lévy processes have been found it becomes even more interesting to find such equations.

5.3.2. Fractional differential equations in relation to Lévy processes

Lévy processes and the corresponding fractional differential equations have been the topic of many recent investigations.[40, 65, 66] Nonnenmacher[40] has shown that the class of one-sided Lévy type probability densities

$$P(x) = \frac{a^{\mu}}{\Gamma(\mu)} x^{-\mu-1} \exp(-a/x) \quad a > 0 \ \ x > 0 \tag{33}$$

is a solution of the fractional integral equation

$$x^{2\mu} P(x) = a^{\mu} \frac{d^{-\mu} P(x)}{dx^{-\mu}}, \tag{34}$$

with Riemann–Liouville definition on the right-hand side. An interesting point of this observation is that the Lévy index appears as an order of the fractional integration in the equation which is also the fractal dimension of the set of points visited by the random walker with eqn (33) as transition probability.

It is clear that eqn (30) can be immediately put formally in the form of fractional differential equation if we use the Weyl definition of the fractional derivative to give

$$\frac{\partial P(x,t)}{\partial t} = -\frac{b}{2\cos(\pi\mu/2)} \left[\frac{d^{\mu} P(x,t)}{[d(x+\infty)]^{\mu}} + \frac{d^{\mu} P(x,t)}{[d(-x+\infty)]^{\mu}} \right]. \tag{35}$$

Getting such an equation, following a different approach, is the aim of Chaves.[66]

In another interesting development, West et al.[65] derive a fractional differential equation for the reduced probability density $\sigma_0(x, t)$ of the variable $x(t)$ in the stochastic differential equation $\dot{x}(t) = \varsigma(t)$. They choose a power-law waiting-time distribution as in eqn (29) and hence power-law behavior in correlation function and arrive at the equation

$$\frac{\partial \sigma_0(x,t)}{\partial t} = C \frac{\partial^2}{\partial x^2} \frac{\partial^{\beta} \sigma_0(x,t)}{\partial t^{\beta}}. \tag{36}$$

The equation is solved giving rise to superdiffusive behavior and with suitable constraint it reduces to the integral eqn (30) for the Lévy processes.

6. Local fractional differential equations

In section 4, we have reviewed the definition of LFD which generalizes usual derivatives to fractional order keeping their local nature intact. In this section, we consider equations[67] in terms of the LFDs called local fractional differential equations (LFDE). These are new kind of equations and unlike FDEs considered in the last section are not just integral equations.

6.1. A simple local fractional differential equation

In order to understand the meaning of these equations we consider a simple equation

$$\mathbb{D}_x^{\alpha} f(x) = g(x). \tag{37}$$

We note[67] that the above equation with $g(x) = \text{const.}$ does not have a finite solution when $0 < q < 1$. Interestingly, the solutions to (37) can exist, when $g(x)$ has a fractal support. For instance, when $g(x) = \chi_C(x)$, the membership function of a cantor set C (i.e. $g(x) = 1$ if x is in C and $g(x) = 0$ otherwise), the solution with initial condition $f(0) = 0$ exists if $q = \alpha \equiv \dim_H C$. Explicitly, generalizing the Riemann integration procedure,

$$f(x) \equiv \frac{P_C(x)}{\Gamma(\alpha+1)} = \lim_{N \to \infty} \sum_{i=0}^{N-1} \frac{(x_{i+1} - x_i)^{\alpha}}{\Gamma(\alpha+1)} F_C^i \tag{38}$$

where x_i are subdivision points of the interval $[x_0 = 0, \ x_N = x]$ and F_C^i is a flag function which takes value 1 if the interval $[x_i, x_{i+1}]$ contains a point of the set C and 0, otherwise. Note that $P_C(x)$ is a Lebesgue–Cantor (staircase) function and satisfies the bounds $ax^{\alpha} \leq P_C(x) \leq bx^{\alpha}$ where a and b are suitable positive constants. The above procedure of integration works only when the box dimension of C is the same as that of Hausdorff dimension.

6.2. *Local fractional Fokker–Planck equation*

Now we again follow the usual procedure[59] to derive the Fokker–Planck equation. But in place of the Taylor expansion in the usual procedure and instead of expansions (18) and (19) in Section 5.2 we use the fractional Taylor expansion (12). This leads us to an equation

$$\mathbb{D}_t^{\alpha} W(x,t) = \mathcal{L}(x,t) W(x,t) \tag{39}$$

where the operator $\mathcal{L}$ is given by

$$\mathcal{L}(x,t) = \mathbb{D}_{x-}^{\beta} A_{\alpha-}^{\beta}(x,t) + \mathbb{D}_{x+}^{\beta} A_{\alpha+}^{\beta}(x,t), \quad 0 < \beta \leq 1$$

and in the latter case we get

$$\mathcal{L}(x,t) = -\frac{\partial}{\partial x} A_{\alpha}^{1}(x,t) + \mathbb{D}_{x-}^{\beta} A_{\alpha-}^{\beta}(x,t) + \mathbb{D}_{x+}^{\beta} A_{\alpha+}^{\beta}(x,t) \quad 1 < \beta \leq 2$$

where

$$A_{\alpha\mp}^{\beta}(x,t) = \lim_{\tau \to 0} \frac{M_{\beta}^{\pm}(x,t,\tau) \Gamma(\alpha+1)}{\tau^{\alpha} \Gamma(\beta+1)}$$

and

$$A_{\alpha}^{\beta}(x,t) = A_{\alpha+}^{\beta}(x,t) + A_{\alpha-}^{\beta}(x,t).$$

Here, corresponding A_{α}s are assumed to exist. The $M_{\beta}^{\pm}(x,t,\tau)$ in the above equation are transitional moments defined by

$$M_{\alpha}^{+}(x,t,\tau) = \int_x^{\infty} dy (y-x)^{\alpha} P(y, t+\tau | x, t) \, a > 0,$$

$$M_{\alpha}^{-}(x,t,\tau) = \int_{-\infty}^{x} dy (x-y)^{\alpha} P(y, t+\tau | x, t) \, a > 0$$

and

$$M_{\alpha}(x,t,\tau) = M_{\alpha}^{+}(x,t,\tau) + M_{\alpha}^{-}(x,t,\tau).$$

Equation (39) can be identified as generalizations of the Fokker–Planck equation in one space variable. It is clear that when $\alpha = 1$ and $\beta = 2$, we get back the usual Fokker–Planck operator.

Now, we consider one specific example of the transition probability and study the corresponding LFFP equation. Let,

$$P(x, t + \tau | x', t) = \frac{1}{\sqrt{\pi \Delta P_C(t, \tau)}} e^{\frac{-(x - x')^2}{\Delta P_C(t, \tau)}} \tag{40}$$

$$= \delta(x - x') \quad \text{if} \ \Delta P_C(t, \tau) = 0, \tag{41}$$

where $\Delta P_C(t, \tau) = P_C(t + \tau) - P_C(t)$. This transition probability describes a nonstationary process which corresponds to transitions occurring only at times which lie on a fractal set. Such a transition probability can be used to model phenomenon where transition is very rare; for instance, diffusion in the presence of traps.

This gives us the following local fractional Fokker–Planck equation (in this case an analog of a diffusion equation).

$$D_t^\alpha W(x, t) = \frac{\Gamma(\alpha + 1)}{4} \chi_C(t) \frac{\partial^2}{\partial x^2} W(x, t). \tag{42}$$

We note that even though the variable t is taking all real positive values the actual evolution takes place only for values of t in the fractal set C. The solution of eqn (42) can easily be obtained as

$$W(x, t) = P_{t - t_0} W(x, t_0) \tag{43}$$

where

$$P_{t - t_0} = \lim_{N \to \infty} \prod_{i=0}^{N-1} \left[1 + \frac{1}{4} (t_{i+1} - t_i)^\alpha F_C^i \frac{\partial^2}{\partial x^2} \right]. \tag{44}$$

Therefore,

$$W(x, t) = \frac{1}{\sqrt{\pi P_C(t)}} e^{\frac{-x^2}{P_C(t)}}. \tag{45}$$

Its consistency can easily be checked by substituting this in Chapman–Kolmogorov equation. This solution satisfies the bounds

$$\frac{1}{\sqrt{\pi b t^\alpha}} e^{\frac{-x^2}{a t^\alpha}} \leq W(x, t) \leq \frac{1}{\sqrt{\pi a t^\alpha}} e^{\frac{-x^2}{b t^\alpha}} \tag{46}$$

for some $0 < a < b$. This is a model solution of a subdiffusive behavior. It is clear that when $\alpha = 1$, we get back the typical solution of the ordinary diffusion equation which is $(\pi t)^{-1/2} \exp(-x^2/t)$.

7. Conclusions

The fractional calculus, a generalization of derivatives and integrals to arbitrary orders, has found many applications in various fields of science. We have reviewed some of the applications of fractional calculus to fractals. Though the first such application appeared in 1968 in the work of Mandelbrot and Van Ness,[2] most of the activity in this direction is not even a decade old. The approach followed by most of the workers in this field is to formulate some fractional differential equation to deal with phenomena related to fractal structures and processes such as Lévy processes, diffusion on fractals, etc. Different authors have used various definitions of fractional derivatives which are nonlocal in nature. Ways to solve these equations have been developed (see, for example, West et al.[65], Glöckle and Nonnemacher[68] and Metzler et al.[69]) which essentially give rise to Fox functions[70] as solutions. The solution of the fractional differential equation should display asymptotic scaling behavior. Therefore, such equations naturally become useful whenever asymptotic scaling is involved; for instance, waiting time distribution in fractal time process, distribution function of Lévy processes, etc. Owing to the nonlocal nature of the definition of the fractional derivatives, they are not suitable to study local scaling properties. Therefore, a suitable local definition of fractional derivative was introduced by modifying a nonlocal definition. Again, the differential equations involving this local fractional derivative have been considered and solved. Such an equation is fundamentally different from the ones discussed above since this equation is intrinsically local. The use of these kinds of equations has been demonstrated by deriving a local fractional analog of the Fokker–Planck equation and considering the example of diffusion in fractal time. Various works like path integral formulation of fractional Brownian motion,[71, 73] fractal time random walks,[74-77] fractional dynamics,[78, 79] apply fractional calculus to fractals.

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