J Indian Inst Sci, Sept–Oct 1998, **78**, 311–372

Differential–difference Kadomtsev–Petviashvili equation: properties and integrability

K. M TAMIZHMANI AND S. KANAGA VEL
Department of Mathematics, Pondicherry University, Pondicherry 605 014, India
email tamizh@yahoo com

Received on October 5, 1998

Abstract

We present a review on certain integrability properties of differential–difference Kadomtsev–Petviashvili (DΔKP) equation We explain the differential–difference version of Sato theory and derive the DΔKP equation as a first nontrivial member in the single-component KP family In this process, we exploit the Sato theory to obtain conservation laws and generalised symmetries of the same We further show that the Wronskian form of the N-soliton solutions and rational solutions follow naturally from this approach Similarity reduction and Painlevé-singularity confinement analysis are performed We also discuss a gauge equivalence of the DΔKP equation and study certain integrability properties of the resulting system as well

Keywords: Integrability, Sato theory, nonlinear systems

1. Introduction

Modern nonlinear dynamics started with a fundamental question—when is the given model integrable? Can one define it precisely? The answer is obviously no. But, at the same time, we try to find a close definition of integrability with the meaning of integrating and find the solution/existence theory like Liouville.[1, 2] In contrast to the linear theory it is well known that there is no general theory to handle nonlinear systems and get their solutions However, the present status is not bad, thanks to the discovery of Inverse Scattering Transform (IST) method[3, 4] through which a large class of nonlinear partial differential equations (NPDEs) had been solved and special solutions called solitons obtained At first, Zabusky and Kruskal observed soliton solution, numerically in Korteweg–de Vries (KdV) equation.[5] Gardner *et al.*[6] developed IST method to solve the initial value problem for the KdV equation This discovery becomes vital since, after KdV equation was solved explicitly, many more nonlinear systems of physical importance were also treated with IST method [3,4] Again, this method was generalized to matrix formalism, notably by Zakharov and Shabat (ZS)[7] and then by Ablowitz, Kaup, Newell and Segur (AKNS) [8] These developments extended the applicability of IST to handle equations with complex potentials and coupled systems as well Lax first reformulated these settings of IST in terms of linear operators, now popularly called Lax pair or L–M pair.[9] These operators satisfy certain linear eigenvalue problem Obtaining a suitable L–M pair for a given nonlinear system is equivalent to say that the given NPDE has been linearized and thus solvability is feasible using IST Those equations solvable through IST are called integrable systems and they possess soliton solutions, in fact N of them. After these pioneering works, many mathematical tools such as Hirota's bilinear formalism,[10–42] Painlevé tests,[43–52] recursion operator, Lie–Backlund symmetries and others[53–73] were developed to identify the integrable systems and study further analytical and algebraic properties of them

In the case of infinite dimensional systems (PDEs), the system is considered integrable (working definition) if it satisfies one of the following criteria.

1 The system is linearized through suitable variable transformation.
2. The system is solvable through IST method,[3] after finding suitable Lax pair or eigenvalue problems.
3 The system possesses infinite number of conserved quantities
4 In view of the symmetry approach, "An equation is integrable if it possesses infinitely many time-independent non-Lie point symmetries" These symmetries are called generalized symmetries or Lie–Backlund transformations.[66]
5 The system is bi- or trilinearizable through suitable dependent variable transformations and admits N-soliton solutions.
6. The system which passes the so-called Painlevé test is a good candidate of an integrable system In this connection, Ablowitz, Ramani and Segur (ARS) formulated the Painlevé conjecture, "Every ordinary differential equation which arises as a reduction of a completely integrable system is of Painlevé type (perhaps after a transformation of variables)".[44] This conjecture provided a most useful integrability detector. Following this conjecture, ARS proposed an algorithm which tests the Painlevé property This property states that a system of ODEs satisfies the necessary condition for the Painlevé property, i.e having no movable critical points other than poles, if all its solutions can be expanded in the Laurent series near every one of their movable singularities This test was extended by Weiss *et al*[47] to PDEs

In the following, we briefly discuss various methods used to study the integrable systems

1.2. *Lax method*

The idea behind this approach is to derive the nonlinear evolution equation which arises as the compatibility condition of two linear equations associated with a spectral problem The spectral problem and the time evolution of the eigenfunction are given by the equations[9]

$$L\psi = \lambda\psi \tag{1}$$

and

$$\frac{\partial\psi}{\partial t} = M\psi, \tag{2}$$

respectively Assume that the spectral parameter λ is independent of t The compatibility condition of these two equations gives us

$$\frac{\partial L}{\partial t} = ML - LM = [M, L] \tag{3}$$

where L and M are linear operators. Equation (3) is called Lax equation. Lax[9] explains how to derive operator M for given L. Once this goal is achieved we can find the solution of the given nonlinear equation through IST method. Numerous generalizations of the eigenvalue problems such as Zakharov and Shabat,[7] Ablowitz, *et al.*,[8] Ablowitz and Haberman,[74] Kaup and Newell,[75] Wadati, *et al.*,[76, 77] Shimizu and Wadati,[78] Ishimori,[79,80] Wadati and Sogo,[81] Konno and Jeffrey [82] are available in literature covering a wide range of evolution equations.

1.3 *Hirota's bilinear method*

Hirota introduced a more direct method[10–42] to derive soliton solutions of nonlinear equations. By introducing a suitable dependent variable transformation, the soliton equation becomes bilinear. Applying the perturbation technique to the resulting bilinear equation one can systematically construct the N-soliton solution. In fact, if the system admits N-soliton solution the series expansion in terms of small parameter in the above perturbative analysis truncates automatically. In this process, we obtain a class of linear partial differential equations which can be solved successively to obtain the exact solution. The soliton solution obtained through this technique is a polynomial in exponential functions. It is also noted that the N-soliton solutions of Hirota's bilinear equation can be written in the form of Wronskian and Grammion determinants and Pfaffians.[21–23, 28–36] The latter formalism is more compact and easy to handle. Apart from finding N-soliton solution using this technique, we can also obtain the rational solutions in a direct way.[83–96] Recently, bilinear approach has been extended to multilinear form, in particular to trilinear case.[97, 98] Many interesting integrable systems have been brought in this framework. In some cases, the trilinear forms can be written in bilinear forms by introducing extra τ-functions.

1.4 *Conservation laws and generalized symmetries*

For a given nonlinear evolution equation of the form

$$u_t = F(u, u_x, u_{xx}, \dots), \tag{6}$$

a conservation law is defined in the form

$$T_t + X_x = 0 \tag{7}$$

which is satisfied by all solutions of (6), where T is the conserved density and X the flux. Here, T_t denotes the total derivative of T with respect to t, likewise, X_x denotes the same for X with respect to x. It is noted that T involves u and its x derivatives only and the terms like $u_t, u_{xt}, \dots$ are replaced by $u, u_x, u_{xx}, \dots$ appropriately using (6), the given equation. The existence of infinite number of conserved quantities which are in involution with respect to a suitable Poisson bracket assures the integrability of the given system.[54, 56, 57, 61, 62] It is a well-established fact that there is a close connection between the symmetries and conserved quantities through the symplectic operators used in Poisson brackets.[60]

1.5. *Painlevé analysis*

Painlevé analysis is used to study the singularity structure of the given nonlinear equation.[43–49] If only the movable critical singularities of all solution of the given system are poles, then we say that the system passes the Painlevé test and hence it could be a good candidate of an integrable system In practice, this analysis is a very effective tool to classify possible integrable systems in a systematic way In recent times, the nature of singularities also dictates the form of the τ-function which is used to bilinearize the given system [49, 99, 100] In addition, there is some evidence to show that this method can also be used to obtain Lax pairs and Backlund transformations.[50] However, it is not yet established completely that one can obtain the last mentioned properties systematically through this approach.

1 6. *Lie symmetries*

The theory of Lie point symmetries goes back to the 19th century Sophus Lie studied the invariance properties of the symmetry groups and used them to solve/classify the differential equation. Using one-parameter Lie group of symmetries one can systematically reduce the order of the ordinary differential equation by one Finding this group of symmetries which leaves the system invariant is tedious and computationally complex, nevertheless, it is very much algorithmic and systematic Due to the algorithmic nature, this technique has been used widely to find special solutions to reduce the dimension of the independent variables especially in NPDEs, to identify integrable systems to one of the six Painlevé equations By this method we can understand the underlying algebraic and geometric structure of the given system.[53–73, 101–110]

1.7 *Sato theory*

Various methods developed so far to investigate the soliton equations indicate the rich mathematical structure of the soliton systems It is Sato who unveiled the algebraic structure behind them using the method of algebraic analysis [111, 112] He noticed that the τ-function of the Kadomtsev–Petviashvili (KP) equation is connected with the Plucker coordinates appearing in the theory of Grassmann manifolds He also noticed that the totality of solutions of the KP equation as well as its generalization constitute an infinite-dimensional Grassmann manifold Ohta *et al* [113] presented a clear description of Sato theory in an elementary way Starting from the pseudo-differential operator they construct a linear homogeneous ordinary differential equation and explicitly explain the connection between the coefficients and the solutions of the same They introduce an infinite number of time variables in the coefficients and impose certain time dependence on the solutions and derive Sato, Lax, and ZS equations and the linear eigenvalue problem associated to the generalized Lax equation. As a consequence, the KP hierarchy was derived in a systematic manner and various reductions of it have been presented. They brought out the connection between the τ-function and the bilinear forms using Young diagram After the development of this grand theory, Date *et al.*[114, 115] and Jimbo and Miwa[116] extended Sato's idea and developed the theory of transformation groups for soliton equations which essentially explains the group-theoretical foundation of Hirota's method and Sato theory. The main aim of this theory is to reveal the intimate relation between the KP hierarchy and the infinite dimensional Lie algebra $gl(\infty)$ using the language of free Fermion operators They indeed had a big program to classify soliton hierarchies written in bilinear form according to various realizations of Lie algebras Using Sato theory, recently a nice method was developed to derive the generalized symmetries and conserved quantities of KP hierarchy [117,119] Hence, it is clear that Sato theory is the most powerful method through which one can obtain systematically integrability properties such as Lax pair, soliton solutions, conservation laws and symmetries of the integrable systems in an unified way [111–113,117–119] Using multicomponent version of this theory one can arrive at Davey–Stewartson equation and nonlinear Schrödinger equation in $2+1$ dimensions [118, 120, 121]

1 8 *Differential–difference framework*

In contrast to the continuous equations wherein enormous amount of research has been done to investigate various aspects of integrability, differential–difference or fully discrete systems have not been studied in depth However, some attempts had been made by Hirota in looking at the bilinear formalism for many known soliton equations in differential–difference settings [12–19] Also, Ablowitz and Ladik introduced differential–difference analogue of AKNS scheme and obtained many differential–difference soliton systems like nonlinear Schrodinger equation They also extended the IST method to differential–difference case [122, 123] In addition, many more important developments have taken place in this area.[115, 124–155] Using group-theoretic techniques, Date *et al.*[115] proposed a method and derived a large class of continuous, semi-continuous, discrete soliton equations More recently, there was a remarkable discovery of proposing singularity structure analysis for nonlinear difference equation by Grammaticos *et al* [156] This technique, now popularly called singularity confinement, is very powerful in identifying discrete integrable systems In particular, it is interesting to see that the discrete versions of Painlevé equations have been obtained through this technique and other properties have been studied [156–171] Soon after the discovery of singularity confinement, Ramani *et al* [137, 138] synthesized both the classical Painlevé analysis and singularity confinement together and proposed the singularity confinement approach to test the nature of singularity in differential–difference equations Again, this has been successfully implemented for several differential–difference systems including integro-differential equations [139, 172] In the same period, following the idea of Maeda,[140–142] Levi and Winternitz proposed Lie symmetry analysis for differential–difference systems[143–145] which was studied by others also [146–150] As in the continuous case, the existence of Lie point symmetries obtained through Lie's one-parameter transformation group for differential–difference equations again becomes very important. Using this theory, as in the continuous case, one could find similarity solutions and use them for reductions Though this method is still in the early stage many interesting results have already been obtained.

1 9 *Present work*

In Section 2, we discuss the derivations of the differential–difference Kadomtsev–Petviashvili (DΔKP) equation, conservation laws, generalized symmetries and solutions.[151, 152] Next, we briefly mention singularity structures and Lie symmetry analysis of DΔKP. Details will be published elsewhere.[154, 155] Finally, we also discuss a gauge equivalence of the DΔKP equation and study certain integrability properties of this system.[153, 155]

In Section 3, preliminary definitions and results needed to develop the differential–difference Sato theory are presented. Also, the Sato equation, generalized Lax equation, Zakharov–Shabat equation, and DΔKP hierarchy are derived. The associated eigenvalue problem is considered and the conserved quantities and generalized symmetries of DΔKP equation are obtained systematically. In Section 4, N-soliton solution and the rational solutions of DΔKP and its hierarchy are given in terms of a Wronskian determinant In Section 5, Lie-point symmetry analysis is performed for the differential–difference KP equation and the Veselov–Shabat equation[173] is obtained as a similarity reduction. Also, singularity structure of the solution of DΔKP equation is analysed using Painlevé-singularity confinement method for the differential–difference equation. In Section 6, we derive a gauge equivalence of DΔKP equation and study certain integrability properties such as Lax pair, conservation laws and generalized symmetries of the resulting system and perform Lie symmetry and singularity structure analysis

2. Differential–difference Sato theory

2.1 *Introduction*

The search for discrete or semi-discrete integrable equations started after the identification of solitons in Toda lattice [135] Toda lattice is a prototype model for the differential–difference soliton equation which possesses all integrability properties such as Lax pair representation, existence of infinite number of conserved quantities and N-soliton solution, etc as other soliton equations in continuous case Thus, various methods used to identify the integrable systems in the continuous case were extended to semi-discrete case too For example, the IST method by Ablowitz and Ladik,[122,123] discrete bilinear forms by Hirota,[12–20] group-theoretic method by Date *et al.*[114, 115] and Jimbo and Miwa,[116] Lax method by Kupershmidt,[136] Lie symmetry method of Maeda *et al* [140–142], Levi and coworkers,[143–146] Quispel and others[147–149] and Gaeta[150] and the Painlevé method by Ramani *et al* [137–138] Since Sato theory unifies all these approaches in the continuous case, it is natural to expect that the Sato theory plays the same role for differential–difference case also. This motivates us to look for the Sato theory for differential–difference integrable equations. Following the work of Ohta *et al.*[113] we formulated a suitable framework to treat the differential–difference equations. In fact, using this approach we have obtained the Lax pair, conservation laws and generalized symmetries of the DΔKP equation systematically [151]

2.2 *Preliminaries*

We start from the definition of the forward difference operator Δ and the shift operator E given by

$$\Delta f(n) = f(n+1) - f(n)$$
$$Ef(n) = f(n+1) \tag{8}$$

for all values of n (real or complex). Here the step size is taken to be one. From (8) it is clear that $\Delta = E - 1$. The Leibniz rule for the difference of product of two functions is given by

$$\Delta^m\big(f(n)g(n)\big) = \sum_{r=0}^{m} \frac{m(m-1)\cdots(m-r+1)}{r!}\big(\Delta^r E^{m-r} f(n)\big)\big(\Delta^{m-r} g(n)\big) \tag{9}$$

or

$$\Delta^m\left(f(n)g(n)\right)=\sum_{r=0}^{m}\frac{m(m-1)\cdots(m-r+1)}{r!}\left(\Delta^r f(n)\right)\left(\Delta^{m-r}E^r g(n)\right) \tag{10}$$

for all integers m Using the Leibniz rule (9) for the difference set-up we arrive at negative and positive powers of Δ in the form.

$$\begin{aligned}
&\cdots\\
\Delta^{-3}(fg)&=\left(E^{-3}f\right)\Delta^{-3}g-3\left(E^{-4}\Delta f\right)\Delta^{-4}g+6\left(E^{-5}\Delta^2 f\right)\Delta^{-5}g+\cdots\\
\Delta^{-2}(fg)&=\left(E^{-2}f\right)\Delta^{-2}g-2\left(E^{-3}\Delta f\right)\Delta^{-3}g+3\left(E^{-4}\Delta^2 f\right)\Delta^{-4}g+\cdots\\
\Delta^{-1}(fg)&=\left(E^{-1}f\right)\Delta^{-1}g-\left(E^{-2}\Delta f\right)\Delta^{-2}g+\left(E^{-3}\Delta^2 f\right)\Delta^{-3}g+\cdots\\
\Delta(fg)&=(Ef)\Delta g+(\Delta f)g\\
\Delta^2(fg)&=\left(E^2f\right)\Delta^2 g+2(E\Delta f)\Delta g+\left(\Delta^2 f\right)g\\
\Delta^3(fg)&=\left(E^3f\right)\Delta^3 g+3\left(E^2\Delta f\right)\Delta^2 g+3\left(E\Delta^2 f\right)\Delta g+\left(\Delta^3 f\right)g\\
&\cdot
\end{aligned} \tag{11}$$

Throughout this paper, we use the following convention

$$\Delta^l f\Delta^m g=\left(\Delta^l f\right)\left(\Delta^m g\right)$$

$$E^l\Delta^m f\Delta^k g=\left(E^l\Delta^m f\right)\left(\Delta^k g\right)$$

$$E^l fE^m g=\left(E^l f\right)\left(E^m g\right)$$

where l, m and k are integers Now we define the formal inner product of the given functions $u(n)$, $v(n)$ in such a way that

$$< u(n), v(n)> = \Delta^{-1}(u(n)v(n)). \tag{12}$$

Also, we assume that $u(n), v(n)\to 0$ as $n\to\infty$ The formal adjoint of the difference operator is defined by

$$(q(n)\Delta^m p(n)) = (-1)^m p(n)\Delta^m E^{-m} q(n) \tag{13}$$

for all functions $p(n)$ and $q(n)$ Throughout this paper we assume that the difference operator Δ and the differential operator $\frac{\partial}{\partial t_k}$ commutes.

2 3. *Pseudo-difference operator*

In the continuous case, the pseudo-differential operator plays a fundamental role in developing Sato theory.[113] By proper manipulation of this operator one can derive Lax pair, conserved quantities and symmetries in a systematic way. So, by analogy, with continuous case it is natural and worth to start with the pseudo-difference operator W, given by

$$W = 1 + w_1\Delta^{-1} + w_2\Delta^{-2} + \quad (14)$$

where w_j, $j = 1, 2, \cdot$ are functions of n We expect that the inverse of the pseudo-difference operator W is also of the same form and is given by

$$W^{-1} = 1 + v_1\Delta^{-1} + v_2\Delta^{-2} + \cdot \quad (15)$$

where v_j, $j = 1, 2,$ are functions of n. Since W and W^{-1} are formal inverse to each other, we have $WW^{-1} = W^{-1}W = 1$.

Using the expressions in eqns (14) and (15) in $WW^{-1} = 1$ and rearranging the terms and comparing the like powers of Δ on both sides we get an infinite number of equations for v_is in terms of w_js, $i, j = 1, 2, \cdot$ which give the relationship between v_is and w_js, $i, j = 1, 2,$. We list the first few of them below.

$$\begin{aligned}
v_1 &= -w_1 \\
v_2 &= w_1E^{-1}w_1 - w_2 \\
v_3 &= -w_1E^{-1}w_1 + w_1E^{-2}w_1 - w_1E^{-1}w_1E^{-2}w_1 + w_1E^{-1}w_2 \\
&\quad + w_2E^{-2}w_1 - w_3
\end{aligned} \quad (16)$$

For convenience, we restrict the operator W to only a finite number of terms say m and thus consider the mth order linear homogeneous ordinary difference equation given by

$$W_m\Delta^m f(n) = (\Delta^m + w_1\Delta^{m-1} + w_2\Delta^{m-2} + \cdot \ + w_m)\, f(n) = 0 \quad (17)$$

which has m linearly independent solutions say, $f^{(1)}(n), f^{(2)}(n),\ , f^{(m)}(n)$ Since these $f^{(j)}(n)$s are solutions of eqn (17) and hence we have a system of m linear equations in m unknowns w_1, $w_2, \cdot, w_m$, given by

$$\begin{array}{ccccccccc}
\Delta^{m-1}f^{(1)}w_1 & + & \Delta^{m-2}f^{(1)}w_2 & + & \cdots & + & f^{(1)}w_m & = & -\Delta^m f^{(1)} \\
\vdots & & \vdots & & \cdot & & \cdot & & \\
\Delta^{m-1}f^{(m)}w_1 & + & \Delta^{m-2}f^{(m)}w_2 & + & \cdot\cdot & + & f^{(m)}w_m & = & -\Delta^m f^{(m)}
\end{array} \quad (18)$$

Solving this system of algebraic equations using Cramer's rule (this is possible because the determinant of the coefficient matrix of the above system (18) is nothing but the Casorati determinant which is nonzero, due to the fact that $f^{(j)}_{(n)}$s are linearly independent), we arrive at

$$w_j = \frac{\begin{vmatrix} \Delta^{m-1}f^{(1)} & \cdot\cdot & -\Delta^m f^{(1)} & \cdots & f^{(1)} \\ \vdots & \cdots & \vdots & \cdot\cdot & \vdots \\ \Delta^{m-1}f^{(m)} & \cdot\cdot & -\Delta^m f^{(m)} & \cdot\cdot & f(m) \end{vmatrix}}{\begin{vmatrix} \Delta^{m-1}f^{(1)} & \cdot\cdot & \Delta^{m-j}f^{(1)} & & f^{(1)} \\ \vdots & \cdots & \vdots & \cdot\cdot & \vdots \\ \Delta^{m-1}f^{(m)} & \cdot\cdot & \Delta^{m-j}f^{(m)} & \cdot & f^{(m)} \end{vmatrix}} \quad (19)$$

for $j = 1, 2, \cdot\cdot, m$. Substituting the values of w_js in (17) and simplifying, we get

$$W_m = \frac{\begin{vmatrix} f^{(1)} & \cdot & f^{(m)} & \Delta^{-m} \\ \vdots & \cdot\cdot & \cdot & \vdots \\ \Delta^{m-1} f^{(1)} & \cdot\cdot & \Delta^{m-1} f^{(m)} & \Delta^{-1} \\ \Delta^m f^{(1)} & \cdots & \Delta^m f^{(m)} & 1 \end{vmatrix}}{\begin{vmatrix} f^{(1)} & \cdots & f^{(m)} \\ \vdots & \cdot & \vdots \\ \Delta^{m-1} f^{(1)} & \cdot & \Delta^{m-1} f^{(m)} \end{vmatrix}}. \tag{20}$$

In eqn (20), the operator Δ^{-j}, $j = 1, 2, \cdot\cdot, m$ has to be put in the rightmost position when we evaluate the determinant of the numerator.

We assume that the set of linearly independent solutions $f^{(j)}(n)$, $j = 1, 2, \cdot\cdot, m$ of the mth order linear difference equation (17) are analytic and hence can be expanded by using Newton–Gregory formula,

$$f^{(j)}(n) = \sum_{r=0}^{\infty} \frac{n^{(r)}}{r!} \xi_r^{(j)}, \tag{21}$$

where $\Delta^r f^{(j)}(0) = \xi_r^{(j)}$ and $n^{(r)} = n(n-1)\cdots(n-r+1)$ Using (18) and (21) we can write the system of linear equations (17) as

$$W_m \Delta^m \left(1, \frac{n^{(1)}}{1!}, \frac{n^{(2)}}{2!}, \cdots\right) \Phi = 0 \tag{22}$$

where

$$\Phi = \begin{pmatrix} \xi_0^{(1)} & \xi_0^{(2)} & \cdots & \xi_0^{(m)} \\ \xi_1^{(1)} & \xi_1^{(2)} & \cdot\cdot & \xi_1^{(m)} \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix} \tag{23}$$

Here $\left(1, \frac{n^{(1)}}{1!}, \frac{n^{(2)}}{2!}, \cdot\cdot\right)$ is an $1 \times \infty$ matrix and Φ is an $\infty \times m$ matrix Let Λ be the shift matrix given by

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdot\cdot \\ 0 & 0 & 1 & 0 & \cdot\cdot \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \cdot & \ddots & \ddots \end{pmatrix}. \tag{24}$$

Using the above matrix, we can write

$$(1+\Lambda)^n = \sum_{r=0}^{\infty} \frac{n^{(r)}}{r!}\Lambda^r$$

$$= \begin{pmatrix} 1 & \frac{n^{(1)}}{1!} & \frac{n^{(2)}}{2!} & \cdot \\ & 1 & \frac{n^{(1)}}{1!} & \cdots \\ & & 1 & \cdots \\ 0 & & & \cdot \end{pmatrix}. \tag{25}$$

Now, we define,

$$H(n) = (1+\Lambda)^n\Phi$$

$$= \sum_{r=0}^{\infty} \frac{n^{(r)}}{r!}\Lambda^r\Phi$$

$$= \begin{pmatrix} f^{(1)} & f^{(2)} & \cdot\cdot & f^{(m)} \\ \Delta f^{(1)} & \Delta f^{(2)} & \cdot\cdot & \Delta f^{(m)} \\ \Delta^2 f^{(1)} & \Delta^2 f(2) & \cdots & \Delta^2 f^{(m)} \\ \vdots & \cdot & \ddots & \cdot \end{pmatrix}. \tag{26}$$

The determinant formed by the first m rows of $H(n)$ is nothing but the denominator in W_m which is the Casorati determinant for the solutions of the difference equation (17).

2 4. *Time dependence*

In this section, we discuss the impact of time dependence in the coefficients of the ordinary difference equation (17). We introduce an infinite number of time variables $t = (t_1, t_2, \ )$ in w_j as $w_j = w_j(n;t), j = 1, 2,$ As a consequence of this, we have

$$f^{(j)} = f^{(j)}(n;t) = f^{(j)}(n, t_1, t_2, \ ) \tag{27}$$

We consider the time evolution of $H(n)$ in the form

$$H(n;t) = \left(\sum_{r=0}^{\infty} \frac{n^{(r)}}{r!}\Lambda^r\right)\exp\eta(t,\Lambda)\Phi \tag{28}$$

where $\eta(t,\Lambda) = \Sigma_{k=1}^{\infty} t_k\Lambda^k$ We write formally,

$$\left(\sum_{r=0}^{\infty} \frac{n^{(r)}}{r!}\Lambda^r\right)\exp\eta(t,\Lambda) = \sum_{k=0}^{\infty} P_k\Lambda^k. \tag{29}$$

Expanding the above expression (29) and comparing the coefficients of like powers of Λ on both sides, we get

$$
\begin{aligned}
P_0 &= 1\\
P_1 &= n + t_1\\
P_2 &= \frac{n(n-1)}{2!} + nt_1 + \frac{1}{2!}\left(t_1^2 + 2t_2\right)\\
P_3 &= \frac{n(n-1)(n-2)}{3!} + \frac{n(n-1)}{2!}t_1 + \frac{1}{2!}\left(t_1^2 + 2t_2\right)n + \frac{1}{3!}\left(t_1^3 + 6t_1t_2 + 6t_3\right)\\
P_4 &= \frac{n(n-1)(n-2)(n-3)}{4!} + \frac{n(n-1)(n-2)}{3!}t_1 + \frac{1}{2!}\left(t_1^2 + 2t_2\right)\frac{n(n-1)}{2!}\\
&+ \frac{1}{3!}\left(t_1^3 + 6t_1t_2 + 6t_3\right)n + \frac{1}{4!}\left(t_1^4 + 12t_1^2t_2 + 12t_2^2 + 24t_3t_1 + 24t_4\right)\\
&\cdots
\end{aligned}
\tag{30}
$$

These polynomials are analogues to Schur polynomials in the continuous case. They have a special property

$$
\frac{\partial P_k}{\partial t_m} = P_{k-m}, \quad P_k = 0, \forall k < 0 \text{ and } \Delta P_k = P_{k-1} \tag{31}
$$

We use the above property (31) to express the function $H(n,t)$ in terms of P_ks, which is written in the form

$$
\begin{aligned}
H(n;t) &= \begin{pmatrix} 1 & P_1 & P_2 & \cdots \\ & 1 & P_1 & \cdots \\ & & 1 & \\ 0 & & & \ddots \end{pmatrix} \begin{pmatrix} \xi_0^{(1)} & \xi_0^{(2)} & \cdot & \xi_0^{(m)} \\ \xi_1^{(1)} & \xi_1^{(2)} & \cdot\cdot & \xi_1^{(m)} \\ \vdots & \cdot & \cdot\cdot & \vdots \end{pmatrix}\\
&= \begin{pmatrix} h_0^{(1)}(n;t) & h_0^{(2)}(n,t) & \cdots & h_0^{(m)}(n;t) \\ h_1^{(1)}(n;t) & h_1^{(2)}(n;t) & \cdots & h_1^{(m)}(n;t) \\ \vdots & \cdot & \cdots & \vdots \end{pmatrix}
\end{aligned}
\tag{32}
$$

where

$$
h_0^{(j)}(n;0) = f^{(j)}(n) \tag{33}
$$

and

$$
h_k^{(j)}(n;t) = \frac{\partial h_0^{(j)}(n;t)}{\partial t_k} = \Delta^k h_0^{(j)}(n,t). \tag{34}
$$

It is easy to infer from (33) and (34) that $h(n;t) = h_0^{(j)}(n;t)$, and $j = 1, 2, \cdot, m$ are solutions of a set of linear partial differential–difference equations

$$\left(\frac{\partial}{\partial t_k} - \Delta^k\right) h(n,t) = 0, \quad k = 1,2,\cdots \tag{35}$$

with the initial value $h(n,0) = f^{(j)}(n)$ Hence, the linear difference equation (17) becomes,

$$W_m \Delta^m h_0^{(j)}(n,t) = \left(\Delta^m + w_1\Delta^{m-1} + w_2\Delta^{m-2} + \cdots + w_m\right) h_0^{(j)}(n;t) = 0, \; j = 1,2,\cdots m. \tag{36}$$

Solving these system of equations (36) as earlier, we get,

$$w_j = \frac{\begin{vmatrix} \Delta^{m-1}h_0^{(1)} & \cdot & -\Delta^m h_0^{(1)} & \cdot & h_0^{(1)} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \Delta^{m-1}h_0^{(m)} & \cdots & -\Delta^m h_0^{(m)} & \cdots & h_0^{(m)} \end{vmatrix}}{\begin{vmatrix} \Delta^{m-1}h_0^{(1)} & \cdots & \Delta^{m-j}h_0^{(1)} & \cdot & h_0^{(1)} \\ \cdot & & \vdots & & \vdots \\ \Delta^{m-1}h_0^{(m)} & \cdot & \Delta^{m-j}h_0^{(m)} & \cdot & h_0^{(m)} \end{vmatrix}} \tag{37}$$

and hence

$$W_m = \frac{\begin{vmatrix} h_0^{(1)} & \cdots & h_0^{(m)} & \Delta^{-m} \\ \cdot & \cdot\cdot & & \vdots \\ \Delta^{m-1}h_0^{(1)} & \cdot\cdot & \Delta^{m-1}h_0^{(m)} & \Delta^{-1} \\ \Delta^m h_0^{(1)} & \cdot\cdot & \Delta^m h_0^{(m)} & 1 \end{vmatrix}}{\begin{vmatrix} h_0^{(1)} & & h_0^{(m)} \\ \vdots & \cdot & \vdots \\ \Delta^{m-1}h_0^{(1)} & & \Delta^{m-1}h_0^{(m)} \end{vmatrix}} \tag{38}$$

Now w_j and W_m are completely given in terms of differential–difference analogues of Schur polynomials P_ks using (30)

2.5. *Sato, Lax, Zakharov and Shabat equations*

It is well known that the integrability of the nonlinear systems is associated with the finding of appropriate Lax or Zakharov–Shabat equations As in the continuous case,[113] the differential–difference version of Sato theory provides the Sato, Lax and Zakharov–Shabat equations naturally. To achieve this goal we proceed as follows: differentiating eqn (36) with respect to t_k, we obtain

$$\frac{\partial W_m}{\partial t_k}\Delta^m h_0^{(j)} + W_m\Delta^m \frac{\partial h_0^{(j)}}{\partial t_k} = 0, \tag{39}$$

since Δ and $\frac{\partial}{\partial t_k}$ commutes Using the relations (34), in (39), we get

$$\left(\frac{\partial W_m}{\partial t_k}\Delta^m + W_m\Delta^m\Delta^k\right)h_0^{(j)}(n,t) = 0 \tag{40}$$

We factorize the operator in (40), as

$$\frac{\partial W_m}{\partial t_k}\Delta^m + W^m\Delta^m\Delta^k = B_k W_m \Delta^m \tag{41}$$

where B_k is a kth-order difference operator. B_ks can be obtained by applying $\Delta^{-m}W_m^{-1}$ from the right of eqn (41)

$$B_k = \frac{\partial W_m}{\partial t_k}W_m^{-1} + W_m\Delta^k W_m^{-1} \tag{42}$$

From eqn (42), we can obtain by multiplying W_m from right,

$$\frac{\partial W_m}{\partial t_k} = B_k W_m - W_m\Delta^k \tag{43}$$

Hence the time evolution of the pseudo-difference operator $W_m(n, t)$ is governed by

$$\frac{\partial W}{\partial t_k} = B_k W - W\Delta^k \tag{44}$$

which is the differential–difference version of the famous Sato equation [113] The B_ks in the Sato equation can be computed from W using the following relation

$$B_k = \left(W\Delta^k W^{-1}\right)^+ \tag{45}$$

where $(\ )^+$ denotes the nonnegative powers of Δ only. We have discarded the first term of eqn (42), because it involves only negative powers of Δ, whereas B_k consists only nonnegative powers of Δ Using (45) we can derive the B_ks explicitly We list below a first few of them:

$$\begin{aligned} B_1 &= \Delta - \Delta w_1 \\ B_2 &= \Delta^2 - \left(2\Delta w_1 + \Delta^2 w_1\right)\Delta + \left(-2\Delta w_1 - 2\Delta^2 w_1 + \Delta w_1\Delta^2 w_1 + 2(\Delta w_1)^2\right. \\ &\quad \left. + w_1\Delta^2 w_1 + 2w_1\Delta w_1 - \Delta^2 w_2 - 2\Delta w_2\right) \end{aligned} \tag{46}$$

Next, we will derive the generalized Lax equation, involving infinite number of time variables. For this, we define

$$L = W\Delta W^{-1}. \tag{47}$$

Substituting the values of W and W^{-1} and rearranging the terms we can write

$$L = \Delta + u_0 + u_1\Delta^{-1} + u_2\Delta^{-2} + \cdots \tag{48}$$

where u_is are expressed in terms of w_js $i = 0, 1, \ j = 1, 2,$ We present some u_is

$$
\begin{aligned}
u_0 &= -\Delta w_1 \\
u_1 &= -\Delta w_1 - \Delta w_2 + w_1 \Delta w_1 \\
u_2 &= -\Delta w_2 - \Delta w_3 + \Delta w_1 E^{-1} w_1 + w_2 \Delta w_1 + E^{-1} w_1 \Delta w_2 - w_1 \Delta w_1 E^{-1} w_1 \\
&\cdot
\end{aligned}
\tag{49}
$$

Differentiating eqn (47) with respect to t_k, we get

$$\frac{\partial L}{\partial t_k} = \frac{\partial W}{\partial t_k} \Delta W^{-1} + W\Delta \frac{\partial W^{-1}}{\partial t_k} \tag{50}$$

The first term on the right-hand side of the above expression (50) will be replaced by the Sato equation (44) whereas for the second term we have to find $\frac{\partial W^{-1}}{\partial t_k}$ For this, differentiating $WW^{-1} = 1$ with respect to t_k we have

$$\frac{\partial W}{\partial t_k} W^{-1} + W \frac{\partial W^{-1}}{\partial t_k} = 0 \tag{51}$$

Operating W^{-1} from the left of the above expression (51) and rearranging the terms we have

$$\frac{\partial W^{-1}}{\partial t_k} = -W^{-1} \frac{\partial W}{\partial t_k} W^{-1}. \tag{52}$$

Using (52) in (50), we obtain

$$\frac{\partial L}{\partial t_k} = \frac{\partial W}{\partial t_k} \Delta W^{-1} + W\Delta \left(-W^{-1} \frac{\partial W}{\partial t_k} W^{-1} \right) \tag{53}$$

Substituting the value of $\frac{\partial W}{\partial t_k}$ from the Sato equation (44), we have

$$
\begin{aligned}
\frac{\partial L}{\partial t_k} &= \left(B_k W - W\Delta^k\right)\Delta W^{-1} - W\Delta W^{-1}\left(B_k W - W\Delta^k\right)W^{-1} \\
&= B_k W\Delta W^{-1} - W\Delta^{k+1} W^{-1} - W\Delta W^{-1} B_k + W\Delta^{k+1} W^{-1} \\
&= B_k L - LB_k \\
&= \left[B_k, L\right]
\end{aligned}
\tag{54}
$$

Thus, we have the generalized Lax equation

$$\frac{\partial L}{\partial t_k} = \left[B_k, L\right], \quad k = 1,2,\cdots \tag{55}$$

It is noted from (47) that

$$L^k = W\Delta^k W^{-1} \tag{56}$$

and hence

$$B_k = (L^k)^+. \tag{57}$$

From $B_k = (L^k)^+$, it is now immediate that

$$B_1 = \Delta + u_0$$
$$B_2 = \Delta^2 + (2u_0 + \Delta u_0)\Delta + (\Delta u_0 + u_0^2 + 2u_1 + \Delta u_1)$$
$$B_3 = \Delta^3 + (3u_0 + 3\Delta u_0 + \Delta^2 u_0)\Delta^2 + (2\Delta^2 u_0 + 3\Delta u_0 + 3u_0^2 + 3u_0\Delta u_0 + (\Delta u_0)^2$$
$$+3u_1 + 3\Delta u_1 + \Delta^2 u_1)\Delta + (\Delta^2 u_0 + 5u_1 u_0 + 3u_0\Delta u_0 + u_0^3 + (\Delta u_0)^2 + \Delta u_0 \Delta u_1$$
$$+3u_0\Delta u_1 + u_1\Delta u_0 + u_1 E^{-1} u_0 + 2\Delta^2 u_1 + 3\Delta u_1 + 3u_2 + 3\Delta u_2 + \Delta^2 u_2) \tag{58}$$

We can show that using (55) and (56),

$$\frac{\partial L^m}{\partial t_k} = [B_k, L^m], m, k = 1,2,\cdots \tag{59}$$

is also true Now, we will obtain the Zakharov–Shabat equation. From (59) we can show that

$$\frac{\partial L^m}{\partial t_k} - \frac{\partial L^k}{\partial t_m} = [B_k, L^m] - [B_m, L^k] \tag{60}$$

holds true We denote $B_k^c = B_k - L^k$ which contains only terms with $\Delta^{-j}, j > 0$. Employing this relation in eqn (60), we arrive at

$$\begin{aligned}
\frac{\partial L^m}{\partial t_k} - \frac{\partial L^k}{\partial t_m} &= [B_k, L^m] - [B_m, L^k] \\
&= [L^k + B_k^c, L^m] - [B_m, B_k - B_k^c] \\
&= L^k L^m + B_k^c L^m - L^m L^k - L^m B_k^c - B_m B_k + B_m B_k^c + B_k B_m - B_k^c B_m \\
&= B_k^c L^m - L^m B_k^c - B_m B_k + B_m B_k^c + B_k B_m - B_k^c B_m \\
&= B_k^c B_m - B_k^c B_m^c - B_m B_k^c + B_m^c B_k^c - B_m B_k + B_m B_k^c + B_k B_m - B_k^c B_m \\
&= B_k B_m - B_m B_k - B_k^c B_m^c + B_m^c B_k^c \\
&= [B_k, B_m] - [B_k^c, B_m^c]
\end{aligned} \tag{61}$$

But, from $L^k = B_k - B_k^c$, we have

$$\frac{\partial B_m}{\partial t_k}-\frac{\partial B_m^c}{\partial t_k}-\frac{\partial B_k}{\partial t_m}+\frac{\partial B_k^c}{\partial t_m}=\left[B_k,B_m\right]-\left[B_k^c,B_m^c\right] \tag{62}$$

Equating the difference part on both sides of (62), we find that

$$\frac{\partial B_m}{\partial t_k}-\frac{\partial B_k}{\partial t_m}=\left[B_k,B_m\right] \tag{63}$$

which is the Zakharov–Shabat equation [113]

2 6. *Differential–difference KP equation*

In the previous section, we derived the Sato, Lax and Zakharov–Shabat equations in a systematic way. In this section, we obtain a hierarchy of differential–difference soliton equations using Sato theory Since the first non-trivial member in this hierarchy is DΔKP equation, we call this as DΔKP hierarchy. Consider the L operator and the operators B_k, $k = 1, 2,$ Using the generalized Lax equation (55), for a given B_k, we can derive a set of infinite number of equations involving u_0, u_1, So, it is possible to generate infinite set of infinite number of equations for u_0, u_1, . By appropriately choosing the equations in different sets, we can derive integrable nonlinear differential–difference equations We wish to remind the reader that not every member in these sets is individually integrable. For example, taking $k = 1$ in eqn (55) we get an infinite set of equations given by

$$\begin{aligned}
&\frac{\partial u_0}{\partial t_1}=\Delta u_1\\
&\frac{\partial u_1}{\partial t_1}=\Delta u_1+\Delta u_2+u_0u_1-u_1E^{-1}u_0\\
&\frac{\partial u_2}{\partial t_1}=\Delta u_2+\Delta u_3+u_0u_2+u_1E^{-1}u_0-u_1E^{-2}u_0-u_2E^{-2}u_0\\
&\quad .
\end{aligned} \tag{64}$$

Also, for $k = 2$, we have

$$\begin{aligned}
\frac{\partial u_0}{\partial t_2}=&\ \Delta^2u_1+2\Delta u_2+\Delta^2u_2+u_1\Delta u_0+2u_0\Delta u_1+\Delta u_0\Delta u_1+u_0u_1-u_1E^{-1}u_0\\
\frac{\partial u_1}{\partial t_2}=&\ \Delta^2u_1+2\Delta u_2+2\Delta^2u_2+2\Delta u_3+\Delta^2u_3+2u_0\Delta u_1+\Delta u_0u_1+2u_0u_2\\
&+u_2\Delta u_0+2u_0\Delta u_2+\Delta u_2\Delta u_0+u_1\Delta u_0+u_1u_0^2+u_1^2+u_1\Delta u_1\\
&-u_1E^{-2}u_0+u_1E^{-1}u_0-u_1\left(E^{-1}u_0\right)^2-u_1E^{-1}u_1-u_2E^{-1}u_0-u_2E^{-2}u_0\\
&\quad .
\end{aligned} \tag{65}$$

Now, we consider the first two equations from the set of equations given in (64) and the first equation in (65). Solving these equations for u_0 we arrive at the DΔKP equation

$$\Delta\left(\frac{\partial u_0}{\partial t_2}+2\frac{\partial u_0}{\partial t_1}-2u_0\frac{\partial u_0}{\partial t_1}\right)=(2+\Delta)\frac{\partial^2 u_0}{\partial t_1^2} \tag{66}$$

This equation was first derived by Date *et al* through group of transformations approach.[115]

2.7 *Conserved quantities*

Once we have the Lax pair, it is natural to ask for the existence of infinite number of conservation laws, a basic property of integrable systems Again we utilize the Sato's framework to derive them Matsukidaira *et al*[117] developed a method to derive conservation laws of the KP equation through Sato theory and the same method was implemented[119] to derive the conserved quantities of Toda lattice. We follow a similar approach and derive the infinite number of conserved quantities for DΔKP equation through Sato theory. For this purpose, we first consider the linear eigenvalue problem associated with the generalized Lax equation (55)

$$L\psi=\lambda\psi$$
$$\frac{\partial\psi}{\partial t_k}=B_k\psi \tag{67}$$

and $\lambda_{t_k}=0$ Using $B_k=L^k+B_k^c$, we rewrite eqn (67), as

$$\frac{\partial\psi}{\partial t_k}=\left(L^k+B_k^c\right)\psi. \tag{68}$$

We recall that B_k^c consists only the negative powers of Δ. Now we will express Δ^{-j}, $j=1, 2,$ in terms of L^{-j}. For this purpose first we find L^{-1} We assume that L^{-1} is of the form

$$L^{-1}=\Delta^{-1}+q_2\Delta^{-2}+q_3\Delta^{-3}+\cdots. \tag{69}$$

Using $L^{-1}L=1$ we can determine q_js and we list some of them:

$$q_2=-E^{-1}u_0$$

$$q_3=E^{-1}u_0-E^{-2}u_0-E^{-1}u_1+E^{-1}u_0E^{-2}u_0$$

$$\begin{aligned}q_4=&-E^{-1}u_0+2E^{-2}u_0-E^{-3}u_0+E^{-1}u_1-E^{-2}u_1-E^{-1}u_2-2E^{-1}u_0E^{-2}u_0\\&+E^{-1}u_0E^{-3}u_0+E^{-1}u_0E^{-2}u_1+E^{-2}u_0E^{-3}u_0+E^{-1}u_1E^{-3}u_0\\&-E^{-1}u_0E^{-2}u_0E^{-3}u_0\end{aligned}$$

$$\begin{aligned}q_5=&E^{-1}u_0-3E^{-2}u_0+3E^{-3}u_0-E^{-4}u_0-E^{-1}u_1+2E^{-2}u_1-E^{-3}u_1+E^{-1}u_2\\&-E^{-2}u_2-E^{-1}u_3+3E^{-1}u_0E^{-2}u_0-3E^{-1}u_0E^{-3}u_0-2E^{-1}u_0E^{-2}u_1\\&+2E^{-1}u_0E^{-3}u_1+E^{-1}u_0E^{-2}u_2-3E^{-2}u_0E^{-3}u_0-3E^{-1}u_1E^{-3}u_0\\&+3E^{-1}u_0E^{-2}u_0E^{-3}u_0+E^{-2}u_0E^{-4}u_0+3E^{-1}u_1E^{-4}u_0-E^{-1}u_0E^{-2}u_0E^{-4}u_0\end{aligned}$$

$$
\begin{aligned}
&-E^{-1}u_0E^{-3}u_1+E^{-2}u_0E^{-3}u_1+E^{-1}u_1E^{-3}u_1-E^{-1}u_0E^{-2}u_0E^{-3}u_1\\
&+E^{-1}u_0E^{-4}u_0+E^{-3}u_0E^{-4}u_0-E^{-1}u_1E^{-4}u_0+E^{-2}u_1E^{-4}u_0+E^{-1}u_2E^{-4}u_0\\
&-E^{-1}u_0E^{-3}u_0E^{-4}u_0-E^{-1}u_0E^{-2}u_1E^{-4}u_0-E^{-2}u_0E^{-3}u_0E^{-4}u_0\\
&-E^{-1}u_1E^{-3}u_0E^{-4}u_0+E^{-1}u_0E^{-2}u_0E^{-3}u_0E^{-4}u_0\\
&\ldots
\end{aligned}
\tag{70}
$$

Using the Leibniz rule (9) and (69) we can derive the higher powers of L^{-j}, $j = 1, 2, \ldots$ We list some of them below:

$$
\begin{aligned}
L^{-2} &= \Delta^{-2}+\left(E^{-1}q_2+q_2\right)\Delta^{-3}+\left(-E^{1}q_2+E^{-2}q_2+E^{-1}q_3+q_2E^{-2}q_2+q_3\right)\Delta^{-4}\\
&\quad+\left(E^{-1}q_2-2E^{-2}q_2+E^{-3}q_2-E^{-1}q_3+E^{-2}q_3+E^{-1}q_4-2q_2E^{-2}q_2\right.\\
&\quad\left.+2q_2E^{-3}q_2+q_2E^{-2}q_3+q_3E^{-3}q_2+q_4\right)\Delta^{-5}+\ldots\\
L^{-3} &= \Delta^{-3}+\left(E^{-2}q_2+E^{-1}q_2+q_2\right)\Delta^{-4}+\left(-2E^{-2}q_2+2E^{-3}q_2+E^{-2}q_3+q_2E^{-3}q_2\right.\\
&\quad\left.+E^{-1}q_2E^{-3}q_2-E^{-1}q_2+E^{-2}q_2+E^{-1}q_3+q_2E^{-2}q_2+q_3\right)\Delta^{-5}+\ldots\\
L^{-4} &= \Delta^{-4}+\left(E^{-3}q_2+E^{-2}q_2+E^{-1}q_2+q_2\right)\Delta^{-5}+\ldots\\
L^{-5} &= \Delta^{-5}+\ldots
\end{aligned}
\tag{71}
$$

Using the expressions for L^{-j} and (70), we can express Δ^{-j}, $j = 1, 2, \ldots$:

$$
\begin{aligned}
\Delta^{-1} &= L^{-1}+E^{-1}u_0L^{-2}+\left(E^{-1}u_0^2-E^{-1}u_0+E^{-2}u_0+E^{-1}u_1\right)L^{-3}+\left(E^{-1}u_0^3-2E^{-1}u_0^2\right.\\
&\quad+2E^{-1}u_0E^{-1}u_1+E^{-2}u_0^2+E^{-2}u_0E^{-1}u_1+E^{-1}u_0-2E^{-2}u_0+E^{3}u_0-E^{-1}u_1\\
&\quad\left.+E^{-2}u_1+E^{-1}u_2+E^{-1}u_0E^{-2}u_0\right)L^{-4}+\left(E^{-1}u_0^4+E^{-1}u_0^3E^{-3}u_0-3E^{-1}u_0^3\right.\\
&\quad-E^{-1}u_0^3E^{-2}u_0+3E^{-1}u_0^2E^{-1}u_1-E^{-1}u_0^2E^{-2}u_0E^{-3}u_0+3E^{-1}u_0^2-4E^{-1}u_0E^{-2}u_0\\
&\quad-4E^{-1}u_0E^{-1}u_1+2E^{-1}u_0E^{-1}u_2+E^{-2}u_0^3+E^{-2}u_0^2E^{-1}u_1-E^{-1}u_0^2E^{-2}u_0^2\\
&\quad+2E^{-1}u_0E^{-2}u_0E^{-1}u_1-E^{-1}u_1E^{-2}u_0+2E^{-1}u_0E^{-4}u_0-4E^{-2}u_0E^{-4}u_0\\
&\quad+4E^{-1}u_0E^{-2}u_0E^{-4}u_0-4E^{-4}u_0E^{-1}u_1+E^{-2}u_0E^{-3}u_0+E^{-3}u_0E^{-1}u_1\\
&\quad+E^{-1}u_1E^{-2}u_1+E^{-1}u_0^2E^{-2}u_0+E^{-1}u_1^2+E^{-3}u_0^2+E^{-3}u_0E^{-2}u_1+E^{-3}u_0E^{-1}u_2\\
&\quad-3E^{-2}u_0^2+E^{-2}u_0E^{-1}u_2+E^{-1}u_0E^{-2}u_0^2-E^{-1}u_0+3E^{-2}u_0-3E^{-3}u_0+E^{-4}u_0\\
&\quad+E^{-1}u_1-2E^{-2}u_1+E^{-3}u_1-E^{-1}u_2+E^{-2}u_2+E^{-1}u_3+4E^{-1}u_0E^{-3}u_0
\end{aligned}
$$

$$+E^{-1}u_0E^{-2}u_1\big)L^{-5}+\cdots \tag{72}$$

$$\Delta^{-2}=L^{-2}+\big(E^{-2}u_0+E^{-1}u_0\big)L^{-3}+\big(E^{-2}u_0^2+E^{-1}u_0E^{-2}u_0+E^{-1}u_0^2-E^{-1}u_0$$
$$+E^{-1}u_1-E^{-2}u_0+2E^{-3}u_0+E^{-2}u_1\big)L^{-4}+\big(E^{-2}u_0^3-2E^{-2}u_0^2+2E^{-2}u_0E^{-3}u_0$$
$$+2E^{-2}u_0E^{-2}u_1-2E^{-2}u_0E^{-1}u_0+2E^{-2}u_0E^{-1}u_1+E^{-1}u_0^3+E^{-1}u_0^2E^{-3}u_0$$
$$-E^{-1}u_0E^{-2}u_0E^{-3}u_0+E^{-1}u_0E^{-2}u_1-2E^{-1}u_0^2+2E^{-1}u_0E^{-1}u_1+2E^{-3}u_0^2$$
$$+E^{-3}u_0E^{-2}u_1+2E^{-1}u_0E^{-3}u_0-E^{-2}u_1+2E^{-3}u_1+E^{-1}u_0-E^{-1}u_1+E^{-1}u_2$$
$$-2E^{-3}u_0+E^{-4}u_0+E^{-2}u_2\big)L^{-5}+\cdots$$

$$\Delta^{-3}=L^{-3}+\big(E^{-3}u_0+E^{-2}u_0+E^{-1}u_0\big)L^{-4}+\big(E^{-3}u_0^2+E^{-2}u_0E^{-3}u_0+E^{-1}u_0E^{-3}u_0$$
$$+E^{-2}u_0^2+E^{-1}u_0E^{-2}u_0+E^{-1}u_0^2-E^{-3}u_0+3E^{-4}u_0-E^{-2}u_0-E^{-1}u_0+E^{-3}u_1$$
$$+E^{-2}u_1+E^{-1}u_1\big)L^{-5}+\cdot$$

Hence, eqn (68) becomes

$$\frac{\partial\psi}{\partial t_k}=\Big(L^k+\sigma_1^{(k)}L^{-1}+\sigma_2^{(k)}L^{-2}+\cdots\Big)\psi \tag{73}$$

where $\sigma_j^{(k)}$s are functions of u_is for all j, $k=1,2,$, $i=0,1,\cdots$. On using $L^j\psi=\lambda^j\psi$ in (73), we obtain

$$\frac{\partial\psi}{\partial t_k}=\left(\lambda^m+\frac{\sigma_1^{(k)}}{\lambda}+\frac{\sigma_2^{(k)}}{\lambda^2}+\cdots\right)\psi. \tag{74}$$

From this, we get

$$\frac{\partial}{\partial t_k}\log\psi=\lambda^k+\sum_{j=1}^{\infty}\frac{\sigma_j^{(k)}}{\lambda^j}. \tag{75}$$

We denote $\sigma^{(k)}=\Sigma_{j=1}^{\infty}\sigma_j^{(k)}\lambda^{-j}$ and hence eqn (75) becomes

$$\sigma^{(k)}=\frac{\partial(\log\psi)}{\partial t_k}-\lambda^k \tag{76}$$

Differentiating eqn (76) with respect to the time variable t_m, we will arrive at the conservation laws

$$\frac{\partial \sigma^{(k)}}{\partial t_m} = \frac{\partial}{\partial t_k}\left(\frac{\partial \log \psi}{\partial t_m}\right), m, k = 1,2, \tag{77}$$

Notice that $\sigma^{(k)}$ and $\frac{\partial \log \psi}{\partial t_m}$ correspond to conserved quantity and flux, respectively. We list the first few of the $\sigma_j^{(1)}$s :

$$\begin{aligned}
\sigma_1^{(1)} &= -u_1 \\
\sigma_2^{(1)} &= -u_1 E^{-1} u_0 - u_2 \\
\sigma_3^{(1)} &= -u_1 \left(E^{-1} u_0\right)^2 + u_1 E^{-1} u_0 - u_1 E^{-2} u_0 - u_1 E^{-1} u_1 - u_2 E^{-2} u_0 \\
&\quad -u_2 E^{-1} u_0 - u_3
\end{aligned} \tag{78}$$

It is known from Section 2.6 that the Lax equation with $k = 1$ gives

$$\begin{aligned}
\frac{\partial u_0}{\partial t_1} &= \Delta u_1 \\
\frac{\partial u_1}{\partial t_1} &= \Delta u_1 + \Delta u_2 + u_0 u_1 - u_1 E^{-1} u_0 \\
\frac{\partial u_2}{\partial t_1} &= \Delta u_2 + \Delta u_3 + u_0 u_2 + u_1 E^{-1} u_0 - u_1 E^{-2} u_0 - u_2 E^{-2} u_0
\end{aligned} \tag{79}$$

From the above equations (79), we can express $u_1, u_2, u_3,$ in terms of u_0, and we list the first few of u_js for $j = 1, 2, \cdots$

$$\begin{aligned}
u_1 &= \Delta^{-1} \frac{\partial u_0}{\partial t_1} \\
u_2 &= \Delta^{-2} \frac{\partial^2 u_0}{\partial t_1^2} - \Delta^{-1} \frac{\partial u_0}{\partial t_1} - E^{-1} u_0 \Delta^{-1} \frac{\partial u_0}{\partial t_1} + \Delta^{-1}\left(u_0 \frac{\partial u_0}{\partial t_1}\right) \\
u_3 &= \Delta^{-3} \frac{\partial^3 u_0}{\partial t_1^3} - 2\Delta^{-2} \frac{\partial^2 u_0}{\partial t_1^2} + \Delta^{-1} \frac{\partial u_0}{\partial t_1} - E^{-1} u_0 \Delta^{2} \frac{\partial^2 u_0}{\partial t_1^2} + E^{-2} u_0 \Delta^{-2} \frac{\partial^2 u_0}{\partial t_1^2} \\
&\quad + 2E^{-1} u_0 \Delta^{-1} \frac{\partial u_0}{\partial t_1} - E^{-2} u_0 \Delta^{-1}\left(u_0 \frac{\partial u_0}{\partial t_1}\right) + \Delta^{-1}\left(\left(\frac{\partial u_0}{\partial t_1}\right)^2 + u_0 \frac{\partial^2 u_0}{\partial t_1^2}\right) \\
&\quad - \Delta^{-1}\left(E^{-1} \frac{\partial u_0}{\partial t_1} \Delta^{-1} \frac{\partial u_0}{\partial t_1} + E^{-1} u_0 \Delta^{-1} \frac{\partial^2 u_0}{\partial t_1^2}\right) + \Delta^{-1}\left(u_0 \Delta^{-1} \frac{\partial^2 u_0}{\partial t_1^2}\right) \\
&\quad + \Delta^{-1}\left(E^{-1} u_0 \Delta^{-1} \frac{\partial^2 u_0}{\partial t_1^2}\right) - 2\Delta^{-1}\left(u_0 \frac{\partial u_0}{\partial t_1}\right) + \Delta^{-1}\left(u_0 E^{-1} u_0 \frac{\partial u_0}{\partial t_1}\right)
\end{aligned} \tag{80}$$

$$+\Delta^{-1}\left(u_0^2\frac{\partial u_0}{\partial t_1}\right)+\Delta^{-1}\left(u_0E^{-1}u_0\Delta^{-1}\frac{\partial u_0}{\partial t_1}\right)-\Delta^{-1}\left(E^{-2}u_0E^{-1}u_0\Delta^{-1}\frac{\partial u_0}{\partial t_1}\right)$$

Now substituting these values in $\sigma_j^{(1)}$, we obtain the conserved densities of the DΔKP equation (66). We list below some of them:

$$\begin{aligned}
\sigma_1^{(1)} &= -\Delta^{-1}\frac{\partial u_0}{\partial t_1}\\
\sigma_2^{(1)} &= -\Delta^{-2}\frac{\partial^2 u_0}{\partial t_1^2}+\Delta^{-1}\frac{\partial u_0}{\partial t_1}-\Delta^{-1}\left(u_0\frac{\partial u_0}{\partial t_1}\right)\\
\sigma_3^{(1)} &= -\left(E^{-1}u_0\right)^2\Delta^{-1}\frac{\partial u_0}{\partial t_1}+E^{-1}u_0\Delta^{-1}\frac{\partial u_0}{\partial t_1}-E^{-2}u_0\Delta^{-1}\frac{\partial u_0}{\partial t_1}\\
&\quad -\Delta^{-1}\frac{\partial u_0}{\partial t_1}E^{-1}\Delta^{-1}\frac{\partial u_0}{\partial t_1}-E^{-2}u_0\Delta^{2}\frac{\partial^2 u_0}{\partial t_1^2}+E^{-2}u_0\Delta^{-1}\frac{\partial u_0}{\partial t_1}\\
&\quad +E^{-1}u_0E^{-2}u_0\Delta^{-1}\frac{\partial u_0}{\partial t_1}-E^{-2}u_0\Delta^{-1}\left(u_0\frac{\partial u_0}{\partial t_1}\right)
\end{aligned}\tag{81}$$

2.8 *Generalized symmetries*

Another important feature of the integrable system is that it admits infinitely many time-independent non-Lie point symmetries called generalized symmetries Again it is simple to derive the generalized symmetries using the theory of Sato. In fact, Matsukidaira *et al.*[117] proposed a method to derive the generalized symmetries of the KP equation using Sato theory They explicitly obtained the eigenfunction of the associated linear eigenvalue problem and showed that the squared eigenfunction generates generalized symmetries. We show that this strategy can also be adopted here and derive the generalized symmetries for DΔKP equation Before doing so, we give a brief review of the basic notions in this theory. We consider an evolution equation

$$u_t = K(u) \tag{82}$$

where K is a functional of u. We call the functional $S(u)$ a symmetry of (82) if it satisfies the linearized equation given by

$$S_t = K'(u)[S], \tag{83}$$

where the Fréchet derivative $K'(u)$ is defined by

$$K'(u)[S] = \frac{\partial}{\partial \epsilon}K(u+\epsilon S)\bigg|_{\epsilon=0}. \tag{84}$$

It can be shown that a symmetry S must satisfy

$$[S, K] \equiv S'[K] - K'[S] = 0. \tag{85}$$

This means that any symmetry S commutes with $K(u)$.

We will show that the eigenfunction of the linear eigenvalue problem (67) and its adjoint generate the symmetries of the DΔKP equation Notice that $L = W\Delta W^{-1}$. Hence, we rewrite eqn (67) in the form

$$L\psi = \lambda\psi$$

$$\text{i e. } W\Delta W^{-1}\psi = \lambda\psi \tag{86}$$

Applying W^{-1} on both sides of (86), we have

$$\Delta W^{-1}\psi = W^{-1}\lambda\psi = \lambda W^{-1}\psi. \tag{87}$$

By taking $\psi_0 = W^{-1}\psi$ in the above equation (88), we arrive at

$$\Delta\psi_0 = \lambda\psi_0. \tag{88}$$

The above equation (88) is just a first-order ordinary linear difference equation, whose solution is given by

$$\psi_0 = g(t_1, t_2 \cdots, \lambda)(1+\lambda)^n \tag{89}$$

where $g(t_1, t_2, \ ; \lambda)$ is an integration function. From this result it follows that

$$W^{-1}\psi = \psi_0 = g(\lambda, t_1, t_2, \ )\,(1+\lambda)^n \tag{90}$$

and hence the eigenfunction is given by

$$\begin{aligned}\psi &= Wg(t_1, t_2,\cdot \ ; \lambda)(1+\lambda)^n \\ &= (1 + w_1\Delta^{-1} + w_2\Delta^{-2} + \ )(1+\lambda)^n g(t_1, t_2, \cdots, \lambda).\end{aligned} \tag{91}$$

To find the eigenfunction we need the value of $\Delta^{-j}(1+\lambda)^n$. For this purpose, first we derive $\Delta^{-1}(1+\lambda)^n$. Now we compute $\Delta(1+\lambda)^n$.

$$\begin{aligned}\Delta(1+\lambda)^n &= (1+\lambda)^{n+1} - (1+\lambda)^n \\ &= (1+\lambda)^n(1+\lambda-1) \\ &= (1+\lambda)^n\lambda\end{aligned} \tag{92}$$

Operating $\frac{1}{\lambda}\Delta^{-1}$ on both sides of (92), we arrive at

$$\Delta^{-1}(1+\lambda)^n = \frac{1}{\lambda}(1+\lambda)^n. \tag{93}$$

Applying Δ^{-1} repeatedly on (93), we have

$$\Delta^{-j}(1+\lambda)^n = \frac{(1+\lambda)^n}{\lambda^j}, \quad j = 1, 2, \cdots. \tag{94}$$

Using (94) in (91), we have

$$\psi=\left(1+\frac{w_1}{\lambda}+\frac{w_2}{\lambda^2}+\cdots\right)(1+\lambda)^n g(t_1,t_2,\cdots;\lambda). \tag{95}$$

But, we have,

$$\frac{\partial\psi}{\partial t_k}=\left(L^k+q_1L^{-1}+q_2L^{-2}+\cdots\right)\psi \tag{96}$$

where q_js are appropriate functions of λ, t_1, t_2, . On using $L^j\psi=\lambda^j\psi$, $j=1, 2,$, we obtain

$$\frac{\partial\psi}{\partial t_k}=\left(\lambda^k+\frac{q_1}{\lambda}+\frac{q_2}{\lambda^2}+\cdot\right)\psi. \tag{97}$$

This implies

$$\frac{\partial\log\psi}{\partial t_k}=\lambda^k+\frac{q_1}{\lambda}+\frac{q_2}{\lambda^2}+ \tag{98}$$

This is true for any integer $k>0$. On integrating the set of equations, we finally find,

$$\psi=\exp\left(\sum_{j=1}^{\infty}\lambda^j t_j+t_0+\sum_{j=1}^{\infty}\tilde{q}_j\lambda^{-j}\right) \tag{99}$$

where $\tilde{q}_j$ is again appropriate functions Comparing eqn (99) with (95) at $t_j=0$, $\forall j=1, 2,\cdot\cdot$ we get $\tilde{q}_j=w_j$, $\forall j=1, 2, \cdot\cdot$, and

$$g(t_1,t_2,\cdot\cdot,\lambda)=\exp\left(\sum_{j=1}^{\infty}\lambda^j t_j\right) \tag{100}$$

Hence

$$\psi=\left(1+\frac{w_1}{\lambda}+\frac{w_2}{\lambda^2}+\cdots\right)(1+\lambda)^n\exp\left(\sum_{j=1}^{\infty}\lambda^j t_j\right). \tag{101}$$

We will obtain the formal adjoint of ψ as

$$\psi^*=\left(1+\frac{w_1^*}{\lambda}+\frac{w_2^*}{\lambda^2}+\cdots\right)(1+\lambda)^{-n}\exp\left(-\sum_{j=1}^{\infty}\lambda^j t_j\right). \tag{102}$$

To find the value of ψ^* we have to determine w_j^*, $\forall j=1, 2, \cdot$. For this purpose, we consider

$$L^*\psi^*=\lambda\psi^*$$

$$\left(W\Delta W^{-1}\right)^{*}\psi^{*}=\lambda\psi^{*}$$

$$\left(W^{-1}\right)^{*}\left(-\Delta E^{-1}\right)\left(W^{*}\right)\psi^{*}=\lambda\psi^{*} \tag{103}$$

$$-\Delta E^{-1}W^{*}\psi^{*}=\lambda W^{*}\psi^{*}$$

Taking $W^{*}\psi^{*}=\psi_0^{*}$ we arrive at a linear difference equation

$$-\Delta E^{-1}\psi_0^{*}=\lambda\psi_0^{*} \tag{104}$$

Solving the above equation (104) we arrive at

$$\psi_0^{*}=W^{*}\psi^{*}=h(t_1,t_2,\cdots,\lambda)(1+\lambda)^{-n} \tag{105}$$

and hence we have

$$\psi^{*}=\left(W^{-1}\right)^{*}h(t_1,t_2,\cdots,\lambda)(1+\lambda)^{-n} \tag{106}$$

Using (13) and (15) in (106), we have

$$\psi^{*}=\left(1+v_1\Delta^{-1}+v_2\Delta^{-2}+\cdots\right)^{*}h(t_1,t_2,\cdots,\lambda)(1+\lambda)^{-n}$$

$$=\left(1-\Delta^{-1}E_{v1}+\Delta^{-2}E^{2}v_2+\cdots\right)h(t_1,t_2,\cdots,\lambda)(1+\lambda)^{-n} \tag{107}$$

Expanding the above equation (107), using Leibniz rule (10), we have

$$\psi^{*}=\left(1-E\left(v_1\Delta^{-1}-\Delta v_1E\Delta^{-2}+\Delta^{2}v_1E^{2}\Delta^{-3}+\cdots\right)\right.$$

$$+E^{2}\left(v_2\Delta^{-2}-2\Delta v_2E\Delta^{-3}+\cdots\right) \tag{108}$$

$$\left.-E^{3}\left(v_3\Delta^{-3}+\cdots\right)+\cdots\right)h(t_1,t_2,\cdots,\lambda)(1+\lambda)^{-n}$$

$$\psi^{*}=\left(1-Ev_1E\Delta^{-1}+\left(E\Delta v_1+E^{2}v_2\right)E^{2}\Delta^{-2}\right.$$

$$\left.-\left(E\Delta^{2}v_1+2\Delta E^{2}v_2+E^{3}v_3\right)E^{3}\Delta^{-3}+\cdots\right)h(t_1,t_2,\cdots,\lambda)(1+\lambda)^{-n} \tag{109}$$

Now, we compute $\Delta E^{-1}(1+\lambda)^{-n}$.

$$\Delta E^{-1}(1+\lambda)^{-n}=\Delta(1+\lambda)^{-n+1}$$

$$=(1+\lambda)^{-n-1+1}-(1+\lambda)^{-n+1}$$

$$=(1+\lambda)^{-n}\left(1-(1+\lambda)\right) \tag{110}$$

$$=-\lambda(1+\lambda)^{-n}$$

Operating $-\frac{1}{\lambda}\Delta^{-1}E$ on both sides of (110), we have

$$\Delta^{-1}E(1+\lambda)^{-n} = -\frac{(1+\lambda)^{-n}}{\lambda} \tag{111}$$

and hence

$$\Delta^{-j}E^{j}(1+\lambda)^{-n} = \frac{(-1)^{j}(1+\lambda)^{-n}}{\lambda^{j}}, \forall j = 1,2,\cdots. \tag{112}$$

Using the above result (112) in (109), we have,

$$\psi^{*} = \left(1+\frac{Ev_1}{\lambda}+\frac{(E\Delta v_1+E^2v_2)}{\lambda^2} + \frac{(E\Delta^2 v_1+2\Delta E^2 v_2+E^3v_3)}{\lambda^3}+\cdots\right)h(t_1,t_2,\cdots\lambda)(1+\lambda)^{-n}. \tag{113}$$

Comparing eqns (113) and (101), we have

$$h(t_1,t_2,\cdots;\lambda) = \exp\left(-\sum_{j=1}^{\infty}\lambda^{j}t_j\right) \tag{114}$$

and the w_j^* s are given by

$$\begin{aligned} w_1^* &= Ev_1 \\ w_2^* &= E\Delta v_1 + E^2 v_2 \\ w_3^* &= E\Delta^2 v_1 + 2\Delta E^2 v_2 + E^3 v_3 \\ w_4^* &= E\Delta^3 v_1 + 3E^2\Delta^2 v_2 + 3E^3\Delta v_3 + E^4 v_4 \\ &\ \ \vdots \end{aligned} \tag{115}$$

On using (16) in (115), we get

$$\begin{aligned} w_1^* &= -Ew_1 \\ w_2^* &= -E\Delta w_1 + Ew_1E^2w_1 - E^2w_2 \\ w_3^* &= -E\Delta^2 w_1 - 2E^3w_2 + 3E^2w_1E^3w_1 + 2E^2w_2 - 2E^2w_1Ew_1 - E^3w_1Ew_1 \\ &\quad + E^3w_1E^2w_2 - E^3w_1E^2w_1Ew_1 + E^3w_2Ew_1 - E^3w_3 \end{aligned} \tag{116}$$

Now, we will show that the eigenfunction and its adjoint in terms of w and w^* can be used to generate generalized symmetries of the DΔKP equation. For this purpose, we adopt the procedure developed in Matsuidaira [117] Using the eigenvalue problem (67)

$$\frac{\partial \psi}{\partial t_1} = B_1 \psi$$

$$\frac{\partial \psi}{\partial t_2} = B_2 \psi \tag{117}$$

and its adjoint eigenvalue problem

$$\frac{\partial \psi^*}{\partial t_1} = -B_1^* \psi^*$$

$$\frac{\partial \psi^*}{\partial t_2} = -B_2^* \psi^* \tag{118}$$

we have

$$\begin{aligned}
\frac{\partial \psi}{\partial t_1} &= \Delta \psi + u_0 \psi \\
\frac{\partial \psi}{\partial t_2} &= \Delta^2 \psi + (2u_0 + \Delta u_0)\Delta \psi + \left(\Delta u_0 + u_0^2 + (2+\Delta)\Delta^{-1} \frac{\partial u_0}{\partial t_1} \right) \psi \\
\frac{\partial \psi^*}{\partial t_1} &= \Delta E^{-1} \psi^* - u_0 \psi^* \\
\frac{\partial \psi^*}{\partial t_2} &= -\Delta^2 E^{-2} \psi^* + \Delta E^{-1}\left(2u_0 \psi^* + \Delta u_0 \psi^*\right) \\
&\quad - \left(\Delta u_0 + u_0^2 + (2+\Delta)\Delta^{-1} \frac{\partial u_0}{\partial t_1} \right) \psi^*
\end{aligned} \tag{119}$$

Using (119), we can see that $s = \psi\psi^*$ satisfies

$$\frac{\partial s}{\partial t_2} - 2u_0 \frac{\partial s}{\partial t_1} + 2\frac{\partial s}{\partial t_1} - (2+\Delta)\Delta^{-1} \frac{\partial^2 s}{\partial t_1^2} = 0 \tag{120}$$

Using the definition of Fréchet derivative (83), the linearized DΔKP equation is given by

$$\frac{\partial S}{\partial t_2} - 2u_0 \frac{\partial S}{\partial t_1} - 2S\frac{\partial u_0}{\partial t_1} + \frac{\partial S}{\partial t_1} - (2+\Delta)\Delta^{-1} \frac{\partial^2 S}{\partial t_1^2} = 0. \tag{121}$$

From (120) and (121) it is obvious that if s satisfies (120), then $S = \frac{\partial s}{\partial t_1}$ satisfies the linearized DΔKP equation (121) Hence, it is immediate that $\frac{\partial}{\partial t_1}\left(\psi\psi^*\right)$ satisfies the linearized DΔKP equation (121). Using (101) and (102), we have

$$\psi\psi^* = \left(1+\frac{w_1}{\lambda}+\frac{w_2}{\lambda^2}+\cdots\right)\left(1+\frac{w_1^*}{\lambda}+\frac{w_2^*}{\lambda^2}+\cdots\right)=\sum_{k=0}^{\infty} s_k\lambda^{-k} \tag{122}$$

where

$$s_k = \sum_{j=0}^{k} w_j w_{k-j}^* \tag{123}$$

with $w_0 = 1$ and $w_0^* = 1$ Since $\psi\psi^*$ is a polynomial expression in λ, and λ is independent of the time variables, we have

$$S_k = \frac{\partial}{\partial t_k} s_k, \quad k = 0,1,2,\cdots \tag{124}$$

which are solutions of (121) and hence generalized symmetries for the DΔKP equation (66) We present below the first few generalized symmetries.

$$\begin{aligned}
S_0 &= \frac{\partial}{\partial t_1}(1) = 0 \\
S_0 &= \frac{\partial}{\partial t_1}(u_0) = \frac{\partial u_0}{\partial t_1} \\
S_2 &= \frac{\partial}{\partial t_1}\left(-u_0 + u_0^2 + (2+\Delta)\Delta^{-1}\frac{\partial u_0}{\partial t_1}\right) = \frac{\partial u_0}{\partial t_2} + \frac{\partial u_0}{\partial t_1} \\
S_3 &= \frac{\partial}{\partial t_1}\left(\frac{\partial u_0}{\partial t_2} + \Delta^{-1}\frac{\partial^2 u_0}{\partial t_1^2} + 3\Delta^{-2}\frac{\partial^2 u_0}{\partial t_1^2} - 4\Delta^{-1}\frac{\partial u_0}{\partial t_1} + 3u_0\Delta^{-1}\frac{\partial u_0}{\partial t_1}\right) \\
&\quad +3\Delta^{-1}\left(u_0\frac{\partial u_0}{\partial t_1} + u_0\frac{\partial u_0}{\partial t_1} + u_0 - 2u_0^2 + u_0^2\right)
\end{aligned} \tag{125}$$

3. Wronskian and rational solutions

3.1 *Introduction*

It is well known that many IST solvable nonlinear evolution equations exhibit multisoliton solutions When we use Hirota's bilinear method[10–14] these N-soliton solutions can be expressed as an Nth order polynomial in N exponentials Perhaps, a more convenient representation of such a solution, however, is in terms of the Wronskian of N exponential functions. The N-soliton solution for soliton equations written in the Wronskian form was first introduced by Satsuma[28] and further developed by Freeman and Nimmo[29] and Nimmo and Freeman[30] This procedure has been applied to the KP,[29,31] the Boussinesq,[30,32] and other soliton systems[33–36]

It is also known that the N-soliton solution of KP hierarchy can be derived through Sato theory,[113] which is expressed by an appropriate τ-function. This τ-function can be expressed in the form of generalized Wronskian determinant defined on the infinite-dimensional Grassmann manifold. In this framework, Hirota's bilinear forms arise naturally as Plücker relations. Using the Laplace expansion of the determinant, we can easily verify that the τ-function satisfies the given Hirota's bilinear form.

It has been recognized that integrable systems, in the sense of IST, possess other class of solutions as well, called rational solutions [3, 83–96] The rational solutions of the soliton equations can be obtained through various means On the other hand, Sato theory provides a systematic approach to find the rational solutions of KP hierarchy [91] The fundamental ones are represented in terms of Schur polynomials which satisfy a certain set of linear differential equations

3.2 *N-Soliton solution*

Now, we consider the DΔKP equation (66) in the form

$$\Delta\left(\frac{\partial u}{\partial t_2}+2\frac{\partial u}{\partial t_1}-2u\frac{\partial u}{\partial t_1}\right)=(2+\Delta)\frac{\partial^2 u}{\partial t_1^2} \tag{126}$$

where $u=u(t_1, t_2, n)$. Now, using the dependent variable transformation

$$u(t_1,t_2,n)=\frac{\partial}{\partial t_1}\log\frac{\tau_{n+1}}{\tau_n} \tag{127}$$

in eqn (126), we arrive at

$$\tau_n\frac{\partial\tau_{n+1}}{\partial t_2}-\tau_{n+1}\frac{\partial\tau_n}{\partial t_2}+2\tau_n\frac{\partial\tau_{n+1}}{\partial t_1}-2\tau_{n+1}\frac{\partial\tau_n}{\partial t_1}+2\frac{\partial\tau_{n+1}}{\partial t_1}\frac{\partial\tau_n}{\partial t_1}-\tau_n\frac{\partial^2\tau_{n+1}}{\partial t_1^2}-\tau_{n+1}\frac{\partial^2\tau_n}{\partial t_1^2}=0. \tag{128}$$

We represent this equation in the Hirota's bilinear form, which can be written in terms of Hirota's bilinear operators These operators are defined by the following rule.[14]

$$D_t^m D_x^k a\cdot b=\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^m\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^k a(x,t)b(x',t')\Big|_{\substack{x'=x\\ t'=t}} \tag{129}$$

where m and k are arbitrary nonnegative integers. Using the above definition (129), we can find

$$D_{t_1}\tau_{n+1}\cdot\tau_n=\tau_n\frac{\partial\tau_{n+1}}{\partial t_1}-\tau_{n+1}\frac{\partial\tau_n}{\partial t_1}$$

$$D_{t_2}\tau_{n+1}\cdot\tau_n=\tau_n\frac{\partial\tau_{n+1}}{\partial t_2}-\tau_{n+1}\frac{\partial\tau_n}{\partial t_2} \tag{130}$$

$$D_{t_1}^2\tau_{n+1}\ \tau_n=\tau_n\frac{\partial^2\tau_{n+1}}{\partial t_1{}^2}-2\frac{\partial\tau_{n+1}}{\partial t_1}\frac{\partial\tau_n}{\partial t_1}+\tau_{n+1}\frac{\partial^2\tau_n}{\partial t_1{}^2}$$

Now, we can easily see that eqn (128) can be written in the bilinear form

$$\left(D_{t_2}+2D_{t_1}-D_{t_1}^2\right)\tau_{n+1}\ \tau_n=0. \tag{131}$$

Next, we prove that the solutions of the DΔKP equation can be represented in the form of Wronskian (Casorati) determinant

$$\tau_n=W\left(f_n^{(1)},f_n^{(2)},\cdots,f_n^{(N)}\right)$$
$$=\begin{vmatrix} f_n^{(1)} & \Delta f_n^{(1)} & \cdots & \Delta^{N-1}f_n^{(1)} \\ f_n^{(2)} & \Delta f_n^{(2)} & \cdots & \Delta^{N-1}f_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_n^{(N)} & \Delta f_n^{(N)} & \cdots & \Delta^{N-1}f_n^{(N)} \end{vmatrix}=\begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} & \cdots & f_{n+N-1}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix}. \tag{132}$$

The determinant τ_n in (132) is nothing but the denominator in the expression (38) given in Section 2. The entries in the determinant (132), $f_n^{(j)}=f^{(j)}(t_1,t_2,n)$, $j=1,2,\ldots,N$ are the solutions of a set of linear partial differential–difference equations

$$\frac{\partial f_n^{(j)}}{\partial t_1}=\Delta f_n^{(j)},$$
$$\frac{\partial f_n^{(j)}}{\partial t_2}=\Delta^2 f_n^{(j)},\quad j=1,2,\ldots,N \tag{133}$$

One of the particular solutions of (133) is readily given by

$$f_n^{(j)}=\left(1+p_j\right)^n\exp\left(p_jt_1+p_j^2t_2\right),\qquad j=1,2,\ldots,N \tag{134}$$

To obtain N-soliton solution in the Wronskian form, it is well known that $f_n^{(j)}$ can be chosen in the form

$$f_n^{(j)}=\exp\eta_j+\exp\xi_j \tag{135}$$

with η_j and ξ_j given by

$$\eta_j=p_jt_1+p_j^2t_2+n\log\left(1+p_j\right)+\eta_{j0}$$
$$\xi_j=q_jt_1+q_j^2t_2+n\log\left(1+q_j\right)+\xi_{j0} \tag{136}$$

Following Freeman and Nimmo's notation,[29, 30] we denote τ_n in (132) as

$$\tau_n = |0,1,2,\ldots,N-1| = \begin{vmatrix} f_n^{(1)} & f_{n+1}^{(1)} & \cdots & f_{n+N-1}^{(1)} \\ f_n^{(2)} & f_{n+1}^{(2)} & \cdot & f_{n+N-1}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ f_n^{(N)} & f_{n+1}^{(N)} & \cdots & f_{n+N-1}^{(N)} \end{vmatrix} \tag{137}$$

The terms involved in (128) may then easily be computed as

$$\begin{aligned}
\tau_{n+1} &= |1,2,\ldots,N| \\
\frac{\partial \tau_n}{\partial t_1} &= |0,1,2,\ldots,N-2,N| - N|0,1,2,\ldots,N-1| \\
\frac{\partial \tau_{n+1}}{\partial t_1} &= |1,2,3,\ldots,N-1,N+1| - N|1,2,\ldots,N| \\
\frac{\partial \tau_n}{\partial t_2} &= N|0,1,2,\ldots,N-1| - |0,1,2,\ldots,N-3,N-1,N| \\
&\quad + |0,1,2,\ldots,N-2,N+1| - 2|0,1,2,\ldots,N-2,N| \\
\frac{\partial \tau_{n+1}}{\partial t_2} &= N|1,2,\ldots,N| - |1,2,\ldots,N-2,N,N+1| \\
&\quad + |1,2,\ldots,N-1,N+2| - 2|1,2,\ldots,N-1,N+1| \\
\frac{\partial^2 \tau_n}{\partial t_1^2} &= N^2|0,1,2,\ldots,N-1| - 2N|0,1,2,\ldots,N-2,N| \\
&\quad + |0,1,2,\ldots,N-3,N-1,N| + |0,1,2,\ldots,N-2,N+1| \\
\frac{\partial^2 \tau_{n+1}}{\partial t_1^2} &= N^2|1,2,\ldots,N| - 2N|1,2,\ldots,N-1,N+1| \\
&\quad + |1,2,\ldots,N-2,N,N+1| + |1,2,\ldots,N-1,N+2|
\end{aligned} \tag{138}$$

wherein the linear equations (133) are used in the derivation of the above results. Using the above expressions (138) in (128), we get

$$\begin{aligned}
&\tau_n \frac{\partial \tau_{n+1}}{\partial t_2} - \tau_{n+1}\frac{\partial \tau_n}{\partial t_2} + 2\tau_n \frac{\partial \tau_{n+1}}{\partial t_1} - 2\tau_{n+1}\frac{\partial \tau_n}{\partial t_1} + 2\frac{\partial \tau_{n+1}}{\partial t_1}\frac{\partial \tau_n}{\partial t_1} \\
&-\tau_n \frac{\partial^2 \tau_{n+1}}{\partial t_1^2} - \tau_{n+1}\frac{\partial^2 \tau_n}{\partial t_1^2} = 2|0,1,2,\ldots,N-1|\,|1,2,\ldots,N-2,N,N+1| \\
&\qquad -2|0,1,2,\ldots,N-2,N|\,|1,2,\ldots,N-1,N+1| \\
&\qquad +2|0,1,2,\ldots,N-2,N+1|\,|1,2,\ldots,N|
\end{aligned} \tag{139}$$

which is the Laplace expansion of the $2N \times 2N$ determinant[21]

$$\begin{vmatrix} 0 & \hat{N-2} & O & N-1 & N & N+1 \\ 0 & O & \hat{N-2} & N-1 & N & N+1 \end{vmatrix} \tag{140}$$

where $\hat{N-2} = |1, 2, \ldots, N-2|$ and O denotes the $(N-2)\times(N-2)$ zero matrix. Since the above determinant (140) is zero it indeed verifies that τ_n satisfies the bilinear equation (128) identically. Thus we have proved that the τ-function defined by (132) gives the N-soliton solution of the DΔKP equation (126).[152]

3.3. *Rational solutions*

In this section, our aim is to describe the method of finding a class of rational solutions for the DΔKP equation. For this purpose, we consider the set of linear partial differential–difference equations (133) with (134) as particular solution. Notice that $f_n^{(j)}$ in (134) can be expressed as a formal power series in p_j and hence we have

$$\left(1+p_j\right)^n \exp\left(p_j t_1 + p_j^2 t_2\right) = \left(1+n^{(1)} p_j + \frac{n^{(2)}}{2!} p_j^2 + \cdots\right)\left(1+\left(p_j t_1 + p_j^2 t_2\right) + \frac{1}{2}\left(p_j t_1 + p_j^2 t_2\right)^2 + \cdots\right) = \sum_{m=0}^{\infty} P_m p_j^m \tag{141}$$

From (141), we have a set of polynomials in the variables n, t_1 and t_2. They can be expressed in a compact way as

$$P_m = \sum_{\substack{\alpha_0,\alpha_1,\alpha_2\ge 0 \\ \alpha_0+\alpha_1+2\alpha_2=m}} \frac{n(n-1)(n-2)\cdots(n-\alpha_0+1)t_1^{\alpha_1} t_2^{\alpha_2}}{\alpha_0!\,\alpha_1!\,\alpha_2!} \tag{142}$$

where $P_m = 0,\ \forall m \le 0$. These P_ms are called the differential–difference analogues of Schur polynomials. Also, one can see that they satisfy the following equations.

$$\begin{aligned} \Delta P_m &= P_{m-1} \\ \frac{\partial P_m}{\partial t_1} &= \Delta P_m \\ \frac{\partial P_m}{\partial t_2} &= \Delta^2 P_m \end{aligned} \tag{143}$$

From the above equations (143), we see that the P_ms are solutions of the equations in (133). But we have already shown that the Wronskian formed by any solution of (133), satisfies the bilinear form of DΔKP equation (128). Thus P_ms are also solutions of bilinear DΔKP equation

(128) Therefore, the polynomials P_m can be used to generate a class of rational solutions for (126). Consider the Wronskian formed by the P_ms

$$P_{l_1 l_2 \cdots l_N} = \begin{vmatrix} P_{l_1} & P_{l_2} & \cdots & P_{l_N} \\ P_{l_1-1} & P_{l_2-1} & \cdots & P_{l_N-1} \\ \vdots & \vdots & \ddots & \vdots \\ P_{l_1-N+1} & P_{l_2-N+1} & & P_{l_N-N+1} \end{vmatrix} \tag{144}$$

where $l_1, l_2, \cdots, l_N$ are distinct integers We list below first few rational solutions generated using (144):

$$P_0 = 1$$

$$P_1 = n + t_1$$

$$P_2 = \frac{n(n-1)}{2!} + \frac{t_1^2}{2!} + nt_1 + t_2$$

$$P_3 = \frac{n(n-1)(n-2)}{3!} + \frac{t_1^3}{3!} + \frac{n(n-1)}{2!}t_1 + \frac{nt_1^2}{2!} + nt_2 + t_1 t_2$$

$$P_{12} = \frac{n}{2} + \frac{n^2}{2} - t_2 + nt_1 + \frac{t_1^3}{2}$$

$$P_{13} = \frac{-n}{3} + \frac{n^3}{3} + n^2 t_1 + nt_1^2 + \frac{t_1^3}{3}$$

$$P_{23} = \frac{-n^2}{12} + \frac{n^4}{12} - nt_2 + t_2^2 - \frac{nt_1}{3} + \frac{n^3 t_1}{3} + \frac{n^2 t_1^2}{2} + \frac{nt_1^3}{3} + \frac{t_1^4}{12}$$

$$P_{123} = \frac{n}{3} + \frac{n^2}{2} + \frac{n^3}{6} - nt_2 + \frac{nt_1}{2} + \frac{n^2 t_1}{2} - t_2 t_1 + \frac{nt_1^2}{2} + \frac{t_1^3}{6}$$

Next, we construct a more general form of rational solutions For this purpose, we consider the τ-function given by

$$\tau_n = W\left(f_n^{(1)}, f_n^{(2)}, \ldots, f_n^{(N)}\right) \tag{145}$$

where the $f_n^{(j)}$s are given by

$$f_n^{(j)} = \left(\frac{\partial}{\partial p_j}\right)^{m_j} \exp\left[\eta(p_j)\right] \equiv P_{m_j}(p_j)\, \exp\left[\eta(p_j)\right], \quad j = 1,2, \ldots, N, m_j \geq 0 \tag{146}$$

and they satisfy eqns (133) with

$$\eta(p_j) = (n + n_j)\log(1 + p_j) + p_j\left(t_1 + \tilde{t}_{1_j}\right) + p_j^2\left(t_2 + \tilde{t}_{2_j}\right) \tag{147}$$

where n_j, $\tilde{t}_{1_j}$ and $\tilde{t}_{2_j}$ are arbitrary phase constants From (146), we have

$$P_{m_j}(p_j) = m_j! \sum_{\substack{\alpha_0,\alpha_1,\alpha_2,\geq 0\\ \alpha_0+\alpha_1+2\alpha_2=m_j}} \prod_{k=0}^{m_j} \frac{\left(\theta_k(p_j)\right)^{\alpha_k}}{\alpha_k!} \tag{148}$$

where

$$\theta_k(p_j) = \frac{1}{k!}\frac{\partial^k}{\partial p_j{}^k}\eta(p_j). \tag{149}$$

These polynomials $P_{m_j}(p_j)$ are the differential–difference analogues of the generalized Schur polynomials Again, it should be noted that the Wronskian formed by these generalized Schur polynomials are also rational solutions for the DΔKP equation (126). But this time the entries in the determinant are arbitrary linear combinations of the generalized Schur polynomials (148) It is easy to derive the N-soliton solutions and the rational solutions of DΔKP hierarchy. If we introduce the infinite number of time variables in the functions $f_n^{(j)}$ in such a way that they satisfy the linear equations

$$\frac{\partial}{\partial t_m} f_n^{(j)} = \Delta^m f_n^{(j)}, \quad j = 1,2, \quad , N, \quad m = 1,2, \tag{150}$$

then the Wronskian formed by these functions is the N-soliton solution of the DΔKP hierarchy. The rational solutions of the DΔKP hierarchy can be obtained as before.

4. Lie point symmetries and Painlevé-singularity confinement analysis

4 1 *Introduction*

In this section, we discuss the underlying Lie point symmetries of the DΔKP and also study the singularity structures of the solutions of this equation. These two aspects played important role in integrable systems for many years. The first one helps us to find special class of solutions in terms of new variables called similarity variables Using these variables we can also reduce the equation to a lower dimensional system. Furthermore, the structure of the symmetries reveals the nature of the associated Lie algebra of symmetry vector fields. The classification of Lie algebras of symmetry vector fields in turn brings out the associated solutions As far as the second part, it is well recognised that the Painlevé-singularity analysis played a vital role for several years in identifying possible integrable systems both in ODEs and PDEs. This is more direct and simple and yet a powerful approach to identify integrable systems though the nature of the singularities appears in the solutions. In the following discussion we give brief introduction to both these methods and apply the techniques to DΔKP. Detailed analysis will be published elsewhere.[154]

A Lie point symmetries

The concept of symmetry is extremely general, and the precise meaning of the term depends to a larger extent on the context we deal with When we use this concept in connection with differential equations, we reserve the word symmetry for group of transformations which leave the given system of differential equations invariant For computing the symmetries we adopt infinitesimal analogues of the transformation However, there are methods to recover the full group from infinitesimal symmetries In particular, the importance of Lie's invariance analysis lies in the fact that it is a systematic approach to discover a class of solutions, reductions to simpler equations through a new set of variables called similarity variables and similarity functions [54, 72] Numerous equations were analysed using this powerful tool Over the years, the method of Lie has been generalized in many directions Though there was intense activity on the symmetry analysis for continuous systems, it is surprising that until recently this theory had no impact on differential–difference systems and discrete equations as well However, it is worth mentioning that Maeda was the first one to apply the theory of symmetries to discrete systems in the variational formalism [140–142] Due to resurgence of interest in the integrability of discrete and differential-difference systems, the symmetry approach again becomes vital to look for symmetries, special solutions and reductions In this background, Levi and Winternitz, and later Quispel, Capel and Sahadevan developed the Lie symmetry method for differential–difference equations [143–150] The Lie point symmetries for the fully discrete equations were also initiated [146] Symmetry analysis for fully discrete systems is yet to be developed as an efficient tool as in the continuous case In view of the importance of Lie theory itself and the nontrivial applicability, we derive the Lie point symmetries of the DΔKP equation in this section, and use them for reduction process

4 2 *Lie's method*

Let us consider a function $u(\mathbf{x}, n)$, $u \in R$, $\mathbf{x} \in R^p$, $n \in Z$ We consider the differential–difference equation of the form

$$F(\mathbf{x}, n, u_n, u_{n,\mathbf{x}}, u_{n+1}, u_{n+l,\mathbf{x}}, \ \ ) = 0 \tag{151}$$

where $l \in Z$ We say that the Lie point symmetry group of transformation[145]

$$\tilde{n} = n, \tilde{\mathbf{x}} = \Lambda_g(\mathbf{x}), \tilde{u}_n = \omega_g(\mathbf{x}, n, u_n) \tag{152}$$

where g denotes the group parameters, λ_g and ω_g are invertible smooth functions, is admitted by the system (151) if $u_n(\mathbf{x})$ is a solution of (151), then $\tilde{u}_n(\tilde{\mathbf{x}})$ is also a solution of (151) The power behind the Lie group of transformation technique lies in the infinitesimal formulation of the group Lie's first fundamental theorem explicitly gives the connection between the infinitesimal transformation and the Lie group of transformation.[54] The infinitesimal one-parameter Lie point transformation corresponding to (152) is given by

$$\begin{aligned} \tilde{n} &= n \\ \tilde{\mathbf{x}} &= x + \in \xi(\mathbf{x}, n, u_n) \\ \tilde{u}_n &= u_n + \in \phi_n(\mathbf{x}, n, u_n) \end{aligned} \tag{153}$$

and the vector field corresponding to the infinitesimal transformation (153) is given by

$$\hat{X} = \sum_{i=1}^{p} \xi_i(\mathbf{x})\partial_{x_i} + \phi_n(\mathbf{x}, n, u_n)\partial_{u_n}. \tag{154}$$

The vector field (154) should be expanded to a larger space based on the order of the given equation (151) For example, if (151) is of order k then (154) should be extended (or prolonged) to $\mathrm{Pr}^{(k)}\,\widetilde{X}$ defined by

$$\mathrm{Pr}^{(k)}\,\hat{X}=\sum_{i=1}^{p}\xi_i(\mathbf{x})\partial_{x_i}+\sum_{l}\phi_{n+l}(\mathbf{x},n,u_{n+l})\partial_{u_{n+l}}$$

$$+\sum_{i=1}^{p}\sum_{l}\phi_{n+l}^{x_i}(\mathbf{x},n+l,u_{n+l}\cdots)\partial_{n+l,x_i}+\cdots \tag{155}$$

with

$$\phi_{n+l}^{x_i}=D_{x_i}\phi_{n+l}-\sum_{j=1}^{p}\left(D_{x_i}\xi_j\right)u_{n+l,x_j}$$

$$\phi_{n+l}^{x_i x_k}=D_{x_k}\phi_{n+l}^{x_i}-\sum_{j=1}^{p}\left(D_{x_k}\xi_j\right)u_{n+l,x_i x_j} \tag{156}$$

where

$$D_{x_i}\psi=\frac{\partial\psi}{\partial x_i}+\sum_{l}\frac{\partial\psi}{\partial u_n}\frac{\partial u_n}{\partial x_i}+\cdots \tag{157}$$

denotes the total derivative operator as in the continuous case.[54] Now, the invariance condition is given by

$$\mathrm{Pr}^{(k)}\,\hat{X}\cdot F\big|_{F=0}=0. \tag{158}$$

The main difference between the continuous and differential–difference case is the summation over l in (155) The number of terms we have to keep depends on the discrete order of the equation. As in the continuous case, eqn (151) is invariant under the action of (152) if the condition (158) holds good. Equation (158) gives the invariant condition from which we have to find the infinitesimal generators of the symmetry group (153) To do this, we expand eqn (158), use eqn (151) and equate the coefficients of the various derivatives of u_{n+l} to zero. This results in an over-determined system of linear equations for the infinitesimal generators of the group. We can solve these determining equations in a closed form and obtain symmetries. These symmetries are then used to find similarity solutions and reductions, etc In order to derive the similarity solutions of the system (151) we use the symmetries ξ_is and ϕ in the characteristic equation

$$\frac{dx_1}{\xi_1}=\frac{dx_2}{\xi_2}=\cdots=\frac{dx_p}{\xi_p}=\frac{du_{n+l}}{\phi_{n+l}} \tag{159}$$

After solving the above equation we arrive at $p-1$ new independent variables called similarity variables. The new dependent variable is the function of similarity variables, called the similarity function Substituting the value of u_n in terms of similarity function in the system (151), and simplifying we arrive at a new system which has the number of independent variables reduced by one compared to the given system. Obtaining lower dimensional equations using this procedure is called similarity reduction. We can also use infinitesimal generators to classify the solutions It can be shown that vector fields associated with infinitesimal generators form Lie algebras

4.3 *Symmetries and similarity reduction of the DΔKP equation*

In this section, we present the classical Lie point symmetries for the DΔKP equation. Using these symmetries we find similarity solution and similarity reduction of the DΔKP equation. As a similarity reduction we obtain Veselov–Shabat equation.[173]

For this purpose, we start with DΔKP equation in the form

$$\bar{u}_{xx} + u_{xx} - 2(1-\bar{u})\bar{u}_x + 2(1-u)u_x - \bar{u}_t + u_t = 0 \tag{160}$$

where we have used $u_n = u$ and $\bar{u}_{n+1} = \bar{u}$ Let us assume that the infinitesimal Lie group of transformation as

$$\begin{aligned} \tilde{n} &= n \\ \tilde{x} &= x + \in \xi(n,x,t,u) \\ \tilde{t} &= t + \in \tau(n,x,t,u) \\ \tilde{u} &= u + \in \phi(n,x,t,u). \end{aligned} \tag{161}$$

The vector field corresponding to (161) is given by

$$\hat{X} = \xi(n,x,t,u)\partial_x + \tau(n,x,t,u)\partial_t + \phi(n,x,t,u)\partial_u \tag{162}$$

Since the order of DΔKP equation is two we consider the second prolongation of the vector field (162), which is given by

$$\begin{aligned} \mathrm{Pr}^{(2)}\,\hat{X} = {} & \xi\partial_x + \tau\partial_t + \phi\partial u + \bar{\phi}\partial_{\bar{u}} + \phi^x\partial_{u_x} + \bar{\phi}^x\partial_{\bar{u}_x} + \phi^t\partial_{\bar{u}_t} + \bar{\phi}^t\partial_{\bar{u}_t} + \phi^{xx}\partial_{u_{xx}} \\ & + \bar{\phi}^{xx}\partial_{\bar{u}_{xx}} + \phi^{xt}\partial_{u_{xt}} + \bar{\phi}^{xt}\partial_{\bar{u}_{xt}} + \phi^{tt}\partial_{u_{tt}} + \bar{\phi}^{tt}\partial_{\bar{u}_{tt}}. \end{aligned} \tag{163}$$

We get the invariant

$$\bar{\phi}^t - \phi^t + 2\bar{\phi}^x - 2\phi^x - 2\bar{u}\bar{\phi}^x - 2\bar{\phi}\bar{u}_x + 2u\phi^x + 2\phi u_x - \bar{\phi}^{xx} - \phi^{xx} = 0 \tag{164}$$

on using the invariant condition (158) and applying (163) on eqn (160) To evaluate this expression we need $\phi^x, \bar{\phi}^x, \phi^t, \bar{\phi}^t, \phi^{xx}, \bar{\phi}^{xx}$ and we can explicitly find them using (156). They are listed as below:

$$\begin{aligned} \phi^x &= \phi_x + (\phi_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t \\ \bar{\phi}^x &= \bar{\phi}_x + (\bar{\phi}_{\bar{u}} - \bar{\xi}_x)\bar{u}_x - \bar{\tau}_x\bar{u}_t - \bar{\xi}_{\bar{u}}\bar{u}_x^2 - \bar{\tau}_{\bar{u}}\bar{u}_x\bar{u}_t \end{aligned}$$

$$
\begin{aligned}
\phi^{t} &= \phi_{t} - \xi_{t}u_{x} + (\phi_{u} - \tau_{t})u_{t} - \xi_{u}u_{x}u_{t} - \tau_{u}u_{t}^{2} \\
\overline{\phi}^{t} &= \overline{\phi}_{t} - \overline{\xi}_{t}\overline{u}_{x} + (\overline{\phi}_{uup} - \overline{\tau}_{t})\overline{u}_{t} - \overline{\xi}_{\overline{u}}\overline{u}_{x}\overline{u}_{t} - \overline{\tau}_{\overline{u}}\overline{u}_{t}^{2} \\
\phi^{xx} &= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_{x} - \tau_{xx}u_{t} + (\phi_{uu} - 2\xi_{xu})u_{x}^{2} \\
&\quad -2\tau_{xu}u_{x}u_{t} - \xi_{uu}u_{x}^{3} - \tau_{uu}u_{x}^{2}u_{t} + (\phi_{u} - 2\xi_{x})u_{xx} \\
&\quad -2\tau_{x}u_{xt} - 3\xi_{u}u_{x}u_{xx} - \tau_{u}u_{xx}u_{t} - 2\tau_{u}u_{x}u_{xt} \\
\overline{\phi}^{xx} &= \overline{\phi}_{xx} + (2\overline{\phi}_{x\overline{u}} - \overline{\xi}_{xx})\overline{u}_{x} - \overline{\tau}_{xx}\overline{u}_{t} + (\overline{\phi}_{\overline{u}\overline{u}} - 2\overline{\xi}_{x\overline{u}})\overline{u}_{x}^{2} \\
&\quad -2\overline{\tau}_{x\overline{u}}\overline{u}_{x}\overline{u}_{t} - \overline{\xi}_{\overline{u}\overline{u}}\overline{u}_{x}^{3} - \overline{\tau}_{\overline{u}\overline{u}}\overline{u}_{x}^{2}\overline{u}_{t} + (\overline{\phi}_{\overline{u}} - 2\overline{\xi}_{x})\overline{u}_{xx} \\
&\quad -2\overline{\tau}_{x}\overline{u}_{xt} - 3\overline{\xi}_{\overline{u}}\overline{u}_{x}\overline{u}_{xx} - \overline{\tau}_{\overline{u}}\overline{u}_{xx}\overline{u}_{t} - 2\overline{\tau}_{\overline{u}}\overline{u}_{x}\overline{u}_{xt}.
\end{aligned}
\tag{165}
$$

Now solve eqn (160) for $\overline{u}_{xx}$ and hence we have

$$\overline{u}_{xx} = \overline{u}_{t} - u_{t} - 2u_{x} + 2uu_{x} + 2u_{x} - 2\overline{u}\overline{u}_{x} - u_{xx}. \tag{166}$$

In order to get the determining equations for the infinitesimal generators we substitute the values of (165) in (164) and using (166), replacing $\overline{u}_{xx}$ in the resulting expression, we have an expression in $u_x, u_t, \overline{u}_x, \overline{u}_t, u_{xx}, u_{xt}, \overline{u}_{xt}$. Equating the coefficient of various powers of the derivatives of u and $\overline{u}$ in the resulting expression to zero we arrive at a linear homogeneous system of partial differential–difference equations. Solving this overdetermined system we obtain the symmetries

$$
\begin{aligned}
\xi &= \frac{1}{2}xf(t) + g(t) \\
\tau &= f(t) \\
\phi &= -\frac{1}{2}f(t)u - \frac{1}{4}f(t)x + \frac{1}{2}f(t) - \frac{1}{2}g(t).
\end{aligned}
\tag{167}
$$

In order to perform similarity reduction first we have to solve the characteristic equation

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\phi} \tag{168}$$

and derive the similarity variable and similarity function. On integrating (168), we relate u to the similarity function $F(\zeta, n)$ through

$$u = -\frac{f(t)}{4f(t)^{\frac{1}{2}}}\int \frac{g(t)}{f(t)^{\frac{3}{2}}}dt - \frac{\zeta f(t)}{4f(t)^{\frac{1}{2}}} + 1 - \frac{g(t)}{2f(t)} + \frac{F(\zeta, n)}{f(t)^{\frac{1}{2}}} \tag{169}$$

where the similarity variable ζ is given by

$$\zeta = \frac{x}{f(t)^{\frac{1}{2}}} - \int \frac{g(t)}{f(t)^{\frac{3}{2}}}dt. \tag{170}$$

Substituting the value of u in DΔKP equation we get the reduced equation

$$\overline{F}_\zeta + F_\zeta + \overline{F}^2 - F^2 = \alpha(n) \tag{171}$$

which is the Veselov–Shabat equation [173] The above system can be derived using l-reduction technique in Sato theory, and moreover this equation can be identified with delay Painlevé equations [153, 172]

B Painlevé-singularity confinement analysis

Even before the discovery of solitons, we had a remarkable theory to test the integrability of ODEs called singularity-analysis first proposed by Kowalevski [174, 175] The motivation for her discovery emerged from the fact that the critical singularities of a linear ODE are fixed.[45, 46, 52] This means that the location of singularities of the solutions of a linear ODE is determined entirely by the coefficients of the ODE This is certainly not the case in nonlinear systems. The structure of the singularity in nonlinear equations is more complicated. While the singularities are fixed for linear ODEs, in the case of nonlinear differential equations, their location (in the complex plane) depends on the initial conditions These singularities are called movable. Painlevé started asking for nonlinear ODEs with fixed critical singularities and attempted to classify all the second-order equations that belong to this class In particular, he examined equations of the form

$$w'' = f(w', w, z) \tag{172}$$

with f polynomial in w', rational in w and analytic in z This classification was completed by Gambier [52, 176] Thus came the discovery of the famous six Painlevé equations.[52]

The Painlevé equations are integrable in principle, however, their integration could not be performed with the methods available at that time. This situation has changed after the discovery of IST Ablowitz and Segur[177] showed that the IST technique could be used to linearize the Painlevé equations Soon after, Ablowitz, Ramani and Segur (ARS)[44] proposed the following conjecture "Every ODE which arises as a reduction of a completely integrable PDE is of Painlevé type (perhaps after a transformation of variables)" The integrable systems also possess what is called the Painlevé property. If all movable singularities of all solutions of an ODE are poles then we say that the system possesses Painlevé property ARS also provided an algorithm to test this property for ODEs. The ARS approach turned out to be the most powerful ool to isolate good candidates of integrable systems [3] Improvements made to it by Weiss *et* *zl* [47] and Gibbon and Tabor[178] to treat PDEs directly without the constraint of considering reluctions have resulted in several new equations The Painlevé test is undoubtedly powerful but t does not have the rigour of a theorem.

In recent years there has been a growing interest in the study of discrete equations In modrn science discrete equations play an important role [133, 134] With the advancement of highpeed computing, discretisation becomes unavoidable. Quite often, discrete models are more ealistic than continuous ones to understand the physics of the problem better However, we an clearly see a close parallel behaviour between the properties of the continuous systems and their discrete analogues.[162] At the same time, it is not obvious to find the discrete analogues of all integrable equations. In the past the focus was not much in this domain, but, has changed very recently due to the appearance of discrete Painlevé equations.[156,157]

Although there was some progress in discrete systems, no singularity-structure analysis (Painlevé method) existed for such systems until the discovery of singularity confinement (the equivalent of Painlevé analysis for discrete systems) by Grammaticos *et al.*[156] As in the Painlevé method for continuous systems, singularity confinement method becomes a powerful tool to detect possible discrete integrable systems. The singularity confinement was complemented by pre-image nonproliferation conditions which means that at each point the mapping will have a single pre-image. In the case of mapping, if no unique pre-image exists then there is no need to use singularity confinements.[170] The most striking use of singularity confinement is the discovery of discrete Painlevé equations.[159] It also plays a vital role in getting other integrability properties of discrete systems.[158–169]

4.4 *Algorithm*

The principle of singularity confinement can be stated as follows. In a rational mapping, singularity may appear spontaneously due to a particular choice of initial condition. In analogy with the continuous case we call this singularity 'movable'. The conjecture states[156] that in integrable systems this singularity must disappear after a few iterations. This is what is meant by 'confinement'. Also, memory of initial conditions must be recovered beyond singularity. We can present the method of implementing singularity confinement in the following way as Painlevé analysis in the ARS method[44] (details in Appendix I):

1) Find all possible singularities and check that they are movable.
   ARS: Find all possible leading behaviours
2) Determine when, at the earliest, the singularities can disappear.
   ARS: Find the resonances
3) Check that fine cancellations ensure that they actually disappear (gives constraints on the parameters)
   ARS: Check compatibility conditions at resonances

For the purpose of dealing with differential–difference equations, neither Painlevé method nor singularity confinement is enough to capture singularities. But Ramani *et al.*[137, 138] have shown how a nice combination of these two methods will allow us to treat differential–difference equations. In fact, it goes beyond in treating integro-differential equations as well. The basic idea is to consider the effect of a singularity in the continuous variable on the discrete evolution. For Painlevé property the singularity must be a pole, as well as the subsequent ones and in addition, this must disappear after a few iterations (in the discrete variable). This idea is very fruitful in dealing delay-differential equations. As an application of this method a few delay-Painlevé equations have also been obtained.[172]

4.5 *Painlevé-singularity confinement analysis for DΔKP*

We illustrate the Painlevé singularity confinement technique on DΔKP and study the singularity structure of the solutions. We introduce the following notations in our discussion $t_1 = x$,

$t_2 = t,\ E^{-1}u_0 = \underline{u},\ u_0 = u,\ Eu_0 = \overline{u}, E^2u_0 = \overline{\overline{u}}, \frac{\partial u}{\partial t} = u_t, \frac{\partial u}{\partial x} = u_x, \frac{\partial^2 u}{\partial x^2} = u_{xx}, \cdots$ Let us write the DΔKP equation as

$$u_t + 2(1-u)u_x - u_{xx} = \underline{u}_t + 2(1-\underline{u})\underline{u}_x + \underline{u}_{xx}. \tag{173}$$

According to singularity-confinement analysis, we assume that a given $\underline{u}$ is regular and using the above equation, we should study the propagation of singularities that appear for u The leading behaviour around the free singularity manifold $\phi(x, t) = 0$ is

$$u = \frac{\phi_x}{\phi} \tag{174}$$

To simplify the calculations, we apply Kruskal's ansatz, i e we put $\phi(x, t) = x + \psi(t)$ without loss of generality In this situation, $\underline{u}$ has a Taylor expansion and thus u be expressed in the Laurent series

$$u = \sum_{j=0}^{\infty} a_j(t)\phi^{j-1} \tag{175}$$

where $a_0 = 1$. Using the expansions (174) and (175) in (173) and performing the usual Painlevé analysis we find that ARS-resonances are $j = -1$, 2 and the compatibility condition for $j = 2$ is automatically satisfied

This is not enough to test integrability through singularity confinement. For this purpose we have to consider the first and second iterations of the recursion (173) and perform the usual Painlevé analysis and check the passing of the test The iterations of eqn (173) are

$$\overline{u}_t + 2(1-\overline{u})\overline{u}_x - \overline{u}_{xx} = u_t + 2(1-u)u_x + u_{xx}, \tag{176}$$

and

$$\overline{\overline{u}}_t + 2\left(1-\overline{\overline{u}}\right)\overline{\overline{u}}_x - \overline{\overline{u}}_{xx} = \overline{u}_t + 2(1-\overline{u})\overline{u}_x + \overline{u}_{xx} \tag{177}$$

We apply the nature of the singularity of the solution from the previous analysis to these above upshifted equations while doing Painlevé analysis and notice that DΔKP equation satisfies the singularity confinement criterion and Painlevé property, thus confirming the integrability of this equation from singularity analysis point of view Details will be published elsewhere [154]

5. A gauge equivalence of differential–difference Kadomtsev–Petviashvili equation

5.1 *Introduction*

In the previous sections we have studied the DΔKP equation in view of Sato theory We derived Lax pair, conserved quantities, generalized symmetries, Wronskian solutions, rational solutions and Lie point symmetries and tested the Painlevé-singularity confinement property.[151, 152, 154, 155] In this section, we discuss a gauge equivalence of the DΔKP equation.[153, 155]

One of the frequent questions asked in the theory of integrable systems is that of the relationship among various eigenvalue problems and of the associated systems. This question has significant implications, and to investigate it, gauge transformation has been introduced which connects one eigenvalue problem to the other and subsequently one integrable equation to the other. Such equivalence of integrable equations has been a subject of intensive research.[179–192] For example, a gauge equivalence of the nonlinear Schrödinger equation and Heisenberg ferromagnet equation was established by Lakshmanan,[179] and later Zakharov and Takhtajan[180] showed a gauge equivalence between the eigenvalue problems. It has been applied by Kundu[181, 182] to many systems in both 1 + 1 and 2 + 1 dimensions. It is well known that three different eigenvalue problems, that is, Ablowitz–Kaup–Newell–Segur (AKNS), Kaup–Newell (KN) and Wadati–Konno–Ichikawa (WKI) are connected through gauge transformation.[80, 81] In view of Sato theory, Kiso derived the modified hierarchies using gauge transformation.[183] There is a close connection among KP, modified KP and Harry–Dym hierarchies which has been established through gauge transformation. An unified approach to gauge transformation and reciprocal links for a broad class of nonlinear evolution equations has also been investigated.[184, 185]

Motivated by these works we discuss a gauge equivalence of the linear eigenvalue problem of DΔKP and derive a differential–difference equation related to DΔKP through a gauge transformation.[153] We find the conserved quantities and generalized symmetries for this system.

5.2 *A gauge equivalence of DΔKP equation*

We start with the pseudo-difference operator

$$\tilde{W} = w_0' + w_1'\Delta^{-1} + w_2'\Delta^{-2} + \cdots \tag{178}$$

where the w_is are functions of $n, t_1, t_2, \ldots$. The formal inverse of $\tilde{W}$ is given by

$$\tilde{W}^{-1} = v_0' + v_1'\Delta^{-1} + v_2'\Delta^{-2} + \cdots \tag{179}$$

Using $\tilde{W}\tilde{W}^{-1} = \tilde{W}^{-1}\tilde{W} = 1$, we get

$$\begin{aligned} 1 &= \left(w_0' + w_1'\Delta^{-1} + w_2'\Delta^{-2} + \cdots\right)\left(v_0' + v_1'\Delta^{-1} + v_2'\Delta^{-2} + \cdots\right) \\ &= w_0'\left(v_0' + v_1'\Delta^{-1} + v_2'\Delta^{-2} + \cdots\right) + w_1'\left(E^{-1}v_0'\Delta^{-1} - E^{-2}\Delta v_0'\Delta^{-2} + \right. \\ &\quad \left. + E^{-1}v_1'\Delta^{-2} + \cdots\right) + w_2'\left(E^{-2}v_0\Delta^{-2} + \cdots\right) + \cdots \end{aligned} \tag{180}$$

Rearrange the terms on the right-hand side of the above expression (180) and compare the like powers of Δ on both sides of (180). This results in an infinite number of equations for v_i's in terms of w_j's, $i, j = 1, 2, \ldots$.

$$\begin{aligned} v_0' &= \frac{1}{w_0'} \\ v_1' &= \frac{-w_1'}{w_0' E^{-1} w_0'} \end{aligned} \tag{181}$$

$$v_2' = -\frac{w_2'}{w_0'E^{-2}w_0'} - \frac{w_1'}{w_0'E^{-2}w_0'} + \frac{w_1'}{w_0'E^{-1}w_0'} + \frac{w_1'E^{-1}w_1'}{w_0'E^{-1}w_0'E^{-2}w_0'}$$

Now we introduce a gauge transformation for the L operator (48) defined in Section 2 by

$$\widetilde{L} = \phi^{-1}L\phi \tag{182}$$

where ϕ^{-1} means $\frac{1}{\phi}$ and the resulting expression for $\widetilde{L}$ is given by

$$\widetilde{L} = u'\Delta + u_0' + u_1'\Delta^{-1} + u_2'\Delta^{-2} + \cdots \tag{183}$$

with

$$\begin{aligned} u' &= \frac{E\phi}{\phi} \\ u_0' &= \frac{E\phi - \phi + u_0\phi}{\phi} \\ u_1' &= \frac{u_1E^{-1}\phi}{\phi} \\ u_2' &= \frac{u_1E^{-2}\phi - u_1E^{-1}\phi + u_2E^{-2}\phi}{\phi} \\ &\ \ \vdots \end{aligned} \tag{184}$$

It is possible to decompose $\widetilde{L}$ in (183) as

$$\widetilde{L} = \widetilde{W}\Delta\widetilde{W}^{-1} \tag{185}$$

Expanding the right-hand side of equation (185), and comparing it with eqn (183), we arrive hat the u_i's can be expressed in terms of w_j's for all $i, j = 0, 1, 2,$ We list the first few of hem.

$$\begin{aligned} u' &= \frac{w_0'}{Ew_0'} \\ u_0' &= \frac{1}{w_0'Ew_0'}\left(w_0'^2 - w_0'Ew_0' - w_0'Ew_1' + w_1'Ew_0'\right) \\ u_0' &= \frac{1}{w_0'Ew_0'E^{-1}w_0'}\left(w_1'w_0'Ew_0' - w_0'^2Ew_2' - w_0'^2Ew_1' + w_0'w_1'Ew_1' - w_1'^2Ew_0' + w_2'w_0'Ew_0'\right) \\ &\ \ \vdots \end{aligned} \tag{186}$$

ι the continuous case,[184] the B_ks are defined by

$$B_k = (L^k)^+ \tag{187}$$

here $()^+$ denotes the strictly positive powers of ∂ for the modified KP hierarchy. Here also we pect the same and therefore define

$$\widetilde{B}_k = \left(\widetilde{L}^k\right)^+ \tag{188}$$

where $()^+$ involves strictly positive powers of Δ and the generalized Lax equation is given by

$$\frac{\partial \widetilde{L}}{\partial t_k} = \left[\widetilde{B}_k, \widetilde{L}\right], k = 1,2,\cdots. \tag{189}$$

From (188) and (183), we get·

$$\begin{aligned}
&\widetilde{B}_1 = u'\Delta \\
&\widetilde{B}_2 = u'Eu'\Delta^{-2} + \left(u'Eu' - u'^2 + u'Eu'_0 + u'u'_0\right)\Delta \\
&\cdots
\end{aligned} \tag{190}$$

Using the Lax equation (189) for $k = 1$ we have the following set of equations

$$\begin{aligned}
&\frac{\partial u'}{\partial t_1} = u'Eu'_0 - u'u'_0 \\
&\frac{\partial u'_0}{\partial t_1} = u'Eu'_0 - u'u'_0 + u'Eu'_1 - u'_1E^{-1}u' \\
&\cdots
\end{aligned} \tag{191}$$

and for $k = 2$ we have,

$$\begin{aligned}
\frac{\partial u'}{\partial t_2} = {} & u'Eu'E^2u'_0 - u'Eu'Eu'_0 + u'Eu'E^2u'_1 - u'^2Eu'_0 + u'\left(Eu'_0\right)^2 + u'^2u'_0 \\
& - u'u'^2_0 - u'u'_1E^{-1}u' \\
& \cdot
\end{aligned} \tag{192}$$

Solving the above set of equations, we arrive at

$$\frac{\partial u'}{\partial t_2} = \frac{\partial^2 u'}{\partial t_1^2} - 2u'\frac{\partial u'}{\partial t_1} + 2u'\Delta^{-1}\frac{\partial}{\partial t_1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right) + 2\frac{\partial u'}{\partial t_1}\Delta^{-1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right). \tag{193}$$

Next we derive the conserved quantities and generalized symmetries for this system

5.3. *Conserved quantities*

In Section 2, we have derived the conserved quantities and generalized symmetries for DΔKP equation (66) For this purpose we follow the procedure described in Matsukidiara *et al* [117] Here, we adopt the same technique and present the conserved quantities and generalized symmetries of the equation (193) For this purpose, we first consider the linear eigenvalue problem associated with the generalized Lax equation (189)

$$\widetilde{L}\psi = \lambda\psi$$
$$\frac{\partial\psi}{\partial t_k} = \widetilde{B}_k\psi \tag{194}$$

where $\lambda_{t_k} = 0$ We assume that $\widetilde{B}_k^c = \widetilde{B}_k - \widetilde{L}_k$, and hence rewrite eqn (194) as

$$\frac{\partial\psi}{\partial t_k} = \left(\widetilde{L}^k + \widetilde{B}_k^c\right)\psi \tag{195}$$

It is noticed that $\widetilde{B}_k^c$ consists of terms involving Δ^{-j}, $j = 0, 1, 2, \ldots$. Now we will express Δ^{-j}, $j = 1, 2, \ldots$ in terms of $\widetilde{L}^{-j}$ For this purpose, first we find $\widetilde{L}^{-1}$ We assume that $\widetilde{L}^{-1}$ is of the form

$$\widetilde{L}^{-1} = q_1'\Delta^{-1} + q_2'\Delta^{-2} + q_3'\Delta^{-3} + \cdots \tag{196}$$

Using $\widetilde{L}^{-1}\widetilde{L} = 1$ we have

$$\begin{aligned} 1 &= \left(q_1'\Delta^{-1} + q_2'\Delta^{-2} + \cdots\right)\left(u'\Delta + u_0' + u_1'\Delta^{-1} + u_2'\Delta^{-2} + \cdots\right) \\ &= q_1'\left(E^{-1}u' - E^{-2}\Delta u'\Delta^{-1} + E^{-3}\Delta^2 u'\Delta^{-2} + \cdots + E^{-1}u_0'\Delta^{-1} - E^{-2}\Delta u_0'\Delta^{-2} + \right. \\ &\quad \left. + E^{-1}u_1'\Delta^{-2} + \cdots\right) + q_2'\left(E^{-2}u'\Delta^{-1} - 2E^{-3}\Delta^2 u'\Delta^{-2} + \cdots + E^{-2}u_0'\Delta^{-2} + \cdots\right) + q_3'\left(E^{-3}u'\Delta^{-2} + \cdots + \cdots\right) \end{aligned} \tag{197}$$

Comparing the like powers of Δ on both sides of (197) we get

$$\begin{aligned} q_1' &= \frac{1}{E^{-1}u'} \\ q_2' &= \frac{1}{E^{-1}u'E^{-2}u'}\left(E^{-1}u' - E^{-2}u' - E^{-1}u_0'\right) \\ q_3' &= \frac{1}{E^{-1}u'E^{-2}u'E^{-3}u'}\left(-E^{-1}u_1'E^{-2}u' + E^{-1}u'E^{-2}u' - E^{-1}u_0'E^{-2}u'\right. \\ &\quad \left. -2E^{-3}u'E^{-1}u' + E^{-3}u'E^{-2}u' + 2E^{-3}u'E^{-1}u_0' - E^{-2}u_0'E^{-1}u' + E^{-1}u_0'E^{-2}u_0'\right) \end{aligned} \tag{198}$$

Using Leibniz rule (9) and (196), we can derive the higher powers of $\widetilde{L}^{-j}$, $j = 1, 2, \ldots$. We list some of them.

$$\begin{aligned} \widetilde{L}^{-2} &= q_1'E^{-1}q_1'\Delta^{-2} + \left(-q_1'E^{-1}q_1' + q_1'E^{-2}q_1' + q_1'E^{-1}q_2' + q_2'E^{-2}q_1'\right)\Delta^{-3} + \cdots \\ \widetilde{L}^{-3} &= q_1'E^{-1}q_1'E^{-2}q_2'\Delta^{-3} + \cdots \\ &\cdots \end{aligned} \tag{199}$$

Using (198) and (199), we can present the values for Δ^{-j}, $j = 1, 2, \cdots$ in terms of negative powers of L as

$$
\begin{aligned}
\Delta^{-1} &= E^{-1}u'\tilde{L}^{-1} + \left(-E^{-1}u'^2 + E^{-2}u'E^{-1}u' + E^{-1}u'E^{-1}u_0'\right)L^{-2} + \left(-E^{-1}u'3\right.\\
&\quad +2E^{-1}u'^2E^{-3}u' + E^{-1}u'^2E^{-2}u_0' + 2E^{-1}u'^2E^{-1}u_0' + E^{-1}u'E^{-2}u'^2\\
&\quad -2E^{-1}u'E^{-2}u'E^{-3}u' - E^{-1}u'E^{-2}u'E^{-2}u_0' - 2E^{-1}u'E^{-1}u_0'E^{-3}u'\\
&\quad -E^{-1}u'E^{-1}u_0'E^{-2}u_0' - E^{-1}u'E^{-1}u_0'^2 - E^{-1}u_1'E^{-2}u' + E^{-1}u'E^{-2}u'\\
&\quad -E^{-1}u_0'E^{-2}u' - 2E^{-3}u'E^{-1}u' + E^{-3}u'E^{-2}u' + 2E^{-3}u'E^{-1}u_0'\\
&\quad \left.-E^{-2}u_0'E^{-1}u' + E^{-1}u_0'E^{-2}u_0'\right)\tilde{L}^{-3} + \\
\Delta^{-2} &= E^{-1}u'E^{-2}u'\tilde{L}^{-2} - \left(E^{-1}u'^2E^{-2}u' + E^{-1}u'E^{-2}u'^2 - 2E^{-1}u'E^{-2}u'E^{-3}u'\right.\\
&\quad \left.-E^{-1}u'E^{-2}u'E^{-2}u_0' - E^{-1}u'E^{-2}u'E^{-1}u_0'\right)\tilde{L}^{-3} + \cdots\\
\Delta^{-3} &= E^{-1}u'E^{-2}u'E^{-3}u'\tilde{L}^{-3} + \cdots
\end{aligned}
\tag{200}
$$

Now, using these results, we can write down eqn (195) in the form

$$
\frac{\partial \psi}{\partial t_k} = \left(\tilde{L}^k + \sigma_0^{(k)} + \sigma_1^{(k)}L^{-1} + \sigma_2^{(k)}L^{-2} + \cdots\right)\psi \tag{201}
$$

where $\sigma_j^{(k)}$s are functions of u', u_i's for all $i, j = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$ On using $L^j\psi = \lambda^j\psi$ in (201), we obtain

$$
\begin{aligned}
\frac{\partial \psi}{\partial t_k} &= \left(\lambda^m + \sigma_0^{(k)} + \frac{\sigma_1^{(k)}}{\lambda} + \frac{\sigma_2^{(k)}}{\lambda^2} + \cdots\right)\psi\\
\frac{1}{\psi}\frac{\partial \psi}{\partial t_k} &= \left(\lambda^k + \sigma_0^{(k)} + \frac{\sigma_1^{(k)}}{\lambda} + \frac{\sigma_2^{(k)}}{\lambda^2} + \cdots\right)\\
\frac{\partial}{\partial t_k}\log\psi &= \lambda^k + \sum_{j=0}^{\infty}\frac{\sigma_j^{(k)}}{\lambda^j}.
\end{aligned}
\tag{202}
$$

We denote $\sigma^{(k)} = \Sigma_{j=0}^{\infty}\sigma_j^{(k)}\lambda^{-j}$ and hence eqn (202) becomes

$$
\sigma^{(k)} = \frac{\partial(\log\psi)}{\partial t_k} - \lambda^k \tag{203}
$$

Differentiating eqn (203) with respect to the variable t_m, we will arrive at the conservation laws

$$
\frac{\partial \sigma^{(k)}}{\partial t_m} = \frac{\partial}{\partial t_k}\left(\frac{\partial \log\psi}{\partial t_m}\right), \quad m, k = 1, 2, \cdots \quad m \neq k. \tag{204}
$$

We the list first few of the $\sigma_j^{(k)}$s :

$$\begin{aligned}
\sigma_0^{(1)} &= -u_0' \\
\sigma_1^{(1)} &= -u_1' E^{-1} u' \\
\sigma_2^{(1)} &= u_1' E^{-1} u'^2 - u_1' E^{-2} u' E^{-1} u' - u_1' E^{-1} u' E^{-1} u_0' - u_2 E^{-1} u' E^{-2} u'
\end{aligned} \tag{205}$$

$$\begin{aligned}
\sigma_0^{(2)} &= -u' E u_0' + u' u_0' - u' E u_1' - u_0'^2 - u_1' E^{-1} u' \\
\sigma_1^{(2)} &= -u' E u' E^{-1} u' + u' u_1' E^{-1} u' - u' E u_2' E^{-1} u' - u_0' u_1' E^{-1} u' + u_1' E^{-1} u'^2 \\
&\quad - u_1' E^{-1} u' E^{-2} u'
\end{aligned} \tag{206}$$

It is known that the Lax equation with $k = 1$ gives

$$\begin{aligned}
\frac{\partial u'}{\partial t_1} &= u' E u_0' - u' u_0' \\
\frac{\partial u_0'}{\partial t_1} &= u' E u_0' - u' u_0' + u' E u_1' - u_1' E^{-1} u'
\end{aligned} \tag{207}$$

From the above equations (207), we can express $u_0', u_1', \cdot\cdot$, in term of u' and we list the first few of u_j's for $j = 0, 1, 2$,

$$\begin{aligned}
u_0' &= \Delta^{-1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right) \\
u_1' &= \frac{1}{E^{-1}u'}\left(\Delta^{-2}\frac{\partial}{\partial t_1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right) - \Delta^{-1}\left(\frac{\partial u'}{\partial t_1}\right)\right)
\end{aligned} \tag{208}$$

Now substituting the values of $u_0', u_1', \cdot$ in $\sigma_j^{(1)}$ (205), we obtain the conserved densities of the differential–difference equation (193) We list below some of them:

$$\begin{aligned}
\sigma_0^{(1)} &= -\Delta^{-1}\left(\frac{1}{u'}\frac{\partial u'}{t_1}\right) \\
\sigma_1^{(1)} &= -\Delta^{-2}\left(-\frac{1}{u'^2}\frac{\partial u'^2}{\partial t_1} + \frac{1}{u'}\frac{\partial^2 u'}{\partial t_1{}^2}\right) + \Delta^{-1}\frac{\partial u'}{\partial t_1} \\
\sigma_2^{(1)} &= E^{-1}u'\Delta^{-2}\frac{\partial}{\partial t_1}\left(\frac{1}{u}\frac{\partial u'}{\partial t_1}\right) - E^{-1}u'\Delta^{-1}\frac{\partial u'}{\partial t_1} - E^{-2}u'\Delta^{-2}\frac{\partial}{\partial t_1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right)
\end{aligned}$$

$$
\begin{aligned}
&+E^{-2}u'\Delta^{-1}\frac{\partial u'}{\partial t_1}-\Delta^{-2}\frac{\partial}{\partial t_1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right)E^{-1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right)\\
&+\Delta^{-2}\frac{\partial}{\partial t_1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right)\Delta^{-1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right)+\Delta^{-1}\frac{\partial u'}{\partial t_1}E^{-1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right)\\
&-\Delta^{-1}\frac{\partial u'}{\partial t_1}\Delta^{-1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right)+\Delta^{-1}\left[E^{-1}u'\Delta^{-1}\frac{\partial}{\partial t_1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right)-E^{-1}u'\frac{\partial u'}{\partial t_1}\right]\\
&-\Delta^{-1}\left[u'\Delta^{-2}\frac{\partial}{\partial t_1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right)-u'\Delta^{-1}\frac{\partial u'}{\partial t_1}\right]\\
&-\Delta^{-1}\left[E^{-2}u'\Delta^{-2}\frac{\partial}{\partial t_1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right)-E^{-2}u'\Delta^{-1}\frac{\partial u'}{\partial t_1}\right]\\
&-\Delta^{-1}\left[E^{-1}u'\frac{\partial}{\partial t_1}\left(\frac{1}{E^{-1}u'}\left(\Delta^{-2}\frac{\partial}{\partial t_1}\left(\frac{1}{u'}\frac{\partial u'}{\partial t_1}\right)-\Delta^{-1}\frac{\partial u'}{\partial t_1}\right)\right)\right]
\end{aligned}
\tag{209}
$$

.

5.4. *Generalized symmetries*

In this section, we derive the generalized symmetries of the differential–difference equation (193).[153, 155] For this purpose, we consider the linear eigenvalue problem

$$
\begin{aligned}
\tilde{L}\psi_n &= \lambda\psi_n\\
\frac{\partial\psi_n}{\partial t_m} &= \tilde{B}_m\psi_n
\end{aligned}
\tag{210}
$$

and the adjoint eigenvalue problem

$$
\begin{aligned}
\tilde{L}^*\psi_n^* &= \lambda\psi_n^*\\
\frac{\partial\psi_n^*}{\partial t_m} &= -B_m^*\psi_n^*
\end{aligned}
\tag{211}
$$

where λ is the spectral parameter and is independent of n and t_m. We follow the same procedure as in Section 2 to compute the eigenfunctions ψ_n and ψ_n^* They are given by

$$
\begin{aligned}
\psi_n &= \left(w_0'+\frac{w_1'}{\lambda}+\frac{w_2'}{\lambda^2}+\cdots\right)(1+\lambda)^n\exp\left(\sum_{i=1}^{\infty}t_i\lambda^i\right)\\
\psi_n^* &= \left(w_0'^*+\frac{w_1'^*}{\lambda}+\frac{w_2'^*}{\lambda^2}+\cdots\right)(1+\lambda)^{-n}\exp\left(-\sum_{i=1}^{\infty}t_i\lambda^i\right)
\end{aligned}
\tag{212}
$$

with

$$w_0'^* = v_0'$$
$$w_1'^* = -Ev_1' \tag{213}$$
$$w_2'^* = -\left(E\Delta^2 v_1' + 2E^2\Delta v_2' + E^3 v_3'\right)$$
$$\vdots$$

For convenience, we denote u' as u_n The equations representing the linear eigenvalue problem and its adjoint are given by

$$\frac{\partial\psi_n}{\partial t_1} = u_n\psi_{n+1} - u_n\psi_n$$
$$\frac{\partial\psi_n}{\partial t_2} = u_n u_{n+1}\psi_{n+2} - u_n u_{n+1}\psi_{n+1} - u_n^2\psi_{n+1} + \frac{\partial u_n}{\partial t_1}\psi_{n+1} + u_n^2\psi_n$$
$$+2u_n\Delta^{-1}\left(\frac{1}{u_n}\frac{\partial u_n}{\partial t_1}\right)\psi_{n+1} - \frac{\partial u_n}{\partial t_1}\psi_n - 2u_n\Delta^{-1}\left(\frac{1}{u_n}\frac{\partial u_n}{\partial t_1}\right)\psi_n \tag{214}$$
$$\frac{\partial\psi_n^*}{\partial t_1} = u_n\psi_n^* - u_{n-1}\psi_{n-1}^*$$
$$\frac{\partial\psi_n^*}{\partial t_2} = u_n u_{n-1}\psi_n^* - u_{n-1}u_{n-2}\psi_{n-2}^* - u_n^2\psi_n^* + \frac{\partial u_n}{\partial t_1}\psi_n^* + u_{n-1}^2\psi_{n-1}^*$$
$$+2u_n\Delta^{-1}\left(\frac{1}{u_n}\frac{\partial u_n}{\partial t_1}\right)\psi_n^* - 2u_{n-1}\Delta^{-1}\left(\frac{1}{u_{n-1}}\frac{\partial u_{n-1}}{\partial t_1}\right)\psi_{n-1}^* - \frac{\partial u_{n-1}}{\partial t_1}\psi_{n-1}^*.$$

By taking

$$\psi_n^* = \Delta E^{-1}\phi_n \tag{215}$$

and

$$s = \psi_n\phi_n \tag{216}$$

one can check that

$$\frac{\partial s}{\partial t_2} = \frac{\partial^2 s}{\partial t_1^2} - 2u_n\frac{\partial s}{\partial t_1} + 2u_n\Delta^{-1}\frac{\partial}{\partial t_1}\left(\frac{1}{u_n}\frac{\partial s}{\partial t_1}\right) + 2\frac{\partial s}{\partial t_1}\Delta^{-1}\left(\frac{1}{u_n}\frac{\partial u_n}{\partial t_1}\right) \tag{217}$$

is consistent with (214). The solution of eqn (217) is derived from

$$\psi\Delta^{-1}E\psi^* = \sum_{m=0}^{\infty} s_m\lambda^{-(m+1)}, \tag{218}$$

where

$$s_m = \sum_{j=0}^{m} w_j w_{m-j}^{*} \tag{219}$$

We denote $S = \frac{\partial s}{\partial t_1}$ and $u_n = u$ Then, eqn (217) becomes

$$\frac{\partial S}{\partial t_2} = \frac{\partial^2 s}{\partial t_1{}^2} - 2u\frac{\partial s}{\partial t_1} - 2S\frac{\partial u}{\partial t_1} + 2u\Delta^{-1}\left(-\frac{2}{u^2}\frac{\partial u}{\partial t_1}\frac{\partial s}{\partial t_1} + \frac{2S}{u^3}\frac{\partial u^2}{\partial t_1} + \frac{1}{u}\frac{\partial^2 S}{\partial t_1{}^2} - \frac{S}{u^2}\frac{\partial^2 u}{\partial t_1{}^2}\right)$$
$$+2S\Delta^{-1}\left(-\frac{1}{u^2}\frac{\partial u^2}{\partial t_1} + \frac{1}{u}\frac{\partial^2 u}{\partial t_1{}^2}\right) + 2\frac{\partial u}{\partial t_1}\Delta^{-1}\left(\frac{1}{u}\frac{\partial S}{\partial t_1} - \frac{S}{u^2}\frac{\partial u}{\partial t_1}\right) + 2\frac{\partial S}{\partial t_1}\Delta^{-1}\left(\frac{1}{u}\frac{\partial u}{\partial t_1}\right) \tag{220}$$

which is nothing but the symmetry invariant equation of (193). The solutions of eqn (220) are the generalized symmetries of eqn (193). We first list a few generalized symmetries of (193):

$$S_0 = \frac{\partial u}{\partial t_1}$$
$$S_1 = \frac{\partial u}{\partial t_2} + \frac{\partial u}{\partial t_1}$$
$$S_2 = \frac{\partial}{\partial t_1}\left(\frac{\partial u}{\partial t_1} + 2u\Delta^{-1}\left(\frac{1}{u}\frac{\partial u}{\partial t_1}\right) - u^2 + u^3 - 3u^2\Delta^{-1}\left(\frac{1}{u}\frac{\partial u}{\partial t_1}\right) + 3u\left(\Delta^{-1}\left(\frac{1}{u}\frac{\partial u}{\partial t_1}\right)\right)^2 \right. \tag{221}$$
$$+3u\Delta^{-2}\left(-\frac{1}{u^2}\frac{\partial u^2}{\partial t_1} + \frac{1}{u}\frac{\partial^2 u}{\partial t_1{}^2}\right) - 3u\Delta^{-1}\frac{\partial u}{\partial t_1} + \frac{\partial^2 u}{\partial t_1{}^2} + 3\frac{\partial u}{\partial t_1}\Delta^{-1}\left(\frac{1}{u}\frac{\partial u}{\partial t_1}\right) - 3u\frac{\partial u}{\partial t_1}$$
$$\left. +3u\Delta^{-1}\left(-\frac{1}{u^2}\frac{\partial u^2}{\partial t_1} + \frac{1}{u}\frac{\partial^2 u}{\partial t_1{}^2}\right)\right)$$

$\cdots$

5.5. *2-Reduction*

In this section, we derive the 2-reduced gauge equivalent DΔKP equation. For this, we consider

$$\tilde{L}^2 = \tilde{B}_2 \tag{222}$$

which will give the constraints

$$u'Eu_0' - u'u_0' + u'Eu_1' + u_0'^2 + u_1'E^{-1}u' = 0$$
$$u'Eu_1' - u'u_1' + u'Eu_2' + u_0'u_1' - u_1'E^{-1}u' + u_1'E^{-2}u' + u_1'E^{-1}u_0' + u_2'E^{-2}u' = 0 \tag{223}$$
$$\cdots,$$

Imposing these on the constraints in eqns (191) and (192) we finally arrive at the reduced system

$$\frac{1}{u'_{n+1}}\frac{\partial u'_{n+1}}{\partial t_1}+\frac{1}{u'_n}\frac{\partial u'_n}{\partial t_1}=u'_n-u'_{n+1}. \tag{224}$$

Here $u'_n = u'$ and eqn (224) is related to the Kac–van Moerbeke system[47] by the transformation $u'_n=\log\left(\frac{v'_{n+1}}{v'_n}\right)$. The Painlevé-singularity confinement analysis and Lie symmetry analysis of (193) will be published elsewhere.[155]

Acknowledgements

In this paper we review some of the results obtained recently with our collaborators Grammaticos, Ramani and Thamizharasi Tamizhmani.

We express our sincere thanks to B.Grammaticos, A Ramani, Thamizharasi Tamizhmani, Y. Ohta, J. Satsuma, J. Hietarinta, J J C Nimmo and W. Oevel for fruitful discussions and encouragement at various stages KMT is supported by the Indo–French Project 1201-1 SK likes to thank the Council for Scientific and Industrial Research (CSIR), India, for financial support

References

1 ARNOLD, V I — *Mathematical methods of classical mechanics*, Springer, 1978.

2 LAKSHMANAN, M — Nonlinear physics Integrability, chaos and beyond, *J Franklin Inst B*, 1997, **334**, 909–969.

3 ABLOWITZ, M J AND CLARKSON, P A — *Solitons, nonlinear evolution equations and inverse scattering*, Cambridge University Press, 1991

4 FADDEEV, L D AND TAKHTAJAN, L A — *Hamiltonian methods in the theory of solitons*, Springer–Verlag, 1987

5 ZABUSKY, N J AND KRUSKAL, M D — Interactions of 'solitons' in a collisionless plasma and the recurrence of the initial states, *Phys. Rev Lett*, 1965, **15**, 240–243

6 GARDNER, C S, GREENE, J M, KRUSKAL, M D AND MIURA, R M — Method for solving the Korteweg–de Vries equation, *Phys Rev Lett*, 1967, **19**, 1095–1097

7 ZAKHAROV, V E AND SHABAT, A. B — Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, *Sov Phys JETP*, 1972, **34**, 62–69

8 ABLOWITZ, M J, KAUP, D J, NEWELL, A C AND SEGUR, H — Method for solving the Sine–Gordon equation, *Phys Rev Lett*, 1973, **30**, 1262–1264

9 LAX, P D — Integral of nonlinear equations of evolution and solitary waves, *Commun Pure Appl Math*, 1968, **21**, 467–490

10 HIROTA, R. — *Direct method in soliton theory*, Iwanami, Tokyo, 1992 (in Japanese)

11 MATSUNO, J — *Bilinear transformation method*, Academic Press, 1984

12. HIROTA, R — Exact N-soliton solutions of a nonlinear lumped-network equation, *J Phys Soc Jap*, 1973, **35**, 286–288

3. HIROTA, R — Exact N-soliton solutions of a nonlinear lumped self-dual network equation, *J. Phys. Soc Jap*, 1973, **35**, 289–294

4 HIROTA, R — *Direct methods in soliton theory* In Solitons (R K Bullough and P J Caudrey (eds)), Springer–Verlag, 1980, pp 157–176

5 HIROTA, R — Nonlinear partial difference equations I A difference analogue of the Korteweg–de Vries equation, *J Phys Soc Jap*, 1977, **43**, 1424–1433

6 HIROTA, R — Nonlinear partial difference equations II. Discrete-time Toda equation, *J Phys Soc Jap*, 1977, **43**, 2074–2078

7 HIROTA, R — Nonlinear partial difference equations III Discrete Sine–Gordon equation, *J Phys Soc Jap*, 1977, **43**, 2079–2086

8 HIROTA, R — Nonlinear partial difference equations IV Backlund transformation for the discrete-time Toda equation, *J Phys Soc Jap*, 1977, **45**, 321–332

9 HIROTA, R — Nonlinear partial difference equations. V Nonlinear equations reducible to linear equations, *J Phys Soc Jap*, 1979, **46**, 312–319

0 HIROTA, R, OHTA, Y AND SATSUMA, J — Solutions of the KP equation and the two dimensional Toda equations, *J Phys Soc Jap*, 1988, **57**, 1901–1904

1 HIROTA, R, OHTA, Y AND SATSUMA, J — Wronskian structures of soliton equations, *Prog Theor Phys Suppl*, 1988, **94**, 59–72

2 HIROTA, R — Soliton solutions to the BKP equations I The Pfaffian technique, *J Phys Soc Jap*, 1989, **58**, 2285–2296

3 HIROTA, R — Soliton solutions to the BKP equation II The integral equation, *J Phys. Soc Jap*, 1989, **58**, 2705–2712

4 HIROTA, R — Discrete analogue of a generalized Toda equation, *J Phys. Soc Jap*, 1981, **50**, 3785–3791

5 HIROTA, R AND SATSUMA, J — Nonlinear evolution equations generated from the Backlund transformation for the Toda lattice, *Prog Theor Phys*, 1976, **55**, 2037–2038

5 HIROTA, R AND SATSUMA, J — Soliton solutions of a coupled Korteweg–de Vries equations, *Phys Lett A*, 1981, **85**, 407–408

7 SATSUMA, J — N-Soliton solution of the two-dimensional Korteweg–de Vries equation, *J. Phys Soc Jap*, 1976, **40**, 286–290

3 SATSUMA, J — A Wronskian representation of N-soliton solutions of nonlinear evolution equations, *J Phys Soc. Jap*, 1979, **46**, 359–360

) FREEMAN, N C AND NIMMO, J J C — Soliton solutions of the Korteweg–de Vries and Kadomtsev–Petviashvili equations The Wronskian technique, *Phys Lett A*, 1983, **95**, 1–3.

) NIMMO, J J C AND FREEMAN, N C — A method of obtaining the soliton solution of the Boussinesq equation in terms of a Wronskian, *Phys Lett A*, 1983, **95**, 4–6

l MIYAKE, S, OHTA, Y AND SATSUMA, J — A representation of solutions for the KP hierarchy and its algebraic structure, *J Phys Soc Jap*, 1990, **59**, 48–55

32 HIROTA, R — Solutions of the classical Boussinesq equation and the spherical Boussinesq equation The Wronskian technique, *J Phys Soc Jap*, 1986, **55**, 2137–2150

33 HIROTA, R, ITO, M AND KAKO, F — Two-dimensional Toda lattice equations, *Prog Theor Phys Suppl*, 1988, **94**, 42–58

34 OHTA, Y AND HIROTA, R — A discrete KdV equation and its Casorati determinant solution, *J Phys Soc Jap*, 1991, **60**, 2095

35 OHTA, Y, HIROTA, R, TSUJIMOTO, S AND IAMI, T. — Casorati and discrete Gram type determinant representations of solutions to the discrete KP hierarchy, *J Phys Soc Jap*, 1993, **62**, 1872–1886

36. OHTA, Y, KAJIWARA, K, MATSUKIDAIRA, J AND SATSUMA, J — Casorati determinant solution for the relativistic Toda lattice equation, *J Math Phys*, 1993, **34**, 5190–5204

37 HIETARINTA, J — A search for bilinear equations passing Hirota's three-soliton condition I KdV-type bilinear equations, *J Math Phys*, 1987, **28**, 1732–1742

38 HIETARINTA, J — A search for bilinear equations passing Hirota's three-soliton condition II MKdV-type bilinear equations, *J Math Phys*, 1987, **28**, 2094–2101

39 HIETARINTA, J — A search for bilinear equations passing Hirota's three-soliton condition III Sine–Gordon-type bilinear equations, *J Math Phys*, 1987, **28**, 2586–2592

40 HIETARINTA, J — A search for bilinear equations passing Hirota's three-soliton condition IV Complex bilinear equations, *J Math Phys*, 1988, **29**, 628–635

41 HIETARINTA, J — Hirota's bilinear method and its generalization, *Int J Mod Phys A*, 1997, **12**, 43–51

42 HIROTA, R — Reduction of soliton equations in bilinear form, *Physica D*, 1986, **18**, 161–170

43 ABLOWITZ, M J AND SEGUR, H — Exact linearization of a Painlevé transcendent, *Phys Rev Lett*, 1977, **38**, 1103–1106

44 ABLOWITZ, M J, RAMANI, A AND SEGUR, H — Nonlinear evolution equations and ordinary differential equations of Painlevé type, *Lett Nuovo Cim*, 1978, **23**, 333–338

45. RAMANI, A, GRAMMATICOS, B AND BOUNTIS, T — The Painlevé property and singularity analysis of integrable and non-integrable systems, *Phys Rep*, 1989, **180**, 159–245

46. LAKSHMANAN, M AND SAHADEVAN, R. — Painlevé analysis, Lie symmetries and integrability of coupled nonlinear oscillators of polynomial type, *Phys Rep*, 1993, **224**, 1

47. WEISS, J, TABOR, M. AND CARNEVALE, G. — The Painlevé property for partial differential equations, *J Math Phys*, 1983, **24**, 522–526

48. CONTE, R — Invariant Painlevé analysis of partial differential equations, *Phys Lett A*, 1989, **140**, 383–390

49 LEVI, D AND WINTERNITZ, P (EDS) — *Painlevé transcendents, their asymptotics and physical applications*, Plenum, 1992

50 CONTE, R. Universal invariance properties of Painlevé analysis and Backlund transformation in nonlinear partial differential equations, *Phys Lett A*, 1988, **134**, 100–104

51 WEISS, J. Backlund transformations and the Painlevé property. In *Partially integrable equations* (R Conte and N Boccara, eds), NATO ASI Series C Mathematical and Physical Sciences, Vol 310, Kluwer, 1990, pp. 375–411

52 INCE, E L. *Ordinary differential equations*, Dover, 1956

53 FOKAS, A S AND SANTINI, P M. The recursion operator of the Kadomtsev–Petviashvili equation and the squared eigenfunctions of the Schrodinger operator, *Stud Appl Math*, 1986, **75**, 179–185

54 OLVER, P J. *Applications of Lie groups to differential equations*, Springer–Verlag, 1986

55 FOKAS, A S AND ANDERSON, R L. On the use of isospectral eigenvalue problems for obtaining hereditary symmetries for Hamiltonian systems, *J Math Phys*, 1982, **23**, 1066–1073

56 FUCHSSTEINER, B AND FOKAS, A S. Symplectic structures, their Backlund transformations and hereditary symmetries, *Physica D*, 1981, **4**, 47–66

57 OEVEL, W AND FUCHSSTEINER, B. The bi-Hamiltonian structure of some nonlinear fifth and seventh-order differential equations and recursion formulas for their symmetries and conserved covariants, *J Math Phys*, 1982, **23**, 358–363

58 KONOPELCHENKO, B. G. *Nonlinear integrable equations, Lecture Notes in Physics*, Vol 270, Springer–Verlag, 1987

59 CHEN, H H, LEE, Y C AND LIU, J E. On a new hierarchy of symmetries for the Kadomtsev–Petviashvili equation, *Physica D*, 1983, **9**, 439–445

60 TAMIZHMANI, K M. *Geometrical, group theoretical and singularity structure aspects of certain nonlinear partial differential equations*, Ph D Thesis, Bharathidasan University, Tiruchirapalli, India, 1986

61 MAGRI, F. A simple model of the integrable Hamiltonian equation, *J Math Phys*, 1978, **19**, 1156–1162

62 DICKEY, L A. *Soliton equations and Hamiltonian systems, Adv Ser Math Phys*, Vol 12, World Scientific, 1991

63 OEVEL, W, ZHANG, H AND FUCHSSTEINER, B. Symmetries, conserved quantities and hierarchies for some lattice systems with soliton structure, *J Math Phys*, 1991, **32**, 1908–1918

64 OEVEL, W, FUCHSSTEINER, B., ZHANG, H AND RAGNISCO, O. Master symmetries, angle variables and recursion operator of the relativistic Toda lattice, *J Math Phys*, 1989, **30**, 2664–2670

65 OEVEL, W, ZHANG, H AND FUCHSSTEINER, B. Master symmetries and multi-Hamiltonian formulations for some integrable lattice systems, *Prog. Theor Phys*, 1989, **81**, 294–308

66 FOKAS, A S. Symmetries and integrability, *Stud Appl Math*, 1987, **77**, 253–299

67 OEVEL, W AND FUCHSSTEINER, B. Explicit formulas for the symmetries and conservation laws of the Kadomtsev–Petviashvili equation, *Phys Lett. A*, 1982, **88**, 323–327

68 OEVEL, W AND FUCHSSTEINER, B — New hierarchies of nonlinear completely integrable systems related to a change of variables for evolution parameters, *Physica A*, 1987, **145**, 67

69 KOSMANN-SCHWARZBACH, Y — Lie algebras of symmetries of partial differential equations In *Differential geometric methods in mathematical physics*, Vol 241, (S Sternberg, ed), D Reidel, 1984

70 STRAMPP, W — Lax-pairs, spectral problems, and recursion operators, *J Math Phys*, 1984, **25**, 2905–2909

71 ZHANG, H , GUI-ZHANG TU, OEVEL, W AND FUCHSSTEINER, B — Symmetries, conserved quantities and hierarchies of some lattice systems with soliton structure, *J Math Phys*, 1991, **32**, 1908

72 STEPHANI, H — *Differential equations Their solution using symmetries*, Cambridge University Press, 1989

73 OVSIANNIKOV, L V — *Group analysis of differential equations*, Academic Press, 1982

74 ABLOWITZ, M J AND HABERMAN, R — Resonantly coupled nonlinear evolution equations, *J Math Phys*, 1975, **16**, 2301–2305

75 KAUP, D. J AND NEWELL, A C — An exact solution for a derivative nonlinear Schrodinger equation, *J Math Phys*, 1978, **19**, 798–801

76 WADATI, M , KONNO, K AND ICHIKAWA, Y H — A generalization of the inverse scattering method, *J Phys Soc Jap*, 1979, **46**, 1965–1966

77. WADATI, M , KONNO, K AND ICHIKAWA, Y H. — New integrable nonlinear evolution equations, *J Phys Soc Jap*, 1979, **47**, 1698–1700

78 SHIMIZU, T AND WADATI, M — A new integrable nonlinear evolution equation, *J Phys Soc Jap*, 1980, **63**, 808–820

79 ISHIMORI, Y — On the modified Korteweg–de Vries equation and the loop soliton, *J Phys Soc Jap*, 1981, **50**, 2471–2472

80 ISHIMORI, Y — A relationship between the Ablowitz–Kaup–Newell–Segur and Wadati-Konno–Ichikawa schemes of the inverse scattering method, *J Phys. Soc Jap*, 1982, **51**, 3036–3041

81. WADATI, M AND SOGO, K — Gauge transformations in soliton theory, *J Phys Soc Jap*, 1983, **52**, 394–398

82. KONNO, K AND JEFFREY, A — The loop soliton. In *Advances in nonlinear waves* (L. Debnath, ed) *Pitman Research Notes in Math*, 1984, **95**, 162–183

83. AIRAULT, H — Rational solutions of Painlevé equations, *Stud. Appl Math*, 1979, **61**, 31–53

84. AIRAULT, H , MCKEAN, H P AND MOSER, J. — Rational and elliptic solutions of the KdV equation and related many-body problems, *Commun Pure Appl Math*, 1977, **30**, 95–198

85 ABLOWITZ, M J AND SATSUMA, J — Solitons and rational solutions of nonlinear evolution equations, *J Math. Phys*, 1978, **19**, 2180–2186.

86 NIMMO, J J C AND FREEMAN, N C — Rational solutions of the Korteweg–de Vries equation in Wronskian form, *Phys Lett A*, 1983, **96**, 443–446

87 MATSUNO, Y — A new proof of the rational *N*-soliton solution for the Kadomtsev–Petviashvili equation, *J Phys Soc Jap*, 1989, **58**, 67–72

88 ADLER, M AND MOSER, J On a class of polynomials connected with the Korteweg–de Vries equation, *Commun Math Phys.*, 1978, **61**, 1–30

89 SATSUMA, J AND ISHIMORI, Y Periodic wave and rational soliton solutions of the Benjamin–Ono equation, *J Phys Soc Jap*, 1979, **46**, 681–687

90 GALKIN, V M, PELINOVSKY, D E AND STEPANYANTS, Y A The structure of the rational solutions to the Boussinesq equation, *Physica D*, 1995, **80**, 246–255

91 PELINOVSKY, D Rational solutions of the Kadomtsev–Petviashvili hierarchy and the dynamics of their poles I New form of a general rational solution, *J Math. Phys*, 1994, **35**, 5820–5830.

92 HU, X B Rational solutions of integrable equations via nonlinear superposition formulae (preprint)

93 ADLER, V E On the rational solutions of the Shabat equation In *Nonlinear physics theory and experiment* (E Alfinito *et al.*, eds), World Scientific, 1996, pp 3–10

94 MATVEEV, V B AND SALLE, M A *Darboux transformations and solitons*, Springer–Verlag, 1991

95 KAJIWARA, K AND OHTA, Y Determinant structure of the rational solutions for the Painlevé IV equation, *J Phys A*, 1998, **31**, 2431–2446

96 KUDRYASHOV, N A AND NIKITIN, V A Painlevé analysis, rational and special solutions of a variable coefficient Korteweg–de Vries equations, *J Phys A*, 1994, **27**, L101

97. MATSUKIDAIRA, J, SATSUMA, J AND STRAMPP, W Soliton equations expressed by trilinear forms and their solutions, *Phys Lett A*, 1990, **147**, 467–471

98 GRAMMATICOS, B, RAMANI, A AND HIETARINTA, J Multilinear operators The natural extension of Hirota's bilinear formalism, *Phys Lett A*, 1994, **190**, 65–70

99 RAMANI, A, GRAMMATICOS, B AND SATSUMA, J Bilinear discrete Painlevé equations, *J Phys A*, 1995, **28**, 4655–4665

100 OHTA, Y, RAMANI, A, GRAMMATICOS, B AND TAMIZHMANI, K M From discrete to continuous Painlevé equations A bilinear approach, *Phys Lett A*, 1996, **216**, 255–261

101 LAKSHMANAN, M. AND KALIAPPAN, P. Lie transformations, nonlinear evolution equations, and Painlevé forms, *J Math Phys*, 1983, **24**, 795–805

102 TAMIZHMANI, K M AND PUNITHAVATHI, P Infinite-dimensional Lie algebraic structure and the symmetry reduction of a nonlinear higher-dimensional equation, *J Phys Soc Jap.*, 1990, **59**, 843–847

103 TAMIZHMANI, K M, RAMANI, A AND GRAMMATICOS, B. Lie symmetries of Hirota's bilinear equations, *J Math Phys*, 1991, **32**, 2635–2659

104 CLARKSON, P A AND KRUSKAL, M D New similarity reduction of the Boussinesq equation, *J Math Phys*, 1989, **30**, 2201–2213

105. CLARKSON, P A AND HOOD, S New symmetry reductions and exact solutions of the Davey–Stewartson system I Reductions to ordinary differential equations, *J Math Phys*, 1994, **35**, 255–283

106 BLUMAN, G W AND KUMEI, S *Symmetries and differential equations*, Springer, 1989

107 CHAMPAGNE, B, HEREMAN, W AND WINTERNITZ, P The computer calculation of Lie point symmetries of large systems of differential equations, *Comp Phys Commun*, 1991, **66**, 319–340

108 HEREMAN, W — New trends in theoretical developments and computational methods (N H Ibragimov, ed), *CRC Handbook of Lie group analysis of differential equations*, Vol 3, pp 367–413, CRC Press, 1996

109 HEAD, A — Lie a MUMATH program for the calculation of the Lie algebra of differential equations, *Comp Phys Commun* , 1993, **77**, 241–248

110 DAVID, D , KAMRAN, N , LEVI, D AND WINTERNITZ, P — Symmetry reduction for the Kadomtsev–Petviashvili equation using a loop algebra, *J Math Phys* , 1986, **27**, 1225–1237

111 SATO, M — Soliton equations as dynamical systems on an infinite dimensional Grassmann manifold, *Publ Res Inst Math Sci* , Kokyoroku, 1981, **439**, 30–46

112 SATO, M AND SATO, Y — Soliton equations as dynamical systems on infinite Grassmann manifold In *Nonlinear partial differential equations in applied science* (H Fujita *et al* , eds), Kinokuniya/North-Holland, 1983, pp 259–271

113 OHTA, Y , SATSUMA, J , TAKAHASHI, D AND TOKIHIRO, T — An elementary introduction to Sato theory, *Prog Theor Phys Suppl* , 1988, **94**, 210–241

114 DATE, E , JIMBO, M , KASHIWARA, M AND MIWA, T — Transformation groups for soliton equations, *Proc RIMS Symp Nonlinear Integrable Systems, Classical and Quantum Theory*, (M Jimbo and T Miwa, eds), Kyoto, 1981, pp 39–119, World Scientific, 1983

115 DATE, E , JIMBO, M AND MIWA, T — Method for generating discrete soliton equations I-V, *J Phys Soc Jap* , 1982, **51**, 4116–4126, 4125–4131, 1983, **52**, 388–393, 761–765, 766–771

116 JIMBO, M AND MIWA, T — Solitons and infinite dimensional Lie algebras, *Publ Res Inst Math Sci Kyoto Univ* , 1983, **19**, 943–1001

117 MATSUKIDAIRA, J , SATSUMA, J. AND STRAMPP, W — Conserved quantities and symmetries of KP hierarchy, *J Math Phys* , 1990, **31**, 1426–1434

118. KAJIWARA, K., MATSUKIDAIRA, J AND SATSUMA, J — Conserved quantities of two component KP hierarchy, *Phys Lett A*, 1990, **146**, 115–118

119. KAJIWARA, K AND SATSUMA, J — The conserved quantities and symmetries of the two-dimensional Toda lattice hierarchy, *J Math Phys* , 1991, **32**, 506–514

120 KONOPELCHENKO, B G AND OEVEL, W — Sato theory and integrable equations in 2+1 dimensions. In *Proc 7th Workshop on Nonlinear Evolution Equations and Dynamical Systems* (NEEDS'91), Baia Verde, Italy, June 19–29, 1991

121 KONOPELCHENKO, B AND OEVEL, W — An *r*-matrix approach to nonstandard classes of integrable equations, *Publs Res Inst Math Sci Kyoto Univ* , 1993, **29**, 581–666

122 ABLOWITZ, M J AND LADIK, J F — Nonlinear differential–difference equations, *J Math Phys.*, 1975, **16**, 598–603

123 ABLOWITZ, M J AND LADIK, J F — Nonlinear differential–difference equations and Fourier analysis, *J Math Phys* , 1976, **17**, 1011–1018

124 UENO, K AND TAKASAKI, K — Toda lattice hierarchy, *Adv Stud Pure Math* , Kinokuniya, Tokyo, 1984, **4**, 1–95

125 OEVEL, W — Poisson brackets for integrable lattice systems In *Algebraic aspects of integrable systems In memory of Irne Dorfman* (A. S Fokas and I M Gelfand, eds), Birkhauser, 1996, pp 261–283

126 BLASZAK, M. AND MARCINIAK, R — R-Matrix approach to lattice integrable systems, *J Math Phys*, 1994, **35**, 4661–4682

127 LEVI, D, PILLONI, L AND SANTINI, P M — Integrable three-dimensional lattices, *J Phys A*, 1981, **14**, 1567–1575

128 WIERSMA, G AND CAPEL, H W — Lattice equations, hierarchies and Hamiltonian structures, *Physica A*, 1987, **142**, 199–244

129 NIJHOFF, F W AND CAPEL, H W — The direct linearisation approach to hierarchies of integrable PDEs in (2+1) dimensions I Lattice equations and the differential-difference hierarchies, *Inv Prob*, 1990, **6**, 567–590

130 NIJHOFF, F W, PAPAGEORGIOU, V G, CAPEL, H W AND QUISPEL, G R W — The lattice Gel'fand-Dikii hierarchy, *Inv Prob*, 1992, **8**, 597–621

131 PAPAGEORGIOU, V, NIJHOFF, F AND CAPEL, H W — Integrable mappings and nonlinear integrable lattice equations, *Phys Lett A*, 1990, **147**, 106–114

132 CAPEL, H W, NIJHOFF, F W AND PAPAGEORGIOU, V G — Complete integrability of Lagrangian mappings and lattices of KdV type, *Phys Lett A*, 1991, **155**, 377–387

133 QUISPEL, G R W., ROBERTS, J A G AND THOMPSON, C J — Integrable mappings and soliton equations I, *Phys Lett A*, 1988, **126**, 419–421

134 QUISPEL, G R W., ROBERTS, J A G AND THOMPSON, C J — Integrable mappings and soliton equations II, *Physica D*, 1989, **34**, 183–192

135 TODA, M — *Nonlinear waves and solitons*, Kluwer, 1989

136 KUPERSHMIDT, B A — Discrete lax equations and differential-difference calculus, *Astérisque*, 1985, **123**, 1

137 RAMANI, A, GRAMMATICOS, B AND TAMIZHMANI, K M — An integrability test for differential-difference systems, *J Phys A*, 1992, **25**, L883–L886

138 RAMANI, A, GRAMMATICOS, B AND TAMIZHMANI, K M — Painlevé analysis and singularity confinement The ultimate conjecture, *J Phys A*, 1993, **26**, L53–L58

139 TAMIZHMANI, K M, RAMANI, A AND GRAMMATICOS, B — Singularity confinement analysis of integro-differential equations of Benjamin-Ono type, *J Phys A*, 1997, **30**, 1017–1022

140 MAEDA, S — Canonical structure and symmetries for discrete systems, *Math Jap*, 1980, **25**, 405–420

141 MAEDA, S — Extension of discrete Noether theorem, *Math Jap*, 1981, **26**, 85–90

142 MAEDA, S — The similarity method for difference equations, *IMA J Appl Math*, 1987, **38**, 129–134

143 LEVI, D AND WINTERNITZ, P — Continuous symmetries of discrete equations, *Phys Lett A*, 1991, **152**, 335–338

144 LEVI, D AND WINTERNITZ, P — Symmetries and conditional symmetries of differential-difference equations, *J Math Phys*, 1993, **34**, 3713–3730

145 LEVI, D AND WINTERNITZ, P — Symmetries of discrete dynamical systems, *J Math Phys*, 1996, **37**, 5551–5576

146 LEVI, D, VINET, L AND WINTERNITZ, P — Lie group formalism for difference equations, *J Phys A*, 1977, **30**, 633–649

147 QUISPEL, G R W, CAPEL, H W AND SAHADEVAN, R — Continuous symmetries of differential-difference equations the Kac-van Moerbeke equation and Painlevé reduction, *Phys Lett A*, 1992, **170**, 379–383

148 QUISPEL, G R W, CAPEL, H W AND SAHADEVAN, R — Continuous symmetries and Painleve reduction of the Kac-van Moerbeke equation, *Proc NATO Workshop on Applications of Analytic and Geometric Methods to Nonlinear Differential Equations* (Clarkson, P A, ed), Kluwer, 1993

149 QUISPEL, G R W AND SAHADEVAN, R — Lie symmetries and the integration of difference equations, *Phys Lett A*, 1993, **184**, 64–70

150 GAETA, G — Lie point symmetries of discrete versus continuous dynamical systems, *Phys Lett A*, 1993, **178**, 376–384

151 KANAGA VEL, S AND TAMIZHMANI, K M — Lax pairs, symmetries and conservation laws of a differential-difference equation—Sato's approach, *Chaos, Solitons Fractals*, 1997, **8**, 917–931

152 THAMIZHARASI TAMIZHMANI, KANAGA VEL, S AND TAMIZHMANI, K M — Wronskian and rational solutions of differential-difference KP equation, *J Phys A*, 1998, **31**, 7627–7633

153 TAMIZHMANI, K M AND KANAGA VEL, S — Gauge equivalence and *l*-reduction of differential-difference KP equation, *Chaos, Solitons Fractals* (to appear)

154 TAMIZHMANI, K M, KANAGA VEL, S, GRAMMATICOS, B AND RAMANI, A — Singularity structure and algebraic properties of the differential-difference Kadomtsev-Petviashvili equation (preprint)

155 KANAGA VEL, S — *On certain integrability aspects of differential-difference Kadomtsev-Petviashvili equation*, Ph. D Thesis, Pondicherry University, India, 1998

156 GRAMMATICOS, B, RAMANI, A AND PAPAGEORGIOU, V G — Do integrable mappings have the Painlevé Property?, *Phys Rev Lett*, 1991, **67**, 1825–1828

157 RAMANI, A, GRAMMATICOS, B AND HIETARINTA, J — Discrete versions of the Painlevé equations, *Phys Rev Lett*, 1991, **67**, 1829–1832

158 PAPAGEORGIOU, V G, NIJHOFF, F W, GRAMMATICOS, B AND RAMANI, A — Isomonodromic deformation problems for discrete analogues of Painlevé equations, *Phys Lett A*, 1992, **164**, 57–64

159 TAMIZHMANI, K M, GRAMMATICOS, B AND RAMANI, A — Schlesinger transforms for the discrete Painlevé IV equation, *Lett Math Phys*, 1993, **29**, 49–54

160 GRAMMATICOS, B AND RAMANI, A — Discrete Painlevé equations Derivation and properties, *NATO ASI C*, 1993, **413**, 299–314

161 RAMANI, A AND GRAMMATICOS, B — Miura transforms for discrete Painlevé equations, *J Phys A*, 1992, **25**, L633–L637

162 GRAMMATICOS, B AND RAMANI, A — Integrability and how to detect it In *Integrability of nonlinear systems, Lecture Notes in Physics*, Vol 495, *Proc CIMPA Int Winter School on Nonlinear Systems* (Y Kosmann-Schwarzbach *et al*, eds) Springer-Verlag, Berlin, 1996, pp 30–94

163 KAJIWARA, K, OHTA, Y, SATSUMA, J, GRAMMATICOS, B AND RAMANI, A — Casorati determinant solutions for the discrete Painlevé-II equation, *J Phys A*, 1994, **27**, 915–922

164 GRAMMATICOS, B AND RAMANI, A — Investigating the integrability of discrete systems, *Int J Mod Phys B*, 1993, **7**, 3551–3565

165 GRAMMATICOS, B , PAPAGEORGIOU, V AND RAMANI, A — Discrete dressing transformations and Painlevé equations, *Phys Lett A*, 1997, **235**, 475–479

166 RAMANI, A , GRAMMATICOS, B , TAMIZHMANI, K M AND LAFORTUNE, S — Again, linearizable mappings, *Physica A*, 1988, **252**, 138–150

167 FOKAS, A S , GRAMMATICOS, B AND RAMANI, A — From continuous to discrete Painlevé equations, *J Math Anal Appl*, 1993, **180**, 342–360

168 GRAMMATICOS, B , OHTA, Y , RAMANI, A , SATSUMA, J AND TAMIZHMANI, K M — A Miura of the Painlevé I equation and its discrete analogs, *Lett Math Phys*, 1997, **39**, 179–186

169 SATSUMA, J , KAJIWARA, K , GRAMMATICOS, B , HIETARINTA, J AND RAMANI, A — Bilinear discrete Painlevé-II and its particular solutions, *J Phys A*, 1995, **28**, 3541–3548

170 GRAMMATICOS, B , RAMANI, A AND TAMIZHMANI, K M — Non-proliferation of preimages integrable mappings, *J Phys A*, 1994, **27**, 559–566

171 BRUSCHI, M , RAGNISCO, O , SANTINI, P M AND TU, G Z — Integrable symplectic maps, *Physica D*, 1991, **49**, 273–294

172 GRAMMATICOS, B , RAMANI, A AND MOREIRA, I C — Delay-differential equations and the Painlevé transcendents, *Physica A*, 1993, **196**, 574–590

173 VESELOV, A P AND SHABAT, A B — Dressing chains and the spectral theory of the Schrodinger operator, *Func Anal Appl*, 1993, **27**, 81–96

174 KOWALEVSKI, S — Sur le problème de la rotation d'un corps solide autour d'un point fixè, *Acta Math*, 1889, **12**, 177–232

175 KOWALEVSKI, S — Sur une propriété d'un système d'équations différentielles qui definit la rotation d'un corps solide autour d'un point Fixè, *Acta Math*, 1889, **14**, 81–93

176 HILLE, E — *Ordinary differential equations in complex domain*, Wiley, 1976

177 ABLOWITZ, M J AND SEGUR, H — Exact linearization of a Painlevé transcendent, *Phys Rev Lett*, 1977, **38**, 1103–1106

178 GIBBON, J D AND TABOR, M — On the one- and two-dimensional Toda lattices and the Painlevé property, *J Math Phys*, 1985, **26**, 1956–1960

179 LAKSHMANAN, M — Continuum spin system as an exactly solvable dynamical system, *Phys Lett A*, 1977, **61**, 53–54

180 ZAKHAROV, V E AND TAKHTAJAN, L A — Equivalence of nonlinear Schrodinger equation and equation of Heisenberg ferromagnet, *Teor Mat Fiz*, 1979, **38**, 26

181 KUNDU, A — Landau–Lifschitz and higher-order nonlinear systems gauge generated from nonlinear Schrodinger type equations, *J Math Phys*, 1984, **25**, 3433–3438

182 KUNDU, A — Exact solutions to higher-order nonlinear equations through gauge transformations, *Physica D*, 1987, **25**, 399–406

183 KISO, K — A remark on the commuting flows defined by Lax equations, *Prog Theor Phys*, 1990, **83**, 1108–1114

184 OEVEL, W AND ROGERS, C — Gauge transformations and reciprocal links in (2+1) dimensions, *Rev Math Phys*, 1993, **5**, 299–330

185	KONOPELCHENKO, B AND OEVEL, W	An r-matrix approach to nonstandard classes of integrable equations, *Publ Res Inst Math Sci*, *Kyoto Univ*, 1993, **29**, 581–666
186	KONOPELCHENKO, B G	*Solitons in multidimensions Inverse spectral transform method*, World Scientific, 1993
187	KONOPELCHENKO, B	The two dimensional matrix spectral problem General structure of the integrable equations and their Backlund transformations, *Phys Lett A*, 1981, **89**, 346–350
188	KONOPELCHENKO, B	On the gauge-invariant description of the evolution equations integrable by Gel'fand–Dikij spectral problems, *Phys Lett A*, 1982, **92**, 323–327
189	KONOPELCHENKO, B	On the general structure of nonlinear evolution equations integrable by the two-dimensional matrix spectral problem, *Commun Math Phys*, 1982, **87**, 105–125
190	GERDJIKOV, V S, IVANOV, M I AND VAKLEV, Y S	Gauge transformations and generating operators for the discrete Zakharov–Shabat system, *Inv Prob*, 1986, **2**, 413.
191	LIPOVSKY, V D AND SHIROKOV, A V	An example of gauge equivalence of multidimensional integrable equations, *Func Anal Appl*, 1989, **23**, 65
192	LEO, R A, MARTINA, L AND SOLIANI, G	Gauge equivalence theory of the noncompact Ishimori model and the Davey–Stewartson equation, *J Math Phys*, 1992, **33**, 1515

Appendix I
Painlevé analysis for PDEs

ARS[44] proposed an algorithm to analyse the Painlevé property of ODEs This has been extended by WTC[47] The Painlevé analysis for PDEs due to WTC can be stated as follows: Let us consider the evolution equation of the form

$$u_t + K(u) = 0, \qquad \text{(A 1)}$$

where $K(u)$ is some nonlinear function of u and its derivatives of order N, in the complex domain We say that an NPDE possesses the generalized Painlevé property[47,60] if the following two conditions are satisfied

(a) The solutions of the NPDE (A 1) must be 'single-valued' about the 'non-characteristic' movable singularity manifold. More precisely, if the singularity manifold is determined by

$$\phi(x, t) = 0, \; \phi_x(x, t) \neq 0 \qquad \text{(A.2)}$$

and $u(x, t)$ is a solution of (A 1), then we seek

$$u = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j \qquad \text{(A 3)}$$

where $\phi = \phi(x, t)$, $u_j = u_j(x, t)$, $u_0 \neq 0$ are analytic functions of (x, t) in a neighbourhood of the manifold and α is a negative integer

(b) Then by Cauchy–Kovalevskaya's theorem, the solution (A 3) should contain N arbitrary functions, one of them being the function ϕ and others coming from the u_js The algorithmic procedure to test the given nonlinear evolution equation for its generalized Painlevé property consists essentially of three steps. We shall describe each of these steps below

Leading order analysis

The analysis starts with the determination of the possible values of α and u_0 in the expansion (A 3) For each values of α, the homogeneous terms with the highest degree may balance each other The terms that balance each other are called leading terms Then all the αs must be negative integers by (a)

For each choice of the α, an algebraic equation for the u_0 in (A 3) is usually obtained by requiring that the coefficient, say A of the dominant term $A\phi^{-d}$ should vanish, where d is the highest degree If u_0 is arbitrary, A should identically vanish.

Resonance analysis

After identifying all the possible branches in the solution (A 3), our next aim is to find the resonances When the coefficient u_j of the term $\phi^{j+\alpha}$ in the expression (A 3) is arbitrary, then we say that the resonance occurs at j in the above series. In order to find the resonance values, we substitute

$$u = u_0\phi^{\alpha} + u_j\phi^{j+\alpha} \tag{A 4}$$

in eqn (A 1), retaining only the most dominant terms, and extracting the coefficient $\tilde{Q}(j) = Q(j)u_j$ of the term $\phi^{j+\alpha-N}$ Then $Q(j) = 0$ is called the resonance equation, in which -1 is always a root, which corresponds to the arbitrary nature of ϕ. Substituting the values of u_0 (obtained earlier in the leading order analysis) in the resonance equation, one can find the remaining roots of $Q(j)$

Arbitrary functions

Having obtained the resonance values, we have to show that necessary arbitrary functions exist at these resonance values in the series without the introduction of any movable critical manifold. Let r_s be the highest of the allowed resonance values Then we substitute

$$u = \sum_{j=0}^{r_s} u_j\phi^{j+\alpha}, \tag{A.5}$$

in the original equation (A 1) and for $j = 0, 1, 2, \ldots, r_s$ requires

$$Q(j)u_j + R_j = 0, \tag{A 6}$$

where the left-hand side of eqn (A 6) is the coefficient of $\phi^{j+\alpha-N}$ and R_j is a polynomial in the partial derivatives of ϕ and u_ks ($k = 0, 1, \ldots, j-1$). Since $Q(j) = 0$, R_j should identically vanish for any resonance j and in which case u_j is arbitrary Suppose if it is not so, we have to introduce logarithmic terms of the form $a_j + b_j \log \phi$ in the series But due to this addition, the logarithmic singularities will appear in the solution manifold. Thus, the condition $R_j = 0$ ensures that the solution is free from movable critical manifolds.

Note: We have noticed rather late in production stage that eqns (4) and (5) do not figure in the paper of K. M. Tamizhmani and S Kanaga Vel, 1998, **78**, 311–372 This omission has probably occurred during revision