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Nonlinear dynamical techniques for analysis and modeling of EEG data

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Abstract

Electroencephalogram (EEG), a record of electrical potentials of the brain, is now analysed as a chaotic signal rather than a stochastic signal in the light of new developments in nonlinear dynamics and chaos The paper reviews the application of nonlinear dynamical techniques to analyse EEG data Earlier studies mostly concerned with calculating the characteristics of the system like correlation dimension and Lyapunov exponents, and use them for various applications like study of different sleep stages, epileptic seizures, depths of anaesthesia, etc One major direction is to develop chaotic, realistic models for EEG generation Their efficacy is demonstrated by data compression as an application

Keywords: Electroencephalogram (EEG), chaos, nonlinear dynamics, EEG analysis, modeling, data compression

1. Introduction

Electroencephalogram (EEG) is a record of electrical potentials in the brain which can be picked up by placing electrodes in the scalp or directly in the cerebral cortex. EEG is one of the commonly used non-invasive techniques to understand brain functions. It provides information on epilepsy, cerebral tumors, cerebral trauma, etc and is also useful to monitor sleep, depth of anaesthesia and cessation of brain function Hans Berger was the first to record the electrical activity of the brain in human subjects.[1] In normal subjects there is a characteristic EEG pattern made of waves of varying frequency which are divided into four EEG rhythms—alpha, beta, theta and delta Other EEG activities include transients which occur spontaneously[2] These transients are called spikes, sharp waves, and spike and wave activities depending on their characteristics.

EEG is a complex and random-looking signal, and hence was analysed until recently as the output of a stochastic process, i.e. generated by a 'black box' or a system driven by some unknown (or white Gaussian) input. Based on this concept, spectral estimation and several other techniques like statistical pattern recognition,[3] segmentation,[4–7] syntactic methods,[8–10] knowledge-based approaches,[11–15] and artificial neural network methods[16, 17] have been developed for analysing EEG. Several models have also been developed based on the assumption that EEG is stochastic.[18, 19]

These techniques bear little or no consideration to the process that generates EEG signal This may be the main reason for the limited success of the automated EEG analysis techniques despite the large number of attempts at automating the EEG interpretation process and ever-increasing sophistication of the methods used.[20]

With recent advances in nonlinear dynamics and chaos, EEG is being considered as an output of a deterministic system rather than a stochastic system. This is based on the concept that even a simple nonlinear deterministic system which is very sensitive to initial conditions, called a chaotic system, can generate outputs which are very complex(-looking). Since EEG is a complex-looking signal, it could be generated by such a nonlinear dynamical system. This concept of chaos introduces a dynamical perspective for understanding brain functions and for analysing EEG.

This paper first reviews how nonlinear dynamical techniques have been applied to analyse EEG data. It then presents various models that have been developed for EEG signal based on chaotic principle Their efficacy is demonstrated by data compression as an application.

To understand what chaos is and the terms associated with nonlinear dynamics and chaos used in the paper, a short overview is given.

2. Overview of nonlinear dynamics and chaos

Any system whose evolution from some initial state is dictated by a set of rules is called a *dynamical system* Consider the following dynamical system which is defined by a set of ordinary differential equations:

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots, x_N) \\ \frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots, x_N) \\ &\vdots \\ \frac{dx_N}{dt} &= f_N(x_1, x_2, \ldots, x_N) \end{aligned} \tag{1}$$

where $\vec{X}(t) = (x_1(t), \ldots, x_N(t))^T$ is the state of the system at time t and $\vec{f} = (f_1, \ldots, f_N)^T : R^N \to R^N$ is called the *vector field.* When the vector field $\vec{f}$ does not contain time explicitly, as given in (1), then we call the system as *autonomous* The dynamical system (1) is *linear* if $\vec{f}$ is linear and *nonlinear* if $\vec{f}$ is nonlinear. A dynamical system does not have to be described by a set of differential equations. Many dynamical systems are described by a set of *difference equations* and they are often referred to as *maps*.

Given the initial condition $\vec{X}(0) = \vec{X}_0$, the solution to (1) is written as $\phi_t(\vec{X}_0)$, and is called the *flow*. We say that the flow $\vec{\phi}_t$ is generated by the vector field $\vec{f}$ The flow ϕ indicates the position of the initial condition $\vec{X}_0$ after time t. The evolution of the dynamical system can be described in its state or phase space which is a Euclidean space whose coordinates are variables that are necessary to completely describe the state of the system at any moment.

To each possible state of the system, there corresponds a point in phase space. The phase space of the autonomous system (1) is a coordinate system with coordinates $x_1, x_2, \ldots, x_N$ Plotting the set of points $\left\{\vec{\phi}_t\left(\vec{X}_0\right)\right\}$ in the phase space, as a function of time, gives the trajectory of the system through $\vec{X}_0$

It can be proved that a trajectory cannot cross itself or no two trajectories can cross each other in the phase space.[21, 22] That is because the crossing point would correspond to a single state from which two different evolutions cannot originate. This is different from two trajectories approaching an equilibrium point as $t \to \infty$, which is allowed

The local rate of expansion or contraction of a dynamical system can be calculated directly from the vector field or difference equation (without explicitly finding any solution). We say a system is *conservative* if the absolute value of the Jacobian of its map equals exactly one or if the divergence of its vector field equals zero for all times and at all points. A physical system is *dissipative* if it is not conservative The phase space of a dissipative dynamical system is continuously shrinking onto a smaller region of phase space called the *attracting set*

The final state or equilibrium state of the evolution of a dynamical system is modeled by *limit sets*, which are the state–space equivalents of the steady state. The asymptotic motions (as $t \to \infty$) of a flow are characterized by four general types of behaviour In order of increasing complexity these are *equilibrium points* or *fixed points, periodic solutions, quasiperiodic solutions* and *chaos*. An equilibrium point or fixed point of a flow is a constant, time-independent solution, i e. $\phi_t\left(\vec{X}_{eq}\right) = \vec{X}_{eq}$ for all t. At an equilibrium point, the vector field vanishes, i.e $\vec{f}\left(\vec{X}\right) = 0$ The limit set corresponding to the equilibrium point is simply the equilibrium point itself A periodic solution of a flow is a time-independent trajectory that precisely returns to itself in time T, called the *period*, i e $\phi_t\left(\vec{X}^*\right)$ is a periodic solution of an autonomous system if, for all t, $\phi_t\left(\vec{X}^*\right) = \phi_{t+T}\left(\vec{X}^*\right)$ for some minimum period $T > 0$ The restriction $T > 0$ is required to prevent the classification of an equilibrium point as a periodic solution. A periodic solution is isolated if its neighborhood possesses no other periodic solution and is called a *limit cycle* The limit set corresponding to a limit cycle is the closed curve traced out by $\phi_t\left(\vec{X}^*\right)$ over one period. A quasiperiodic solution is one formed from the sum of periodic solutions with incommensurate periods. Two periods are incommensurate if their ratio is irrational

A bounded asymptotic motion that is not an equilibrium point, period or quasiperiodic is often called *chaotic* Additionally, the asymptotic solution should possess sensitive dependence on initial conditions: give two distinct initial conditions arbitrarily close to one another and the trajectories emanating from them diverge at a rate characteristic of the system until they become uncorrelated for all practical purposes

The stable asymptotic motions (or limit sets) described above are examples of *attractors*. The unstable limit sets are examples of *repellers* Attractors that are chaotic are called chaotic attractors.

Sensitive dependence on initial conditions has an important implication. If the initial conditions are known exactly, its evolution can be predicted for ever The problem, however, is that one cannot have perfect knowledge of the initial conditions. Instruments can measure the various parameters only approximately There will always be some deviation from the actual ones. They may be very close to each other, but will not be the same. In such a case, even if we know completely the physical laws that govern the system, due to the nature of the underlying attractor, the actual state of the system at a later time can be totally different from the one predicted Due to the nature of the system, initial errors are amplified and, therefore, prediction is limited

The dynamics of a system are dictated by the geometry of the phase space and its attractor This geometry can be quantified by a series of dimensions and Lyapunov exponents There are different types of dimensions; the most familiar ones being the *Euclidean* and *topological dimensions* According to the Euclidean definition, a configuration is called one dimensional if it is embedded on a straight line, two dimensional if it is embedded on a plane and three dimensional if it is embedded on space. But the topological dimension of a point is zero, of curves is one, of surfaces is two and of space is three. None of these dimensions allows non-integer values and none can be used to describe strange attractors The generic term for a dimension that has non-integer dimensions is called a *fractal.* Almost all strange attractors are fractals Three different types of fractal dimension, *the capacity dimension, the information dimension* and *the correlation dimension*, are presented below.

Capacity dimension: The simplest type of dimension is the capacity dimension. Cover an attractor A with volume elements (with Euclidean dimensions of A) with diameter ε Let $N(\varepsilon)$ be the minimum number of volume elements needed to cover A Then $N(\varepsilon)$ is inversely proportional to ε^D where D is the Euclidean dimension of A, i e.

$$N(\varepsilon) = k\varepsilon^{-D}$$

for some constant k. Then the capacity dimension[23] is obtained by solving for D and taking the limit as ε approaches zero

$$D_{\text{cap}} = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln \frac{1}{\epsilon}}.$$

D_{cap} can take non-integer values For example, the capacity dimension of Koch curve (which is generated by considering a straight-line segment of length L and replacing its middle third by two equal segments of side $L/2$ forming part of an equilateral triangle and repeating this procedure many times) is given by $D_{\text{cap}} = \frac{\log 4}{\log 3} = 1.26$ [24]

Information dimension: Capacity dimension is a purely metric concept. It does not utilize the information about the time behaviour of the dynamical system Information dimension, on the other hand, takes it into account Let $N(\varepsilon)$ be the minimum number of volume elements with diameter ε needed to cover attractor A Then the information dimension D_I is defined as[23]

$$D_I = \lim_{\varepsilon \to 0} \frac{H(\varepsilon)}{\ln \frac{1}{\varepsilon}}$$

where $H(\varepsilon) = -\Sigma_{i=1}^{N(\varepsilon)} p_i \ln p_i$. p_i is the relative frequency with which a trajectory enters the ith volume element, and $H(\varepsilon)$ the entropy, i.e. the amount of information needed to specify the state of the system to an accuracy of ε if the state is known to be on the attractor.

Correlation dimension: Correlation dimension is another probabilistic type of dimension. Let $N(\varepsilon)$ be the minimum number of volume elements with diameter ε needed to cover attractor A. Then the correlation dimension D_C is defined as

$$D_C = \lim_{\varepsilon \to 0} \frac{\ln \Sigma_{i=1}^{N(\varepsilon)} p_i^2}{\ln \varepsilon}$$

where p_i is the relative frequency with which a trajectory enters the ith volume element. Now, define the correlation function as

$$C(\varepsilon) = \lim_{N \to \infty} \frac{1}{N^2} \sum_{ij} H\left(\varepsilon - \left\| X_i - X_j \right\|\right)$$

where $H(\theta)$ is the Heaviside function and N the number of points of the trajectory The summation counts the number of pairs of points (X_i, X_j) such that $\|X_i - X_j\| < \varepsilon$. Then[23]

$$D_C = \lim_{\varepsilon \to 0} \frac{\ln C(\varepsilon)}{\ln \varepsilon}.$$

The main reason for finding the dimension of an attractor is to estimate the minimum number of variables needed to describe the steady-state dynamics. So, there seems to be no theoretical reason for choosing one type of dimension over another, except for ease and accuracy of its computation.

Another set of exponents that can characterize the properties of an attractor of a dynamical system is the *Lyapunov exponents*. The Lyapunov exponents are related to the average rates of convergence and/or divergence of nearby trajectories in phase space, and, therefore, measure how predictable or unpredictable the system is. A formal definition of Lyapunov exponents is given below: An attractor embedded in an n-dimensional Euclidean space is considered and a set of initial conditions in the attractor that are confined within an n-dimensional sphere is taken The space is allowed to evolve in time and its long-term evolution monitored The principal axes of this sphere are ordered from the most rapidly to the least rapidly growing, and the mean growth rate λ_i of any given principal axis p_i is computed These growth rates may be defined as follows:

$$\lambda_i = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \frac{d}{dt} \ln\left[\frac{p_i(t)}{p_i(0)}\right]$$

$$= \lim_{T \to \infty} \frac{1}{T} \ln\left[\frac{p_i(T)}{p_i(0)}\right].$$

Here $p_i(0)$ is the radius of the principal axis p_i at $t = 0$ (i e in the initial hypersphere), and $p_i(T)$ is its radius after a long time T The set of λ_is is referred to as the Lyapunov exponent spectrum There are as many Lyapunov exponents as the dimension of the phase space.

When *at least one Lyapunov exponent is positive, then the system at hand is chaotic*, and there will be exponential divergence of nearby points along at least one direction on the attractor This results in an inability to predict the evolution of the trajectory beyond an interval of time approximately the inverse of the divergence rate. When no positive Lyapunov exponent exists, there is no exponential divergence, and the long-term predictability of the system at hand is guaranteed

3. Application to EEG analysis

In the previous section, a short overview of nonlinear dynamics and chaos was given. In this section, we will see how these concepts are useful in the analysis of EEG Since a dynamic model may be preferred to a phenomenological stochastic descriptor, a formalism of chaotic dynamics has found application in analysing the EEG signal and in understanding brain functions. So, EEG is now being considered as the output of a nonlinear dynamic (chaotic) system and efforts are on to calculate the various characteristic parameters of the system discussed in the last section and to develop various applications using them.

Most of the studies made in this direction are towards the evaluation of correlation dimension of EEG signals recorded under different neurophysiological states Since the dimension of the attractor is a characteristic feature of the underlying neuronal processes generating EEG signals, it has been applied to classification of neural activities, feature detection of various brain states, studying the effects of drugs on brain, etc Babloyantz *et al.*[25-27] calculated the correlation dimension of different levels of sleep stages and found it increasing with increase in mental activity They have reported a correlation dimension of about 10 for EEG recorded from mentally active subjects, of about 6 from alert but resting subjects and of around 4 from subjects in sleep stage 4. EEG of Creutzfeld-Jacob disease recorded has a dimension slightly less than sleep stage 4. Other researchers[28-30] have calculated the correlation dimension of alpha waves from normal relaxing subjects with eyes closed

Correlation dimension of EEG signals reveals differences in the state of the neuronal networks as it gets involved in epileptic seizures when it is lower than that of waking state Babloyantz and Destexhe[26, 31] found a correlation dimension of around 2 for EEG recorded during petitmal epileptic seizures Iasemides *et al*[32] performed a phase space analysis of EEG in temporal lobe epilepsy and reported decrease in correlation dimension and increase in Lyapunov exponent during an epileptic attack This indicates that although the attractor has relatively smaller dimension during epileptic attack, the spinning inside the attractor is greater and information on the initial conditions is lost

Scientists at the Center for Nonlinear Studies at Los Alamos believe that it may be possible to develop a computer analysis of EEG recordings to characterize different forms of seizure.[33] In addition, it has been shown[34] that, during epileptic seizures, different areas of the brain from which EEG signals are recorded do not show a reduction of correlation dimension in the same way as in a kindling experiment Indeed, a low correlation dimension value was observed mitially only in entorhinal cortex and fascia dentata In the course of the seizure, the EEG signal recorded from the ipsilateral hippocampus showed a low correlation dimension, but the contralateral hippocampus tended to a low value only in the late seizure, while the value of correlation dimension of the signal recorded from cortex increased appreciably and became more irregular. This time sequence of changes in correlation dimension suggests that the recruitment of different areas into the seizure takes place according to a given sequence: first the cortex and facia dentata, then the ipsilateral hippocampus, and later the contralateral hippocampus, while the cortical changes in the late phase to a much less regular pattern.

The clear transition from a very high to a very low correlation dimension during seizure activity may be used to detect the onset of epileptic seizure activity as well as for the localization of an epileptogenic focus in epileptic patients [35,36] Yaylali *et al.* have calculated the correlation dimension of the unbiased autocorrelation function of the scalp EEG data and used it to identify various types of seizures.[37]

Studies on EEG signals during high-level thinking or creativity show a change in the correlation dimension during intensive cognitive activities [38,39] Trent *et al.*[40] have presented a complexity measure which shows a peak during transition between thinking and non-thinking states. Also, correlation dimension shows a difference between the REM (rapid eye movement) and non-REM vigilance states.[41] Itil[42] has studied the effect of medication on alpha activity in patients and did not find any change in dynamic measures.

Studies[43] show that in schizophrenic patients the dimensional complexity of EEG patterns at the frontal sites is larger than that at the central sites and the reverse is observed in control subjects This indicates that a higher frontal than central dimensional complexity is a characteristic in schizophrenic patients. Zbigniew *et al.*[44] have proposed an algorithm for computation of chaoticity based on local Lyapunov exponents and present possible applications of this method for specific schizophrenic cases. They also show that chaoticity will be able to detect critical transitions which occur in the dynamics of the brain.

Several visualization techniques in phase space have also been proposed for interpretation of clinical EEG records and to characterize clinically significant features such as spikes and seizures.[45–47] Abu-Faraj *et al.*[48] have presented correlation dimension as a topographical mapping across the scalp which allows the comparison of the dimension across different sites of the head visually.

Studies with different levels of anaesthesia[33,49,50] show a decrease in the dimensionality of EEG signals as anaesthetic depth increases. This helps in assessing brain integrity and/or depth of anaesthesia during surgical procedures.

Regarding the absolute values of the attractor dimensions reported in literature, the following point has to be noted. The dimension estimates are dependent on factors like the number of data points considered, the algorithm used for finding them, etc. Hence, instead of focusing on the absolute values of correlation dimension estimates, it may be better to concentrate on the changes in the values of correlation dimension under different psychophysiological conditions.[20] Albano *et al.*[51] believe that though there can be difference in the estimates of correlation dimension obtained with different algorithms and signal sampling protocols, the ratio of correlation dimensions is robust.

From the foregoing discussion it will be noted that significant amount of work has been done on calculating the characteristics of the system, like correlation dimension and Lyapunov exponent, and applying them to analyse different sleep stages, epileptic seizures, depths of anaesthesia, etc. However, very little work has been done beyond this point Calculating the parameters is just another feature extraction process (like the EEG spectral estimate) and is still too much of a phenomenological approach One should go beyond this but where to? One major direction is to develop chaotic, realistic models for EEG generation.

Once a powerful predictive model is in place, one can think of many applications using it, like compression of EEG data, which is of practical importance The next section deals with various nonlinear dynamical models that are available in literature Compression of EEG data using nonlinear dynamical models is also explained

4. Nonlinear dynamical modeling of EEG data

A major work in the direction of modeling the brain dynamics was done by Freeman and his colleagues [52–56] They have modeled the olfactory system of the rabbits as a chaotic system using a set of coupled nonlinear ordinary differential equations [56, 57] Their solutions simulate various EEG patterns observed experimentally and thus establish to a large extent the physiological mechanisms by which these patterns emerge From the model and other observations, they have concluded that chaotic olfactory dynamics in the olfactory bulb supports a global attractor that affords quick access to and dissemination of information.

Lopes da Silva *et al* [58] have developed a lumped parameter model of cortical columns for alpha rhythm and Jansen[20] has demonstrated the presence of chaotic behaviour in it. The model is based on two interacting populations of neurons, one consisting of main cells and the other of local interneurons The population of main cells is characterized by two linear transfer functions representing excitory and inhibitory post-synaptic potentials and a static, nonlinear element which relates the average level of membrane potential to the pulse density of action potentials fired by the neurons The population of interneurons is similarly described by analogous linear and nonlinear functions Finally, interconnectivity constants which represent the average number of synaptic contacts from main cells to interneurons, and from interneurons to main cells for both excitory and inhibitory branches are used The input to the model is the external pulse density which can take on various forms. This model is capable of complex behaviour when it is allowed to operate in a nonlinear mode.

Mukesh and Natkar[59] have used a general regression neural network with different processing elements to model EEG signals during awake, sleep and rapid eye movement stages They have also studied the relation between the number of processing elements required for simulation and the complexity of the EEG pattern They also observed a correlation between the number of processing elements and fractal dimension. Parikh and Pratap[60] propose a map to describe EEG activity and observe that the predictive ability of the model is limited to a few time steps as expected for a chaotic time series Gonzales *et al* [61] have proposed a nonlinear model of the form $X[n+1] = C_0 + C_1X[n] + C_2X[n]^2 + C_3X[n]^3$ to describe EEG dynamics and have studied the prediction of bifurcations due to change of parameters.

A low-order model of sleep stage II EEG, based on gamma kernel, has been proposed by de Silva and de Oliveira.[62] Assuming that the EEG can be obtained as the output of a nonlinear time-invariant system excited with Gaussian white noise during short periods of sleep stage II, they have developed a model for the system based on dispersive tapped delay line, the gamma net,[62] coupled with a feedforward neural net. This model is used to simulate the EEG signal.

Blinowska and Malinowsky[63] have applied a method by Sugihara and May[64] for prediction of chaotic signals to forecast EEG time series and compare the performance with AR model The EEG data is embedded in an E-dimensional plane and prediction of one E-dimensional point is done by keeping track of the movement of neighbors of that point giving them exponential weights depending on their distance.

We have proposed a model for EEG data (PEE model) based on nonlinear dynamical principles which is explained below What we have on hand is one-dimensional time series and we assume that this has come from a nonlinear dynamical system of more than three dimensions which is very sensitive to initial conditions Hence, the first step towards developing a model is to get back the approximate phase space from the one-dimensional time series This can be done using Taken's embedding theorem[65] by generating the E-dimensional vector X, given by $X[n] = [x(n),\ x(n+\tau),\ x(n+2\tau),\ ..,\ x(n+(E-1)\tau)]^T$, from the time series $\{x(t)\}$ Here τ is called the delay time and E the embedding dimension. The process itself is called time delay embedding We have to choose E and τ such that the dynamics is brought out properly

Once E and τ are chosen properly, we can fit a model for the evolution in the E-dimensional state space; we have modeled the dynamics of the brain as

$$X[n+1] = F(X[n]).$$

F is expressed as

$$F(X) = \sum_{I=0}^{M} C^{(I)} \pi^{(I)}(X)$$

where I is E-dimensional vector index, $\pi^{(I)}$ represents the set of orthonormal polynomials which serve as the basis and $C^{(I)}$s are the expansion coefficients $<\pi^{(I)}, \pi^{(J)}> = \delta_{I,J}$ Each element of F is expressed as

$$f_s(X) = \sum_{a_1=0}^{n} \sum_{a_2=0}^{a_1} \cdots \sum_{a_E=0}^{a_{E-1}} C_s^{a_1,a_2,\ldots,a_E} \pi^{a_1,a_2,\ldots,a_E}(X)$$

where n is the order of the polynomial and C_s are the expansion coefficients corresponding to f_S with $s = 1, 2, 3, \ldots, E$. The set of orthonormal polynomials is constructed using Gram–Schmidt's orthogonalization process starting with $\pi^{(0)} = 1$ and finding the other $\pi^{(I)}$s recursively from this For finding the expansion coefficients, we follow a method proposed by Giona[66] The method does not involve multiparameter optimization and it expresses the expansion coefficients in terms of hierarchies of moments and functional moments.

The whole issue can be viewed as an estimation problem We are trying to estimate $X[n+1]$ from $X[n]$ Let us call the estimated signal as $X[n+1]^e$. We can estimate $X[n+2]$, $X[n+3]$, etc from $X[n]$ by the following procedure

$$X[n+1]^e = F(X[n])$$
$$X[n+2]^e = F(X[n+1]^e)$$

and so on In general,

$$X[n+i]^e = F(X[n+i-1]^e) \qquad i = 2,3, \ldots$$

Viewing the estimation in the E-dimensional space, we can write,

$$\begin{array}{ccccc} X[n] & \rightarrow & X[n+1]^e & , & X[n+2]^e \\ \begin{bmatrix} x(n) \\ x(n+\tau) \\ \vdots \\ x(n+(E-1)\tau) \end{bmatrix} & \rightarrow & \begin{bmatrix} x(n+1)^e \\ x(n+1+\tau)^e \\ \vdots \\ x(n+1+(E-1)\tau)^e \end{bmatrix} & , & \begin{bmatrix} x(n+2)^e \\ x(n+2+\tau)^e \\ \vdots \\ x(n+2+(E-1)\tau)^e \end{bmatrix}, \ldots \end{array}$$

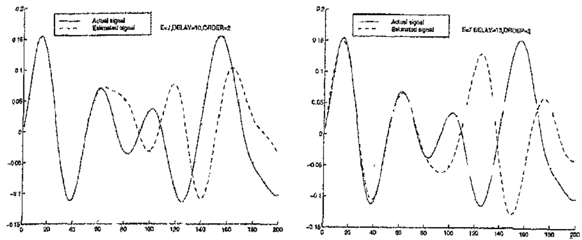
where $x(i)^e$'s represent the estimated data points

Let us look at $X[n+1]^e = [x(n+1)^e, x(n+1+\tau)^e, \ldots, x(n+1+(E-1)\tau)^e]^T$ which is obtained from $X[n] = [x(n), x(n+\tau), \ldots, x(n+(E-1)\tau)]^T$ Since $x(n+1)^e, x(n+1+\tau)^e, \ldots, x(n+1+(E-2)\tau)^e$ are estimated from the past and future values, i e from $x(n), x(n+\tau), \ldots, x(n+(E-1)\tau$, we call the process as smoothening Since $x(n+1+(E-1)\tau)^e$ is estimated from the past values alone, we can call the process as prediction

To get back the one-dimensional time series from the E-dimensional space, we pull out $x(n+1)^e$ from $X[n+1]^e$, $x(n+2)^e$ from $X[n+2]^e$ and so on Now, if we look at the estimated one-dimensional series $\{x(n+i)^e\}$ with $i = 1,2, \ldots$, we can say that $\{x(n+i)^e\}$ with $i = 1,2, \ldots, (E-1)\tau$ are got by smoothening and $\{x(n+i)^e\}$ with $i > (E-1)\tau$ are got by prediction. So, in general, we can call F as a non-causal transformation which takes $X[n]$ to $X[n+1]$

The performance of the model is presented here for deep sleep EEG data (delta wave) Different methods have been applied to find out the minimal E needed and the optimum value of τ,[67–69] an E of 7 or 8 and a τ of 10 to 15 are found to be optimal for the data To test if the model works well, estimation using the model was carried out Figure 1 shows the actual and estimated signals for $E = 7$, $\tau = 10$, $p = 2$ and $E = 7$, $\tau = 13$, $p = 2$, respectively As can be seen from the figures, the model is able to estimate the data well up to around 61 and 81 points, respectively. The performance of the model has been analysed for various values of E, τ and p.[70]

We have also applied a model, proposed by Sugihara and May,[64] to EEG signals and studied its estimation capacity (both smoothening and prediction) The idea behind the model (call it NNA model) is as follows We already have in hand a set of consecutive points of evolution



(a) (b)

FIG 1 Actual and estimated signals for (a) $E = 7$, order $= 2$, $\tau = 10$ and (b) $E = 7$, order $= 2$, $\tau = 13$

of the dynamical system, also called a set of base points, and we want to see where a given point, not included in the base points, moves or evolves in the E-dimensional space. For that, we find closest neighbors to the given point from the set of base points Then to see where the given point has moved in the phase space, we see where the neighbors have moved (which is known since we already have the evolution of them), and add up the evolutions of the neighbors after giving exponential weights depending on the neighbors' distance from the given point This is represented mathematically as follows·

Let $x(1)$, $x(2)$, , $x(N)$ be the one-dimensional time series we have in hand The first step towards modeling is to embed this in the appropriate E-dimensional space Let $X[1]$, $X[2]$,. , $X[N-(E-1)\tau]$ be the set of embedded points (with $X[n]$ being $(x(n), x(n+\tau), .., x(n+(E-1)\tau))^T$ where E is the embedded dimension and τ is the delay time Consider the set of points $X[B_1]$, $X[B_2]$, , $X[B_{NB}]$ as the base and the evolution of some point, say $X[k]$, which does not belong to the base, is to be found It is to be noted that to get back the one-dimensional time series from the E-dimensional point, $x(k+1)$ is pulled out from $X[k+1]$, $x(k+2)$ from $X[k+2]$ and so on So, the evolution of the one-dimensional point $x(k)$, 'f' steps into the future is given by

$$x(k+f) = \sum_{i=1}^{NNB} x^*(k_i+f)\exp\left[-A \ \text{dist}\left(X[k], X^*[k_i]\right)\right]$$

where $k_i + f < NB$, $X^*[k_i]$s are the closest neighbors of $X[k]$, $x^*(k_i)$s are the first coordinates of the E-dimensional points $X^*[k_i]$s, dist( ) the Euclidean distance in E-dimensions, A a constant and NNB the number of neighbors to be considered. This is a non-parametric method which uses no prior information about the model used to generate the time series, it uses the information in the output itself.

It can be looked at as an estimation problem as was done in the previous model. The data points $x(k+1)$, $x(k+2)$, , $x(k+L)$ are estimated from $X[k] = (x(k), x(k+\tau), .., x(k+(E-1)\tau))^T$ and the set of base points, where L is the length up to which estimation is done Let the estimated data points be represented as $x^e(k+1)$, $x^e(k+2)$,.. , $x^e(k+L)$. Since $x^e(k+1)$,

$x^e(k+2)$, , $x^e(k+(E-1)\tau)$ are estimated from the past and future samples, the process is called smoothening Estimation of $x^e(k+(E-1)\tau+1), x^e(k+(E-1)\tau+2)$, , $x^e(k+L)$ is based only on past samples and hence the process is called prediction

We present the performance of the model for deep sleep EEG data The data was first embedded in a 7-dimensional space with a delay of 13 Estimation was carried out, keeping the initial 1,750 data points as the base and considering 200 neighbors with an A of 0 1 One important point to be noted is that the estimated signal follows the actual signal only in shape, up to a certain point, and if we need exact matching with the estimated signal, weight A should be varied till a good matching is obtained, another way is to scale and translate the estimated signal to match with the original signal This is done as follows: first, each data point of the estimated signal is multiplied with a scale factor and then a translation factor is added to it The scale factor and translation factor are given below

$$\text{scale factor} = \frac{\text{range of actual signal}}{\text{range of estimated signal}}$$

$$\text{translation factor} = \text{mean of actual signal} - \text{mean of estimated signal}$$

The scaled and translated signal is shown in Fig 2 As can be seen, there is good matching between the actual and estimated signals up to some point, and then they deviate

4 1. *Compression of EEG data*

The estimation capacity of the models can be effectively used to compress the EEG data The general idea is explained below a model is fitted for the data to be compressed, and the model coefficients are stored Assume that the model is able to estimate well up to L points, taking one data point as input. Then, compression can be achieved in the following way the first data point is stored The next L data points can be estimated from the first data point using the model and so are not stored The next data point to be stored is the L+2nd one. The subsequent L data points, i e. L+3 to 2L+1 are not stored since they can be estimated from the L+2nd data point using the model. This procedure is continued The compressed data set will contain only the data points in intervals of L

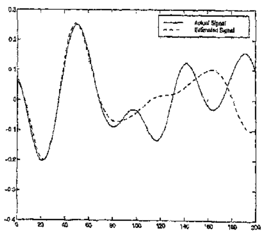


FIG 2 Estimated signal with scaling and translation

The quality of the compression technique depends on how well the model is able to estimate, the compression ratio depends on how long the model is able to estimate well, and the complexity of the compression scheme depends on the complexity of the model To get a high compression ratio and a good-quality decompressed signal, we need a model which can estimate well for a longer duration

When the model based on orthonormal polynomial expansion (PEE model) is used for compressing the data as explained above, we are able to get up to 50 1 compression ratio with very good-quality decompressed signals [70] The decompressed signal is said to be of 'good quality' when it matches well with the original signal, in a scale that is presentable to the physician

We have developed[70] another compression technique using the NNA model. The idea is explained below

Let N be the number of one-dimensional points to be compressed; compression is achieved in the following way

1 The first NB E-dimensional points, $X[1]$ to $X[NB]$, are saved and they will serve as the base for estimation using the model
2 Take the next E-dimensional point $X[k]$, k being $NB + 1$
3 Estimate the E-dimensional point $X[k] = (x(k), x(k+\tau), \ , x(k+(E-1)\tau))^T$ further up to $(E-1)\tau$ points using the model; the neighbors of $X[k]$ are searched within the base points $X[1]$ to $X[NB]$.
4 Scale and translate the estimated signal and check if the estimated signal matches well with the original signal by calculating some parameter (called the cut-off parameter), like the correlation coefficient between the original and the estimated signals
5 If the estimated signal matches well with the original signal, save the initial E-dimensional point $X[k] = (x(k), x(k+\tau), \ ., x(k+(E-1)\tau)^T$ along with the scale factor and the translation factor and do not save the next $(E-1)\tau$ points since they can be estimated from $X[k]$ using the model. Then go to the next E-dimensional point, i e. increment k to $k+(E-1)\tau+1$ and repeat from step 3
6. If the estimated signal does not match well with the original signal, save the τ one-dimensional points $x(k), x(k+1),..., x(k+\tau-1)$ and go to the next E-dimensional point, i.e. increment k to $k+\tau+1$ and repeat from step 3

To demarcate each step, some number, which is quite different from the data points, should be added or saved in the beginning of steps 5 or 6 Different demarcation numbers should be used for steps, say D_1 for step 5 and D_2 for step 6.

The compression ratio is calculated as follows Let Z be the number of times step 5 gets executed and let Y be the number of times step 6 gets executed, i e. Z is the number of times there is good matching between the estimated and the original signals and Y the number of times there is no good matching between the original and the estimated signals. Then the number of points to be stored (N') becomes

$$N' = [(E+2)Z] + [\tau Y] + [Z+Y]$$

$$= [(E+3)Z] + [(\tau+1)Y] \tag{2}$$

The term $(E+2)Z$ corresponds to one E-dimensional point, one scale factor and one translation factor to be stored Z times, the term τY corresponds to τ one-dimensional points to be stored Y times and $Z+Y$ corresponds to the demarcation points to be stored The compression ratio is

$$N \quad N'$$

The following procedure should be followed for uncompressing the data

1 Check if the demarcation number is D_1 or D_2

2 If it is D_1, then the data stored after that will be the E-dimensional point, say $X[p]$, along with the scale and translation factors So, estimate $x(p+1)$, $x(p+2)$, , $x(p+(E-1)\tau)$ from $x[p]$ using the model, $x(p)$ can be extracted from $X[p]$ itself Then scale and translate the estimated signal using the scale and translation factors and go to step 1

3 If the demarcation number is D_2, then τ data points themselves, say $x(r)$, $x(r+1)$,. , $x(r+\tau-1)$, will be stored after D_2 and so nothing needs to be done, go to step 1

The technique gives a moderate compression ratio with a moderate quality of decompressed signal We are able to get around 15.1 compression ratio whereas the method discussed before this is able to give up to 50 1 compression ratio with extremely good-quality decompressed signal But the advantage here is that this method takes less time to decompress, for example, this method takes only 0 9 seconds to decompress 1,000 points[70] whereas the technique discussed earlier takes as much as 13 6 seconds (with $E = 7$ and $p = 2$) The computation was done using a shared memory parallel computer, DEC TurboLaser 8400 with 8 processors, so, the time here represents the CPU time taken by the above-mentioned computer to decompress 1,000 points

5. Conclusions

The paper discusses how nonlinear dynamical techniques are used to analyse EEG data It deals with various models that have been developed for EEG signal based on chaotic principles and about data compression as an application of chaotic modeling EEG being a complex signal, its analysis was based until recently on the assumption that it is the output of a stochastic process. With recent developments in nonlinear dynamics and chaos, it is being considered as an output of a chaotic system This is based on the idea that even a simple nonlinear deterministic system which is very sensitive to initial conditions can generate outputs which are very complex(-looking) Since EEG is a complex-looking signal, it could have been generated by such a nonlinear dynamical system Most of the studies done in this direction are on calculating the characteristics of the system like correlation dimension and Lyapunov exponents, and applying them to analyse different sleep stages, epileptic seizures, depths of anaesthesia, etc Several models, based on chaotic principles, have also been developed for EEG data The models are able to give good estimation of the data and this concept can be used for compressing the data efficiently. Up to 98% compression can be obtained using the chaotic model-based compression techniques.

We expect that a model-driven approach should result in a better understanding of EEG waveform than the conventional phenomenological approach It may help in the construction of physiologically realistic mathematical models that simulate brain activities. Nonlinear dynamical tools have potential application in feature extraction, data compression and analysis of EEG data The techniques have great potential in studying transitional states of brain activity seen in epilepsy, cognitive task performance and various states of consciousness. More studies are required to tap the potential of nonlinear dynamical techniques for these applications to EEG and this paper is an attempt in that direction. They will have a great impact on research and clinical use of EEG data

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