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BOOK REVIEWS

Deterministic chaos: Complex chance out of simple necessity by N Kumar, Universities Press (India) Limited, 3-5-820 Hyderguda, Hyderabad 500 029, 1996, pp 106, Rs 70.

This book is about chaos, a subject which gives insight into a variety of seemingly random phenomena Science pervades of disciplines from physics to biology The most difficult aspect of a book on chaos is how best to introduce concepts deeply rooted in mathematics particularly when the book is addressed to students at school and undergraduate level Professor Kumar has accomplished this task admirably Many concepts are made easily understandable by giving day-to-day examples (for example, phase locking). The choice of words (for instance, 'mere thinking of it') and flow of the language are just right to make the book very readable

In the introductory chapter, Professor Kumar uses every-day examples starting from leaking faucet to laboratory examples of turbulence, Rayleigh–Bénard convection and the oscillatory chemical reaction of Belesov–Zhabotinski This chapter is the most interesting since it draws on his wide knowledge of different areas to drive home the ideas Right in the first chapter, using Bernoulli shift, he introduces concepts such as stretching and folding, sensitive dependence on initial conditions and algorithmic complexity which are characteristic features of chaos All these are done without resorting to any hard mathematics The subtitles in the chapters are themselves catchy.

In the second chapter, the author introduces the concept of phase space, various types of attractors and their stability Poincaré section is introduced, and dissipative and conservative flows are explained The next chapter is devoted to maps Both logistic and circle maps are considered in some detail. Using logistic map, the author discusses period doubling bifurcation in some detail Professor Kumar has put some effort to communicate the excitement of the universality exhibited by the class of maps with quadratic maximum The importance of the two constants which characterize this universality, namely, the Feigenbaum's number and the number associated with metric universality are discussed Tangent bifurcation and intermittency are also discussed. This is followed by a discussion of the circle map Here, concepts of quasiperiodicity and phase locking are well communicated followed by a discussion on devil's staircase and Arnold's tongue A brief mention of different routes to chaos along with known laboratory examples are included at the end of the chapter The fourth chapter deals with continuous time dissipative systems. Lorenz model is taken as a typical example The emergence of a strange attractor as a consequence of stretching and folding of orbits is well illustrated The next chapter deals with Hamiltonian chaos Beginning with Hamiltonian equations, the author introduces integrable and nonintegrable systems After discussing the distinction between rational and irrational invariant tori, the KAM theorem is explained Starting from a two-dimensional integrable system, destruction of the resonant rational tori is dealt with by considering the addition of a nonintegrable perturbation The importance of elliptic and hyperbolic fixed points for the emergence of stochastic motion is discussed Standard map and driven pendulum are discussed as examples. Geometric example of Sinai billiard ball is also mentioned

One of the quantifiers of chaotic dynamics is the fractal dimension of the chaotic attractor A fairly detailed discussion of fractals and multifractals is found in the sixth chapter This chapter starts with typical examples of deterministic fractals such as Koch snow flake and Cantor dust As a measure of such objects, the capacity dimension or the fractal dimension is then introduced Multifractal measures are introduced as a necessary complication meant to quantify objects which could have spectrum of fractal dimensions A simple means of generating a multifractal using an iterative scheme is illustrated There is also a brief account of reconstruction of a strange attractor from a chaotic time series using delay time embedding with a view of stressing how various dimensional invariants can be recovered from the chaotic time series Finally, in the last chapter insights provided by science of chaos in understanding hitherto unresolved problems are mentioned This chapter also mentions synchronization, control of chaos and encoding of signals

Even though the book appears small in size, almost all concepts and to a large extent the essential mathematical details are dealt with In particular, the appendices contain a brief account of Lyapunov exponents, Bernoulli shift, Bakers transformation, Smale horseshoe, etc Global bifurcations such as homo- and heteroclinic bifurcations are also explained in the appendix This will appeal to a reader who is inclined towards quantitative details

Chaos is largely a numerical science and therefore considerable effort normally goes in conveying concepts through figures Professor Kumar has done this without actually using a computer for generating the plots Most of the figures are hand drawn including the butterfly on the cover page The style, content and presentation of the material serves the audience to which it is intended, namely, the undergraduate students Generally, when one attempts to convey concepts deeply rooted in mathematical details, there is always a possibility of committing errors that stem from oversimplifying That has not happened with this book The book also brings out the fact that science of chaos has had profound influence on many disciplines Many would have liked to know what the subject is all about but would have found it hard to read a standard technical book This book will serve this purpose very well since it has the right mixture of concepts, the minimum technical details and a good pedagogical approach

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G ANANTHAKRISHNA

A practical guide to heavy tails: Statistical techniques and applications by R. J. Adler *et al*, Birkhauser Verlag AG, Klosterberg 23, CH-4010 Basel, Switzerland, 1998, pp 534, sFr. 108.

Everyone in the business of science (well, almost everyone) knows about the Gaussian distribution Many reasons have been invoked to justify its use, such as phenomenological justifications that use the central limit theorem, maximum entropy or worst case analysis, or merely the convenience of being able to use its enormous mathematical spin-offs that have evolved over the years Go from random variables to random processes and one finds that a similar special status is accorded to Gaussian processes and processes that are 'driven' in some sense by Gaussian noise (e g diffusions) This Gaussian culture, along with the 'Poisson culture' that is its discrete counterpart, have been a part of the consciousness of generations of probabilists and statisticians to such an extent that anything else appears an aberration No better evidence of this mental make-up than the use of the word 'outliers' to indicate data points significantly away from the typical, the connotation being that they are somehow bad or undesirable Of course, one knew that there are infinitely divisible distributions other than the Gaussian or Poisson, and independent increment processes other than the Brownian motion or Poisson process, but the study of these has largely remained an esoteric topic pursued by a few specialists

But this is no longer so The need to induct such distributions and processes into mainstream research has been thrust forth forcefully by emerging applications, the most notable being internet traffic and finance. In either of these two domains, initial developments used the Guassian/Poisson crutches as one might expect, but the limitations of these soon became apparent It was no longer possible to treat outliers as freak episodes, but was essential to develop models that admit them in a natural way Thus the need for 'heavy tailed distributions', dubbed thus because their probability densities have 'fat' or 'heavy' tails as one moves towards infinity in any direction Contrast this with the rapidly decreasing tails of the bell-shaped Gaussian curve

For the ever-adventurous community of probabilists and statisticians seeking fresh hunting grounds (not to mention funding possibilities), this has been a cornucopia of research problems. There has been a flurry of activity to build a body of work that will hang together as some sort of a general approach to these, along the lines of the existing constructs for the Gaussian case. A group of probabilists and statisticians have got together and put together this collection of articles, with an eye on its pedagogical value Given the current paucity of 'user-friendly' resources for this area, their explicit aim has been to create a 'guide' or 'handbook' of sorts

The articles have been categorized into several subgroups The first grouping of five articles on 'applications' starts on the right footing by describing the issues thrown up by internet and finance applications The six articles in the second group on 'time series' consider the extensions of traditional techniques in time and frequency domain to these processes Typically, the limitations of strict adherence to conventional wisdom (e g the use of the only second-order statistics) is pointed out and alternatives/extensions are proposed Topics such as robust estimation, estimation of long-range dependence and effects of nonlinearities are discussed, with appropriate caveats for actual practice. The three sections that follow, 'heavy tail estimation', 'regression' and 'signal processing', have two articles each and deal with various statistical issues such as bootstrap and parameter estimation The section on 'model structures', with three articles, addresses various theoretical issues about these distributions and processes The final quartet on 'numerical procedures' highlights computational and approximate issues as well as software implementations

The list of authors (too long to be reproduced here) features many of the active workers in the field Together they have put together a valuable collection of articles on an important field in the process of maturing, with immense potential for both theory and applications A very timely contribution

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High dimensional probability by Ernst Eberlein *et al*, Birkhauser Verlag AG, Klosterberg 23, CH-4010 Basel, Switzerland, 1998, pp 344, sFr 128.

'High-dimensional probability' is the new avatar of 'probability on Banach spaces' The book contains 20 articles. Brief descriptions of some of the articles based on the interests of this reviewer are given in this review.

In 'Convergence in law of random elements and random sets', Hoffmann-Jorgensen gives the basic theory of weak convergence of random elements and sets in great generality

M Ledoux (A short proof of the Gaussian isoperimetric inequality) deals with some versions of the following inequality

If A is a Borel set in R, A_r is the set of points whose distance from A is less than r, and μ is the standard Gaussian measure on R^n then $\mu(A_r) \geq \Phi(a+r)$ where a is a number satisfying $\mu(A) \geq \Phi(a)$ and $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy, x \in \mathrm{R}$ This important inequality is a consequence of an inequality due to S

Bobkov and a simple self-contained proof of this last inequality using only the methods of calculus is given in the article The proof is based on an article of the author with D Bakry

In 'The best doob-type bounds for the maximum of Brownian paths' G Peskir proves the inequality

$$E\left(\max_{0\le t\le\tau}|B(t)|\right)^p \le C_{pq}\left(E\left(\int_0^\tau |B(t)|^{q-1}\,dt\right)\right)^{\frac{p}{q-1}}$$

for all stop times τ, for $0<p<q+1$ with $q>0$ They obtain the best constant c_{pq} and their result extends to all non-negative submartingales.

In 'A consequence for random polynomials of a result of De La Pena and Montgomery–Smith' E Gine uses decoupling of tail probabilities of U-statistics to derive decoupling inequalities for general polynomials in any set of independent (not necessarily symmetric) random variables

'Small deviation probabilities of sums of independent random variables' by T Dunker, M. A Lifshits and W Linde In 1986, V M Zolotarev announced a precise description of the behaviour of

$$P\left\{\sum_{j=0}^{\infty}\phi(j)\xi_j^2<r\right\}$$

as $r\to 0$ where $\{\phi(j)\}\in l^1$, $\phi(j)>0\,\forall j$ and $\{\xi_j\}$ is i i d. $N(0, 1)$ The present authors find an inaccuracy in the result of Zolotarev and give a corrected version They also deal with the case in which $\{\xi_j\}$ is replaced by an i i d sequence of non-negative random variables with finite second moment A result in this direction was obtained by R Davis and S Resnick in 1991 and M A Lifshits in 1997, but the present paper focuses on the verifiability of the result in the most important specific examples

W. V Li and G. Pritchard consider the following problem (Central limit theorem for the sock-sorting problem) n different pairs of socks are scrambled in a laundry bag Socks are drawn randomly one at a time from the bag and laid on the table When the mate of a sock appears the two are removed from the table How much table space is required? The authors prove a functional central limit theorem for the table space usage as a function of the number of socks drawn

'Laws of large numbers and continuity of processes' by B Heinkel For a sequence $\{X_n\}$ of independent random variables which are either centered and i i d or symmetric it is shown that the Strong Law of Large Numbers is equivalent to a s continuity of the paths of the process

$$\left\{(1-t)^2\sum_{n\ge 3}t^n\sum_{0<i<n}X_iX_{n-i}\right\}$$

and the fact that

$$\frac{1}{n^2}\sum_{k=1}^{n}X_k^2\to 0 \text{ a s}$$

In 'On random measure processes with application to smooth empirical processes' P Gaenssler *et al* prove a uniform law of large numbers for function-indexed 'random measure processes'. In particular, this gives a uniform law of large numbers for smoothed empirical processes They also prove a functional central limit theorem for smoothed empirical processes under conditions different from the ones found in literature

In 'A sufficient condition for the continuity of high order Gaussian chaos processes', M B Marcus obtains sufficient conditions for continuity and boundedness of mth order Gaussian chaos processes ($m \geq 2$) The results are stated in *Probability in Banach spaces* by M Ledoux and M Talagrand, Springer Verlag, 1991 However, the proof given here is incorrect

The paper 'Optimal tail comparison based on comparison of moments' by I Pinelis is concerned with the problem of maximizing $E\psi(\xi)$ over all random elements ξ satisfying the inequality $E\phi(\xi) \leq E\phi(\eta)$ for all $\phi \in \mathcal{F}$ where η is a given random variable on some measurable space, ψ is a given measurable function and $\mathcal{F}$ is a given set of measurable functions on that measurable space.

On the whole, a useful collection of articles in this area by some of the active researchers with a fair indication of the current themes in the field Of interest more to specialists rather than the broader community of probabilists

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Integration—A functional approach by Klaus Bichteler, Birkhauser Verlag AG, Klosterberg 23, CH-4010 Basel, Switzerland, 1998, pp. 208, sFr 68

This is a modern book dealing with a very ancient topic integration of functions The list of names of star mathematicians associated with the crucial development of the subject at various stages is indeed mind boggling Riemann, Cauchy, Borel, Lebesgue, Radon, Nikodym, Riesz, Carathéodory, Daniel, Stieltjes, etc Apart from other questions, the motivation came from Fourier analysis definition of Fourier coefficients involved integration of periodic functions over the period If the function is continuous, the theory of Riemann integration is adequate for defining its integral. Let us briefly recall the main idea Riemann squeezed the given function from above and below by step functions, functions which are piece-wise constants on sub-intervals Integrals of this latter class of functions are easy to define

$$\sum_{\text{steps}}(\text{step size}) \times (\text{height of the step})$$

It was soon realized that the class of functions integrable in the Riemann sense is too narrow It is not stable under several limit operations. Extensions of the Riemann's idea to higher dimensions posed grave difficulties At the same time, input from physics suggested that many physical quantities may have 'bad' densities but their averaged values make sense perfectly Thus, there was a need to enlarge the concept of the integral Lebesgue entered the scene and resolved many of these problems His idea was the same as that of Riemann except that he approximated by simple functions (instead of step functions) which are piece-wise constants on subsets A which are not necessarily sub-intervals Unlike Riemann who partitioned the domain (physical space) Lebesgue worked with co-domain (state space) and this is the key point for his success Next step in carrying out Lebesgue's idea required measuring the size of subsets A Thus was born Lebesgue measure. In its construction, he admitted countable interval covering of subsets. We recall that finite covering technique was commonly used during that era It is true that the class of functions integrable in Lebesgue sense is strictly larger than Riemann class However, the largest class of functions for which one can define integral which has desired properties is unknown

The works of Carathéodory mark a watershed in the point of view adopted in the theory of integration As we know, there are two objects involved in the integration integrand and the measure In the classical book, Carathéodory generalized the concept of length of interval to the concept of measure on large class of sets His method is very powerful and this gave rise to abstract and axiomatic measure theory The tools he introduced (outer measure, cut condition, etc) are quite ingenious Because of influential books by Halmos and others, the above approach now forms part of the curriculum in various universities around the world This method is also successful in dealing with lower dimensional measures (e g Hausdorff measures) However, its success in stochastic integration is limited

The present book presents a different path known as Daniell's approach to integration It concentrates on the integrand and not on the measure That is why this is referred to as functional approach In this method, one extends the linear functional, namely, the Riemann integral, which is originally defined for compactly supported continuous functions to a large collection By taking characteristic functions associated with sets, one can recover the measure Readers familiar with measure theory can easily compare this situation with Riesz Representation Theorem There are some earlier books where this approach is followed, for example, L H Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, 1953 Traditionally, one defines Daniell's lower and upper means and the integrable class is precisely the one for which their two means coincide The novelty in the present book is that the use of Daniell's lower mean is altogether avoided Having identified the integrable class, the author proceeds to introduce Lebesgue spaces and study their properties Product measures, convolutions, Radon–Nikodym Theorem are some of the other topics covered via functional approach

Given the distribution theory and modern development of partial differential equations, the idea of viewing measures as linear functionals (rather than as a set function) has gained momentum One of the striking results is that any periodic distribution has a well-defined Fourier series which converges to it This has solved to a great extent our initial question on Fourier series with which we started our discussion Another application which comes to my mind is optimal design problems Their initial formulation involves minimization over a class of sets Generically, such problems do not admit a solution Thus, a relaxation procedure is followed in which we replace sets by their characteristic functions Daniell's approach is perhaps the first illustration of such ideas

For reasons cited above, I feel that a balanced view of both approaches (Carathéodory and Daniell) should be taught in any analysis course at M Sc level The present volume, being a modern introduction to Daniell approach, is recommended as a textbook for its lucid style, liberal number of exercises and general presentation.

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The computational beauty of nature by Gary William Flake, The MIT Press, 55 Hayward Street, Cambridge, Mass 02142, USA, 1998, pp 483, $45.

Computers have come to occupy a centre stage in modern scientific exploration For example, computer experiments have played (and continue to play) a crucial role in understanding nonlinear systems where no analytic solutions are possible Further, an indepth appreciation of the beauty of fractals would not have been possible without computers The book under review explores various fields using computation as the basic tool.

The book deals with the following topics—computation, fractals, chaos, complex systems and adaptation Since each of these topics has already been individually covered by other books, the author tries to be different by exploring all of them in one single book. He ties them all together by using the computer as a laboratory to study them He also stresses the unifying (but well-known) theme that simple recursive rules can lead to complicated behaviour and patterns in many of these fields This prevents the book from turning out to be an unconnected collection of topics

In the first part, the author starts with a simple introduction to numbers including concepts of countability He then introduces the method of Godelization where many natural numbers are mapped into one single number Then the Church–Turing thesis is explained briefly The famous halting problem is also discussed This part ends with a brief description of Godel's theorem

The next part deals with fractals Fractals have become very fashionable in recent times This part starts by recollecting some basic properties of fractals including self-similarity and fractional dimension Lindemayer systems that model plant growth using simple recursive rules are dealt with next Many figures are given as examples One nice feature of the book is that it provides the address of an FTP site that contains the source code of programs used in the book for computer experiments. Thus a reader, using these programs, can start by reproducing most of the figures given in the book and continue with further exploration on one's own

Iterated function systems using affine linear transformations are described next The part on fractals ends with an inevitable description of Julia sets and the Mandelbrot set (the so-called master Julia set) A nice touch is the inclusion of a relatively unknown and ill-understood property of the Mandelbrot set where the value of π emerges as a higher-order pattern on its own

The third part of the book deals with the now ubiquitous phenomenon of chaos This part starts, predictably enough, with the example of the logistics map One wishes that the author had used a different example The inclusion of shadowing lemma which has important implications for computability of chaotic systems is more satisfying Another satisfying feature is the inclusion of a "Further exploration" section at the end of each chapter which guides the reader to new unexplored avenues Strange attractors, attractors of chaotic systems which typically have fractional dimensions, are described next Again, familiar examples are given which could have been avoided A nice self-contained explanation of control of chaos is a welcome addition This part ends with a chapter linking together chaos, randomness and incomputability

The fourth part is an exploration of the world of complex systems (autocatalytic sets, ecosystems, economies, etc) The first chapter is an introduction to cellular automata including Wolfram's classification and Conway's famous Game of Life The next chapter deals with self-organization where autonomous agents interact in such a way as to create global order Three examples—Resnick's termites, Langton's virtual ants and Reynolds boids (virtual birds)—are explored using cellular automata Next, concepts of cooperation and competition in a family of interacting agents are explained using game theory In particular, non-zero sum games like the well-known Prisoners Dilemma (the prototypical model of social conflict) are studied in some detail. Applications to ecosystems are also considered The final chapter deals with artificial neural networks (especially the Hopfield neural network)

The final part introduces adaptive systems which adapt to external changes in the environment using feedback mechanisms It starts with a nice description of genetic algorithms and classifier systems (in particular, the zeroth-level classifier systems) The next chapter introduces the concept of Rosenblatt's perceptron, a type of pattern classification device based on a model of visual perception

To summarize, the book nicely ties together different areas like chaos, fractals, computation and complex/adaptive systems using computer-assisted exploration. The availability of source code for various programs and sections on "Further exploration" and "Unifying themes" at the end of each chapter are some of the highlights of the book The use of cliched examples to illustrate various concepts is unwarranted given that this book is preceded by many others covering the same areas

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GOVINDAN RANGARAJAN

Bifurcation of planar vector fields and Hilbert's sixteenth problem by Robert Roussarie, Birkhauser Verlag AG, Klosterberg 23, CH-4010 Basel, Switzerland, 1998, pp 224, sFr 98

The subject matter of the book is dynamical systems One of the goals of the subject is to describe the behaviour of systems for large times What is given is a law governing their evolution in an infinitesimal amount of time. Above situation is usually modelled by a set of ordinary differential equation:

$$\frac{dx(t)}{dt} = X(x(t)). \tag{1}$$

The aim is to study the asymptotic behaviour of the solution as the independent variable, namely, time, goes on infinity One of the examples is the study of the solar system where X is a vector field representing Newton's gravitational force of attraction The unknown x is also a vector with components $(x_1 \ x_N)$ depending on the position and the velocity of the Sun and the planets One of the characteristic features of the example is that X is singular

In general, even if X is a smooth force field and $N \geq 3$, the system represented by x can exhibit 'chaotic' behaviour for large times. There is an abundant literature about chaos in the field of mathematics, physics and other relevant sciences Given this situation, the framework of the present volume at first sight seems to be very modest Indeed, it is devoted to the case $N = 2$ and where X is smooth and even analytic at many times In the study of chaos, such a situation is usually brushed aside by stating that the case is uninteresting and the behaviour is described by Poincaré–Bendixson Theorem One may thus wonder at the relevance of the present book It is true that the case of many bodies ($N \leq 3$) is complicated If we view it as a case of two bodies ($N = 2$), it is clear that the complication is due to the fact that the force exerted on the two bodies by the rest is complex If this force is assumed to be small then we can easily see the relevance of the case $N = 2$ and the system behaviour under perturbations This is precisely the object of this book Starting with Poincaré–Bendixson Theorem for an individual X, it presents various scenarios for the behaviour under perturbations by considering a family (X_λ) There are many intricacies in mathematical analysis and the proofs are surprisingly delicate. Hilbert's sixteenth problem is one aspect of it

In the case of a single vector field X, the possible asymptotic behaviour, roughly speaking, is described by singular points, limit cycles and saddle connections In the case of a family (X_λ), these objects change as the parameter λ varies and at critical values λ^*, qualitative changes (called bifurcations) take place in the behaviour Bifurcation analysis of singular points (e.g. Hopf bifurcation) is classical and can be found in many mathematical texts. The main aim of the present book is to study bifurcation of limit cycles. It is intuitively clear that such bifurcations can occur only if there is an accumulation of limit cycles at the critical value λ^*. This idea has been formalized in this monograph by the introduction of the notion of *limit periodic sets* Γ associated with the family (X_λ) The notion of *cyclicity*, *Cycl*(X_λ, Γ), which denotes the number of limit cycles bifurcating from Γ, is also introduced.

The prototype example of bifurcation analysis done in this book is Bogdanov–Takens family $\left(X_{\lambda}^{N}\right)$ which involves two scalar parameters It is shown how a small limit cycle is created from a singular point and how it grows and finally how it dies at saddle connection

The study of bifurcation of limit cycles is different from that of singular points and it requires different mathematical tools and methods One classical tool used in this context is Poincaré transversal and the associated return map P Depending on the nature of the return map, different methods are employed to obtain the results The aim of this text is to present them in a systematic way If P is smooth (e g. elliptic singular point, periodic orbit) then methods of analytic geometry, theory of Abelian integrals, preparation theorem, etc are used (Ch 4). In the case of homoclinic saddle connection, P is not smooth Asymptotic Dulac series (involving logarithms) is used to analyse this case (Ch 5) Finally, in Chapter 6 (see also Ch 3), the author presents desingularization process (blow-up of singularities) to show how more complicated limit periodic sets can be reduced to the elementary cases treated in the previous two chapters

Hilbert's sixteenth problem presented at Paris Congress in 1901 concerns, in particular, in showing that there are finitely many limit cycles for polynomial planar vector fields A more general conjecture regarding the finiteness of $Cycl(X_{\lambda}, \Gamma)$ is offered in Chapter 2 along with a programme for solving Hilbert's problem Results proved in Chapters 3–5 establish the truth of the general conjecture in a number of cases Thus, the second half of the title of the book is also justified

This text brings together several results from previous articles and presents them in a coherent manner The reader is a little bit inconvenienced by a few (inconsequential) misprints here and there He should also be very careful with the numbering of theorems which is not very standard Otherwise, I feel that this is a treasure house of mathematical ideas and methods in qualitative theory of dynamical systems initiated by Poincaré

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Algebraic aspects of integrable systems by A S Fokas and I M Gelfand, Birkhauser Verlag AG, Klosterberg 23, CH-4010 Basel, Switzerland, 1998, pp 360, sFr 138

This volume is a collection of articles on integrable systems, dedicated to late Irene Dorfman whose works on the algebraic structures in integrable systems are well known Particularly, her discovery that integrability of nonlinear evolution equation has a close connection with the existence of bi-Hamiltonian structures (discovered by Magri at the same time but independently) has inspired a large number of mathematicians and physicists working in this area The papers reflect on the advancements made in the algebraic aspects of integrable systems, both formula and specific, besides some contributions to analytic aspects, and essentially give a flavour of the state of the art Of the 15 contributions, those of Alber *et al* and McKean consider analytic aspects of integrable systems, while the rest deal with algebraic aspects

The articles on algebraic structures deal broadly with the following aspects

(i) Bi-Hamiltonian structures

(ii) Bi-Hamiltonian operators and their related Poisson brackets

(iii) Extension of the above properties to discrete systems

(iv) Master symmetries and infinitely many symmetries

(v) Transformations

The details are as follows.

The 1970s works of Dorfman and Magri clearly show that integrable soliton equations such as Kortegweg–de Vries and nonlinear Schrodinger equations and their generalizations can be constructed using a bi-Hamiltonian method Fokas *et al* describe a basic approach to construct a wide variety of integrable bi-Hamiltonian equations that include nonlinear dispersion, supporting novel type of solitonic solutions Fordy and Harris present a systematic construction of Hamiltonian structures written in stationary manifold coordinates by restricting to isospectral flows Fuchssteiner describes how compatible Hamiltonian pairs play a crucial role in the theory of integrable systems

Many papers are devoted to the construction of bi-Hamiltonian operators, their related Poisson brackets and their connection to Lax operators in the study of algebraic structures In particular, the paper of Oevel discusses the bi-Hamiltonian operators in connection with r-matrices and the modified Yang–Baxter equation, while Dickey constructs the τ-functions of Zakharov–Shabat and other matrix hierarchies of integrable equations Cohen *et al* identify some nice connections between pseudo-differential operators and modular forms

An interesting area of activity in recent times has been the extension of integrability aspects of continuous evolution equations to corresponding discrete equations Discretized versions and their algebraic properties of various equations including the equation $u_t + uu_x = 0$ (Kuperschmidt), certain Schwarzian equation (Nijhoff) and Newman equation (Ragnisco and Suris) are discussed The paper of Semenov–Tian–Shansky and Sevostyanov discusses nonultralocal Poisson brackets for one-dimensional lattice systems

Grunbaum and Haine, in line with the interest of Dorfman on the concept of master symmetries, bring out certain relations between the Toda lattice, Bocher's theorem and orthogonal polynomials Integrable soliton systems are associated with infinitely many symmetries Kodama and Mikhailov discuss nonintegrable effects appearing in higher order corrections of an asymptotic perturbation expansion for a given nonlinear wave equation as extension of theory of normal form approach of Kodama and approximate symmetry approach of Mikhailov, and identify obstacles to asymptotic integrability Santini describes the crucial role played by symmetries in the deviation of multiscale expansion for a class of nonlinear Schrodinger equation hierarchy and in the study of propagation of quasi-monochromatic, nondissipative and weakly nonlinear waves

Further, Ferapontov and Mokhov demonstrate that for an arbitrary number n of primary fields, the equations of associativity can be rewritten in the form of $(n-2)$ pairwise commuting systems of hydrodynamic-type integrable systems, having certain natural Hamiltonian representation Schief and Rogers show that the $(2 + 1)$ dimensional Lowener–Konopelchenko–Rogers integrable system is invariant under Laplace–Darboux-type transformations which is then exploited to derive novel Ernst-type integrable equations

Two papers deal with analytic aspects of integrable systems McKean deals with Dirac and Schrodinger periodic operators to study the canonical 1-form of symplectic geometry in the context of defocusing nonlinear Schrodinger equation and Korteweg–de Vries equation using trace formulas. Considering complex billiard Hamiltonian systems and nonlinear wave systems, Alber *et al.* construct interesting classes of solutions in the context of Riemannian manifolds.

To summarize, the book is a collection of articles written by experts in the field primarily on the various algebraic structures associated with integrable systems They provide a feel of the present status of the field, while providing an opportunity to appreciate the contributions and interests of Irene Dorfman Researchers working in the area of integrable systems will find it a useful addition to literature

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M LAKSHMANAN

Generalized characteristics of first order PDEs by A Melikyan, Birkhauser Verlag AG, Klosterberg 23, CH-4010 Basel, Switzerland, 1998, pp. 310, sFr 148

This book reports on recent progress made on a very classical topic, namely, that of defining unique solution to a system of ordinary differential equations specified by a vector field X If X is Lipschitz, the classical Cauchy—Lipschitz Theorem achieves this task by associating a unique solution to each given initial condition Given initial position and velocity, the above theorem solved the problem of finding the trajectory of a classical particle moving in a smooth potential field The major challenging task ahead is to consider the case when X is not Lipschitz

Let us start with a quick look at a way in which first-order PDEs and their characteristics arise Study of propagation of light waves provides an illustrative example Their behaviour is modelled by PDEs such as *the wave equation* However, in the short-wave limit, the concept of the wave function ϕ is not very useful and is replaced by the notion of families of trajectories. The subtle connection between these two theories is provided by the ansatz of the geometrical optics $\phi = ae^{i\varepsilon^{-1}u}$, where a is the amplitude, u the phase function and ε the wavelength It can be deduced that u satisfies the Hamilton–Jacobi equation

$$F(x, \nabla u(x)) = 0 \tag{1}$$

where $F = F(x, p)$ is the so-called Hamiltonian associated with PDE Since (1) is a first-order PDE, the method of characteristics applies and reduces it to a system of ODEs in the smooth region

$$x = F_p, p = -F_x \tag{2}$$

The solution (when it exists uniquely), called *characteristics*, defines a curve in the phase space (x, p) and its projection in the physical space x defines what is called *ray* These are the appropriate tools to describe linear wave phenomena in the short-wave limit This description is consistent with *Fermat's principle* and *Snell's law* of refraction Typically, even if F is regular, rays tend to focus on *caustic surfaces* Since the value of u is transported along rays, u becomes multivalued on caustics. At first sight, this seems to suggest the failure of the above construction of u

On the other hand, results proving the existence and uniqueness of the wave function ϕ in various situations indicate that the above difficulty can be overcome This is confirmed by means of the ingenious procedure of Maslov and one can globally define characteristics (and hence rays) Morphology (i e form) of structurally stable caustic singularities exhibited by wavefronts is a study of independent interest

It is useful to recall the importance of characteristics in the above example They are carriers of information from initial data They predict the position of singularities of the solution and the speed with which they propagate A question for the future is how to generalize this to nonlinear equations

Study of transport of passive scalar by incompressible fluid flows requires knowledge of streamlines n turbulent regimes, the velocity field is notoriously irregular, oscillating, chaotic, random, etc Defini-ion of unique characteristics (i e streamlines) under these circumstances remains a big challenge How-ever, thanks to the concept of *renormalized solutions*, some understanding of the basic issues involved as been achieved

If we consider compressible fluid flows without dissipation (modelled by nonlinear hyperbolic con-ervation laws), there is a new singularity called *shocks* across which state variable may admit a jump Iow to define unique characteristics in this case? The second law of thermodynamics is imposed in the orm of what is called entropy condition to eliminate certain non-physical shocks and thereby define the haracteristics uniquely These ideas have been successfully implemented in the case of single and double onservation laws to demonstrate uniqueness, asymptotic stability and other qualitative properties of the olution

Another set of ideas to deal with singularities and non-uniqueness is set-valued mappings, generalized radients and *differential inclusions.* This is especially fruitful if the vector field enjoys some monotonic-y properties

In this book, the examples motivating the theory of characteristics come from optimal control and dif-rential games In these cases, the corresponding first-order equation is of the form

$$F(x, u(x), \nabla u(x)) = 0 \tag{3}$$

he function $F = F(x, u, p)$ which has now u-dependence need not be smooth The corresponding classi-il characteristics are modifications of (2) and are given by

$$\dot{x} = F_p, \quad \dot{u} = < p, F_p >, \quad \dot{p} = -F_x - pF_u \tag{4}$$

ie concept of *viscosity solution* $u \in C^0$ which generalizes the entropy condition is the most appropriate ie for (3). In this set-up, it is not clear how to define solution for (4) and validate method of characteris-s which usually demands $F \in C^2$ and $u \in C^2$ However, uniqueness and stability of viscosity solutions dicate that the characteristic system (4) may be suitably modified to cover even the non-smooth cases us is indeed confirmed in a number cases of singularities, mainly codimension one, considered in this ok These singularities are consistent with viscosity condition, they are called equivocal, focal, disper-l, universal singularities (the terminology coming mainly from differential games) The above classifi-tion is essentially based on the behaviour of classical characteristics defined by (4) in the vicinity of igularities (This is analogous to, for instance, Lax entropy condition in the hyperbolic conservation ws) Many of them occur in cases where F is neither convex nor concave The location of these singu-ities is not known a priori Its determination as well as the construction of u are part of the problem To s end, the author proposes what is called *method of singular characteristics* (MSC) The basic idea of construction comes from differential geometry and is presented in Chapter 1 along with an application some free boundary problems In Chapter 2, MSC is applied in the thoery of viscosity solutions apter 3 suggests a survey of various features of Hamiltonians arising in calculus of variations, optimal ntrol and differential games Chapters 4 and 5 are devoted to the study of singularities occurring in ferential games. In Chapter 6, the author investigates a somewhat non-classical situation where the ution u is smooth but not the Hamiltonian F. An independent analysis of MSC along with its direct sentation is attempted in Chapter 7 for two-dimensional problems It is demonstrated in Chapter 8 that SC can be useful for second-order PDE as well Viscosity solutions are not considered here but it is umed that the equation is nothing but the stationarity condition of a functional defined not only on ictions but also on sets which may possibly contain their singularities

In the above discussion, we have presented a glimpse of the contents of the book as well as their importance/significance We have also tried to place the contribution of the book in the overall development of the entire subject One of the goals is to keep track of singularities of solutions, appearance of new ones, their locations, speed and strength Such an information is vital not only for the advancement of the theory but also for the design of efficient numerical schemes Characteristics are carriers of such an information Undoubtedly, MSC presented in this volume is a step in this direction Active researchers in PDE will find it extremely useful

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Differential and integral operators by I Gohberg *et al*, Birkhauser Verlag AG, Klosterberg 23, CH-4010 Basel, Switzerland, 1998, pp. 344, sFr. 148

Operator theory is one branch of mathematics which is used and recognized as important by both pure and applied mathematicians without any dispute. On the one hand, it is a field of mathematics where mathematicians can work on problems which are only of mathematical interest without bothering about applications Here one can study interesting problems in very abstract set up On the other hand, the applications of operator theory are tremendous In simple terms, it unifies a whole lot of interesting practical problems from various branches of science In fact, one can put various systems in a single framework For example, the applications concerned problems from mathematical physics, quantum mechanics, hydrodynamics, astrophysics, network and systems, etc to name a few The basic idea is to view equations describing the physical phenomena as operator equations and then use the beautiful abstract theorems available in operator theory In this respect, differential operators, pseudo-differential operators, integral operators, etc all come under the framework of general operator theory

The book under review is the first in the two volumes of proceedings of the international workshop on 'Operator Theory and Applications' held at the University of Regensberg, Germany, during July 31–August 4, 1995 This workshop was the 8th in a series of workshops on 'Operator Theory' held at various places with a regular two-year interval starting from the 1981 workshop at Santa Monica, California, USA This itself shows the importance of operator theory in the present-day mathematics

The book under review contains 22 articles covering a wide range of latest developments in operator theory and its applications, namely, spectral theory of ordinary and partial differential operators, pseudo-differential and integral operators As it covers a range of specialized topics, a complete analysis of the book means analysing each article separately which is not possible and is beyond the scope of this review

As everyone knows most of the physical systems are modelled by differential equations, it is not surprising that majority of the articles are concerned with differential operators, more specifically partial differential operators Of course, there are other interesting articles from integral operators and pseudo-differential operators For example, in the article by M. M Malamud, the author studies the invariant and hyper-invariant subspaces of direct sum of Volterra operators In the pseudo-differential category, there are two articles, one by V S Rabinovich and the other by E. Schrohe and B. W Schulze. In both of them, Mellin pseudo-differential operator techniques are used Rabinovich studies singular integral operators on some Carleson curves while in the other article boundary-value problems on manifolds with edges are investigated S Albeverio and K A Makarov studies a model of a quantum mechanical system related to the three-body problem which is defined in terms of a pseudo-differential operator with unbounded symbol

Most of the other articles, broadly can be put in the differential operator category A Lifschitz in his article studies a problem related to a three-dimensional quasi-helical plasma equilibria with flow, where he uses the Fourier transform method to reduce the original problem with partial differential operator to a family of spectral problems for ordinary differential operator He also presents some numerical procedure for finding the spectrum This is the only article presenting some numerical results Spectral problems related to generalized string equation and matrix Sturm–Liouville equations are investigated in the article of L A Saknovich Many articles are devoted to the study of Sturm–Liouville operators, viz 'On estimates of the first eigenvalue in some elliptic problems' by Yu V Egorov and V A Kondratiev, 'Non singularity of critical points of some differential and difference operators' by A Fleige and B Najman, and 'Interpolation of some function spaces and indefinite Sturm–Liouville problem' by S G Pyatkov

This book also contains many other articles of interest studying various aspects of partial differential operators, more specifically elliptic operators We do not go into their details

Though all the articles presented in the book come under the purview of operator theory, each one is a specialized topic needing different treatment In other words it is not concentrated on a special area Of course, one can refer to a particular article of his or her choice This can neither be used as a textbook nor as a reference volume to begin or carry out research It is just a collection good articles combined together in the broad area of operator theory and looks more like an issue of a journal

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