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Results concerning probabilistic 2-metric spaces*

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Abstract

In this paper, we have introduced the concept of probabilistic maximal triangle in a subset of 2-menger space and have established its properties in the form of some theorems.

In a recent paper Khan and Zertaj[1] has introduced following concepts.

Definition A. A probabilistic 2-metric space (P-2-M space) is an ordered pair (X, F) consisting of non-empty set X and a mapping F from $X \times X \times X$ to L, the collection of all distribution functions. The value of F at $(u, v, w) \in X \times X \times X$ is represented by F_{uvw}. The functions F_{uvw} are assumed to satisfy the following conditions:

I. $F_{uvw}^{(x)} = 1$ for all $x > o$ if at least two of u, v, w are equal.

II. $F_{uvw}^{(o)} = 0$.

III. F_{uvw} is invariant under all permutations of u, v, w.

IV. If $F_{uvs}^{(x)} = 1$, $F_{usw}^{(y)} = 1$ and $F_{svw}^{(z)} = 1$ then $F_{uvw}^{(x+y+z)} = 1$.

In every 2-metric space (X, d) the 2-metric d induces a mapping F: $X \times X \times X \rightarrow L$ such that $F(u, v, w)(x) = F_{uvw}^{(x)} = H(x-d(u, v, w))$ where H is a distribution function defined by

$$H(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0. \end{cases}$$

With the interpretation of $F_{uvw}^{(x)}$ as the probability that the area of the triangle with vertices u, v, w is less than x, one sees that conditions I, II and III are straight forward generalization of the corresponding conditions of 2-metric spaces. Condition IV is a minimal generalization of triangular area inequality which may be interpreted as follow "it is certain that the area of the triangle with vertices u, v, s is less than x, the are of the triangle with vertices u, s, w is certainly less than y and the area of the triangle with vertices s, v, w is certainly less than z then the area of the triangle with vertices u, v, w must certainly be less than $x + y + z$.

The condition IV is always satisfied in 2-metric spaces where it reduces to triangular area inequality (T.A–inequality). However in those P-2-M spaces in which the equality

$F_{uvw}^{(x)} = 1$ does not hold for $(u \neq v \neq w)$ any finite x, IV will be satisfied only vacously. Therefore a stronger version of generalized T. A-inequality on the pattern of Menger space was introduced.

Definition B. A function T: [0, 1] × [0, 1] × [0, 1] to [0, 1] is called T-2-norm if it satisfies the following

(T_1) T (a, 1, 1) = a, T (0, 0, 0) = 0.

(T_2) T (a, b, c) is invariant under all permutations of a, b, c.

(T_3) T (e, f, g) ≥ T (a, b, c) if e ≥ a, f ≥ b, g ≥ c.

(T_4) T (T(a, b, c,), d, e) ≥ T (a, T (b, c, d), e) = T (a, b, T (c, d, e)).

T = min is the strongest possible universal T for

T (a, b, c) ≤ T (a, 1, 1) = a (i)

T(a, b, c) = T(b, c, a) ≤ T(b, 1, 1) = b (ii)

T(a, b, c) = T(c, a, b) ≤ T(c, 1, 1) = c (iii)

(i), (ii) and (iii) implies T(a, b, c) ≤ Min(a, b, c).

Definition C. A 2-Menger space (X, F, T) is a P-2-M space (X, F) in which T-2-norm satisfies the following condition.

IV M. $F_{uvw}^{(x+y+z)} \geq T\left(F_{uvs}^{(x)}, F_{usw}^{(y)}, F_{svw}^{(z)}\right)$

for all x, y, z ≥ 0 and for all u, v, w, s (distinct or not) ∈ X.

Motivated by R. J. Egbert[2] we have introduced some new concepts and have proved some theorems in this note (for a related concept in Menger space see [3, 4] also). Throughout the discussion we will consider (X, F, T) as 2-Menger space with T-2-norm-continuous in all arguments.

Definition 1: Let A be a non-empty subset of X. The function D_A defined by

$$D_A^{(x)} = \sup_{t<x}\left[\inf_{u,v,w \in A} F_{uvw}^{(t)}\right]$$

will be called the probabilistic maximal triangle in A.

We now establish the properties of the probabilistic area. We omit the proofs requiring only routine calculations.

Theorem 1. The function D_A is a distribution function.

Definition 2. A non-empty subset A of X is bounded if $\sup_x D_A^{(x)} = 1$, semibounded if $0 < \sup D_A^{(x)} < 1$ and unbounded if $D_A = 0$.

Theorem 2. If A is a non-empty subset of X then $D_A = H$ iff A consists of not more than two points.

Theorem 3. If A and B are non-empty subsets of X and $A \subset B$ then $D_A \geq D_B$.

Theorem 4. If A, B and C are three non-empty subsets of X such that $A \cap B \cap C = \phi$ then

$$D_{A\cup B\cup C}^{(x+y+z)} \geq T\left(D_{A\cup B}^{(x)}, D_{A\cup C}^{(y)}, D_{B\cup C}^{(z)}\right). \tag{1.1}$$

Proof. Let x, y and z be given. To establish (1.1) we first show that

$$\inf_{u,v,w\in A\cup B\cup C} F_{uvw}^{(x+y+z)} \geq T\left(\inf_{u,v,w\in A\cup B} F_{uvw}^{(x)}, \inf_{u,v,w\in A\cup C} F_{uvw}^{(y)}, \inf_{u,v,w\in B\cup C} F_{uvw}^{(z)}\right). \tag{1.2}$$

Let

$$\inf_{u,v,w\in A\cup B\cup C} F_{uvw}^{(x+y+z)} = \inf_{\substack{u\in A\\ v\in B\\ w\in C}} F_{uvw}^{(x+y+z)}. \tag{1.3}$$

Now for any quartret of points u, v, w and s in X, we have $F_{uvw}^{(x+y+z)} \geq T\left(F_{uvs}^{(x)}, F_{usw}^{(y)}, F_{svw}^{(z)}\right)$.

Taking the infimum of both sides of this inequality as u ranges over A, v ranges over B, w ranges over C, s ranges over $A \cap B \cap C$ and using (1.3) we have,

$$\inf_{u,v,w\in A\cup B\cup C} F_{uvw}^{(x+y+z)} \geq \inf_{\substack{u\in A\\ v\in B\\ w\in C\\ s\in A\cap B\cap C}} \left(F_{uvs}^{(x)}, F_{usw}^{(y)}, F_{svw}^{(z)}\right).$$

However, since T is continuous in all arguments and non-decreasing we obtain

$$\inf_{u,v,w\in A\cup B\cup C} F_{uvw}^{(x+y+z)} \geq T\left(\inf_{u,v,s\in A\cup B} F_{uvs}^{(x)}, \inf_{u,s,w\in A\cup C} F_{usw}^{(y)}, \operatorname{Inf}_{s,v,w\in B\cup C} F_{svw}^{(z)}\right).$$

If (1.3) does not hold, then infimum $F_{uvw}^{(x+y+z)}$ takenover u, v, w in $A \cup B \cup C$ is either attained with all the variables u, v, w in one of the three sets A, B, C or with any two of u, v, w in one of the sets A, B, C and remaining one variable in any one of the remaining two sets.

Let, $\inf_{u,v,w\in A\cup B\cup C} F_{uvw}^{(x+y+z)} = \inf_{u,v,w\in A} F_{uvw}^{(x+y+z)}$.

We have

$$\inf_{u,v,w\in A\cup B\cup C} F_{uvw}^{(x+y+z)} \geq T\left(\inf_{u,v,w\in A} F_{uvw}^{(x)}, H(y), H(z)\right)$$

$$\geq T\left(\inf_{u,v,w\in A\cup B} F_{uvw}^{(x)}, H(y), H(z)\right)$$

$$\geq T\left(\inf_{u,v,w\in A\bigcup B} F_{uvw}^{(x)}, \inf_{u,v,w\in A\bigcup C} F_{uvw}^{(y)}, \inf_{u,v,w\in B\bigcup C} F_{uvw}^{(z)}\right).$$

The same argument works if infimum $F_{uvw}^{(x+y+z)}$ taken over u, v, w in $A \bigcup B \bigcup C$ is attained with all the variables u, v, w in B or in C.

Now if infimum $F_{uvw}^{(x+y+z)}$, u, v, w ranges over $A \bigcup B \bigcup C$ is neither attained with u in A, v in B and win C nor with all the variables u, v, w in one of the three sets A, B, C then it must be attained with any two of u, v, w in one of the sets A, B, C and remaining one variable in any one of the remaining two sets.

Let, $\inf\limits_{u,v,w\in A\bigcup B\bigcup C} F_{uvw}^{(x+y+z)} = \inf\limits_{\substack{u,v\in A\\ w\in B}} F_{uvw}^{(x+y+z)}$.

Then $\inf\limits_{u,v,w\in A\bigcup B\bigcup C} F_{uvw}^{(x+y+z)} \geq T\left(\inf\limits_{\substack{u,v\in A\\ w\in B}} F_{uvw}^{(x)}, H(y), H(z)\right)$

$$\geq T\left(\inf_{u,v,w\in A\bigcup B} F_{uvw}^{(x)}, \inf_{u,v,w\in A\bigcup C} F_{uvw}^{(y)}, \inf_{u,v,w\in B\bigcup C} F_{uvw}^{(z)}\right).$$

The same argument works for other combinations. Finally using the fact that the cuboid $\{(p, q, r): o \leq p \leq x, o \leq q \leq y, o \leq r \leq z\}$ is contained in the tetrahedron $\{(p, q, r) : p, q, r \geq o, p+q+r < x+y+z\}$.

The inequality (1.2) and continuity of T gives

$$D_{A\bigcup B\bigcup C}^{(x+y+z)} = \sup_{p+q+r<x+y+z}\left(\inf_{u,v,w\in A\bigcup B\bigcup C} F_{uvw}^{(p+q+r)}\right)$$

$$\geq \sup_{\substack{p<x\\ q<y\\ r<z}}\left(\inf_{u,v,w\in A\bigcup B\bigcup C} F_{uvw}^{(p+q+r)}\right)$$

$$\geq T\left(\sup_{p<x}\left(\inf_{u,v,w\in A\bigcup B} F_{uvw}^{(p)}\right), \sup_{q<y}\left(\inf_{u,v,w\in A\bigcup C} F_{uvw}^{(q)}\right), \sup_{r<z}\left(\inf_{u,v,w\in B\bigcup C} F_{uvw}^{(r)}\right)\right)$$

$$= T\left(D_{A\bigcup B}^{(x)}, D_{A\bigcup C}^{(y)}, D_{B\bigcup C}^{(z)}\right).$$

Definition 2–Let A, B, C be non-empty subsets of X.

The probabilistic area of A, B, C is the function F_{ABC} defined by

$$F_{ABC}^{(x)} = \sup_{t<x}\left(T\left(\inf_{p\in A}\left(\inf_{q\in B}\left(\sup_{r\in C} F_{pqr}^{(t)}\right)\right), \inf_{q\in B}\left(\inf_{r\in C}\left(\sup_{p\in A} F_{pqr}^{(t)}\right)\right), \inf_{r\in C}\left(\inf_{p\in A}\left(\sup_{q\in B} F_{pqr}^{(t)}\right)\right)\right)\right).$$

We omit the proofs of the following properties.

Theorem 5. F_{ABC} is a distribution function.

Theorem 6. If A, B, C are non-empty subsets of X, then, F_{ABC} is invariant under all permutations of A, B, C.

Theorem 7. If A, B, C and D are non-empty subsets of X then for any x, y and z

$$F_{ABC}^{(x+y+z)} \geq T\left(F_{ABD}^{(x)}, F_{ADC}^{(y)}, F_{DBC}^{(z)}\right)$$

Proof. Let u, v, w be given. Then for any quartrets of points p, q, r and s in X we have $F_{pqr}^{(u+v+w)} \geq T\left(F_{pqs}^{(u)}, F_{psr}^{(v)}, F_{sqr}^{(w)}\right)$.

Since T is continuous and monotonic

$$\sup_{r\in C} F_{pqr}^{(u+v+w)} \geq T\left(\sup_{s\in D} F_{pqs}^{(u)}, \inf_{s\in D}\left(\sup_{r\in C} F_{psr}^{(v)}\right), \left(\inf_{s\in D}\left(\sup_{r\in C} F_{sqr}^{(w)}\right)\right)\right).$$

Consequently

$$\inf_{p\in A}\left(\inf_{q\in B}\left(\sup_{r\in C} F_{pqr}^{(u+v+w)}\right)\right) \geq T\left(\inf_{p\in A}\left(\inf_{q\in B}\left(\sup_{r\in C} F_{pqs}^{(u)}\right)\right)\right.$$

$$\left.\inf_{p\in A}\left(\inf_{s\in D}\left(\sup_{r\in C} F_{psr}^{(v)}\right)\right), \inf_{s\in D}\left(\inf_{q\in B}\left(\sup_{r\in C} F_{sqr}^{(w)}\right)\right)\right).$$

Similarly

$$\inf_{p\in A}\left(\inf_{r\in C}\left(\sup_{q\in B} F_{pqr}^{(u+v+w)}\right)\right) \geq T\left(\inf_{p\in A}\left(\inf_{s\in D}\left(\sup_{q\in B} F_{pqs}^{(u)}\right)\right),\right.$$

$$\left.\inf_{p\in A}\left(\inf_{r\in C}\left(\sup_{s\in D} F_{psr}^{(v)}\right)\right), \inf_{s\in D}\left(\inf_{r\in C}\left(\sup_{q\in B} F_{sqr}^{(w)}\right)\right)\right),$$

and

$$\inf_{q\in B}\left(\inf_{r\in C}\left(\sup_{p\in A} F_{pqr}^{(u+v+w)}\right)\right) \geq T\left(\inf_{q\in B}\left(\inf_{s\in D}\left(\sup_{p\in A} F_{pqs}^{(u)}\right)\right)\right.$$

$$\left.\inf_{s\in D}\left(\inf_{r\in C}\left(\sup_{p\in A} F_{psr}^{(v)}\right)\right), \inf_{q\in B}\left(\inf_{r\in C}\left(\sup_{s\in D} F_{sqr}^{(w)}\right)\right)\right).$$

Therefore, since T is associative, we have

$$T\left(\inf_{p\in A}\left(\inf_{q\in B}\left(\sup_{r\in C} F_{pqr}^{(u+v+w)}\right)\right), \inf_{p\in A}\left(\inf_{r\in C}\left(\sup_{q\in B} F_{pqr}^{(u+v+w)}\right)\right),\right.$$

$$\inf_{q\in B}\left(\inf_{p\in A}\left(\sup_{r\in C}F_{pqr}^{(u+v+w)}\right)\right)\geq T\left(T\left(\inf_{p\in A}\left(\inf_{q\in B}\left(\sup_{s\in D}F_{pqs}^{(u)}\right)\right),\right.\right.$$

$$\left.\inf_{p\in A}\left(\inf_{s\in D}\left(\sup_{q\in B}F_{pqs}^{(u)}\right)\right),\inf_{q\in B}\left(\inf_{s\in D}\left(\sup_{p\in A}F_{pqs}^{(u)}\right)\right)\right),$$

$$T\left(\inf_{p\in A}\left(\inf_{s\in D}\left(\sup_{r\in C}F_{psr}^{(v)}\right)\right),\inf_{p\in A}\left(\inf_{r\in C}\left(\sup_{s\in D}F_{psr}^{(v)}\right)\right),\right.$$

$$\left.\inf_{s\in D}\left(\inf_{r\in C}\left(\sup_{p\in A}F_{psr}^{(v)}\right)\right)\right),T\left(\inf_{s\in D}\left(\inf_{q\in B}\left(\sup_{r\in C}F_{sqr}^{(w)}\right)\right),\right.$$

$$\left.\left.\inf_{s\in D}\left(\inf_{r\in C}\left(\sup_{q\in B}F_{sqr}^{(w)}\right)\right),\inf_{q\in B}\left(\inf_{r\in C}\left(\sup_{s\in D}F_{sqr}^{(w)}\right)\right)\right)\right).$$

Now arguing as in the last step of the proof of theorem 4 we have

$$F_{ABC}^{(x+y+z)}=\sup_{u+v+w<x+y+z}T\left(\inf_{p\in A}\left(\inf_{q\in B}\left(\sup_{r\in C}F_{pqr}^{(u+v+w)}\right)\right)\right.$$

$$\left.\inf_{p\in A}\left(\inf_{r\in C}\left(\sup_{q\in B}F_{pqr}^{(u+v+w)}\right)\right),\inf_{q\in B}\left(\inf_{r\in C}\left(\sup_{p\in A}F_{pqr}^{(u+v+w)}\right)\right)\right),$$

$$\geq\sup_{\substack{u<x\\ v<y\\ w<z}}T\left(\inf_{p\in A}\left(\inf_{q\in B}\left(\sup_{r\in C}F_{pqr}^{(u+v+w)}\right)\right)\right.$$

$$\left.\inf_{p\in A}\left(\inf_{r\in C}\left(\sup_{q\in B}F_{pqr}^{(u+v+w)}\right)\right),\inf_{q\in B}\left(\inf_{r\in C}\left(\sup_{p\in A}F_{pqr}^{(u+v+w)}\right)\right)\right).$$

$$T\left\{\sup_{u<x}T\left(\inf_{p\in A}\left(\inf_{q\in B}\left(\sup_{s\in D}F_{pqs}^{(u)}\right)\right)\right),\inf_{p\in A}\left(\inf_{s\in D}\left(\sup_{q\in B}F_{pqs}^{(u)}\right)\right),\right.$$

$$\inf_{q\in B}\left(\inf_{s\in D}\left(\sup_{p\in A}F_{pqs}^{(u)}\right)\right),\sup_{v<y}T\left(\inf_{p\in A}\left(\inf_{s\in D}\left(\sup_{r\in C}F_{psr}^{(v)}\right)\right),\right.$$

$$\left.\inf_{p\in A}\left(\inf_{r\in C}\left(\sup_{s\in D}F_{psr}^{(v)}\right)\right),\inf_{s\in D}\left(\inf_{r\in C}\left(\sup_{p\in A}F_{psr}^{(v)}\right)\right)\right),$$

$$\sup_{w<z} T\left(\inf_{s\in D}\left(\inf_{q\in B}\left(\sup_{r\in C} F_{sqr}^{(w)} \right)\right), \inf_{s\in D}\left(\inf_{r\in C}\left(\sup_{q\in B} F_{sqr}^{(w)} \right)\right),\right.$$

$$\left. \inf_{q\in B}\left(\inf_{r\in C}\left(\sup_{s\in D} F_{sqr}^{(w)} \right)\right)\right\}.$$

$$= T\left(F_{ABD}^{(x)}, F_{ADC}^{(y)}, F_{DBC}^{(z)} \right).$$

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