

Quantum Mechanics of Dissipative Systems*

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Abstract

It is well-known that there is no Hamiltonian mechanics to describe a dissipative system either in Classical or Quantum mechanics. Probabilistic models are introduced to describe the effect of the environment that leads to dissipation. It is argued in the context of such models that one is forced to use non-commutative stochastic processes to give a satisfactory description.

Introduction

Most of us are familiar with the standard theories in classical and Quantum Mechanics which are primarily designed to deal with a conservative Hamiltonian system. Non-conservative or dissipative systems have also been dealt with in literature¹, though in a somewhat phenomenological spirit. From a physical point of view, it is clear that dissipative motion of a system must result from its interaction with its environment (or heat-bath), the dynamics of which we may not be interested in. But if we look at the total system viz. the system and its environment, the dynamics is again expected to be conservative. In other words, the dissipative motion is believed to be the direct consequence of our inability to observe or of the lack of interest in the motion of the total system. Here we first briefly study a classical stochastic model for the damped harmonic oscillator and then the quantum stochastic theory of dissipative system under the influence of a potential.

2. Classical damped harmonic oscillator

If we choose the constants like the mass of the particle and the spring constant suitably, then the equation of motion of interest here is:

$$\frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + \nu^2 x = 0, \quad (1)$$

where $\alpha > 0$ is the friction coefficient and ν is the frequency of the oscillator. We rewrite (1) as a pair of first order equations in analogy with the Hamilton's equation (with $\delta^2 = \nu^2 - \alpha^2 > 0$):

$$\frac{dx}{dt} = p - \alpha x, \quad \frac{dp}{dt} = -\delta^2 x - \alpha p, \quad (2)$$

a more symmetric version of which is

$$\frac{d\tilde{x}}{dt} = \tilde{\delta}\tilde{p} - \alpha\tilde{x}, \quad \frac{d\tilde{p}}{dt} = -\tilde{\delta}\tilde{x} - \alpha\tilde{p}, \quad (2')$$

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where we have set $\tilde{x} = x$ and $\tilde{p} = \delta^{-1}p$. Clearly equations (2) or (2') cannot be obtained as Hamiltonian equation for the pair (x, p) or (\tilde{x}, \tilde{p}) for any real Hamiltonian. Also note that if we solve any of these equations and compute the Poisson bracket between $x(t)$ and $p(t)$, we get $\{x(t), p(t)\} = -e^{-2\alpha t}$ even though $\{x(0), p(0)\} = -1$.

Next we want to consider a model of evolution where the particle is coupled to classical Brownian motion $\omega(t)$ representing the environment in which the particle moves. In terms of the variables (\tilde{x}, \tilde{p}) this is given by the stochastic differential equations.

$$\left. \begin{aligned} d\tilde{x} &= (2\alpha)^{(1/2)} \tilde{p} d\omega + (-\alpha\tilde{x} + \delta\tilde{p}) dt \\ d\tilde{p} &= -(2\alpha)^{(1/2)} \tilde{x} d\omega + (-\alpha\tilde{p} - \delta\tilde{x}) dt. \end{aligned} \right\} \quad (3)$$

Rewriting equation (3) in matrix-form with $\xi = \begin{pmatrix} \tilde{x} \\ \tilde{p} \end{pmatrix}$, $A = \begin{pmatrix} -\alpha & \delta \\ -\delta & -\alpha \end{pmatrix} = -\alpha I + i\delta\sigma$, $\sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ we have:

$$d\xi = A\xi dt + i(2\alpha)^{(1/2)} \sigma \xi d\omega. \quad (3')$$

This matrix equation (3') can be explicitly solved:

$$\xi(t) = \exp\left\{i\left[(2\alpha)^{(1/2)} \omega(t) + \delta t\right]\sigma\right\} \xi(0) \text{ or} \quad (4)$$

$$\left. \begin{aligned} x(t) &= x(0) \cos\left(\delta t + (2\alpha)^{(1/2)} \omega(t)\right) + \delta^{-1} p(0) \sin\left(\delta t + (2\alpha)^{(1/2)} \omega(t)\right) \\ p(t) &= p(0) \cos\left(\delta t + (2\alpha)^{(1/2)} \omega(t)\right) - \delta x(0) \sin\left(\delta t + (2\alpha)^{(1/2)} \omega(t)\right) \end{aligned} \right\} \quad (4')$$

Two conclusions are immediate from (4) and (4') (i) $\{x(t), p(t)\} = \{x(0), p(0)\} = -1$ for all t and almost all ω . In fact, we see from (4') that $(x(0), p(0)) \mapsto (x(t), p(t))$ is a symplectic flow. (ii) If we denote by $\langle \cdot \rangle$ the expectation w.r.t. the Brownian motion, then we get from (3) that $\frac{d\langle x \rangle}{dt} = -\alpha \langle x \rangle + \langle p \rangle$, $\frac{d\langle p \rangle}{dt} = -\alpha \langle p \rangle - \delta^2 \langle x \rangle$ which is identical to (2). Thus the expectation of the observables, in which all the effects of the environment is washed out, satisfy exactly the equation of motion of the damped harmonic oscillator though the total evolution of the same sets of observables in interaction with its environment remains symplectic.

For a system whose conservative version is governed by a more general Hamiltonian $H = \frac{1}{2}p^2 + V(x)$, the same procedure leads to some difficulties. Here the equations (1) and (2) are replaced respectively by

$$\frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + V'(x) = 0 \text{ and} \quad (5)$$

$$\frac{dx}{dt} = \{H, x\} - \alpha x, \quad \frac{dp}{dt} = \{H, p\} - \alpha p + \alpha^2 x. \quad (6)$$

One may try to include the extra term $\alpha^2 x$ in (6) in the Hamiltonian by writing a new Hamiltonian $H' = \frac{1}{2}p^2 + V(x) - \frac{1}{2}\alpha^2 x^2$ so that (6) becomes

$$\frac{dx}{dt} = \{H', x\} - \alpha x, \quad \frac{dp}{dt} = \{H', p\} - \alpha p, \quad (7)$$

which can be dilated to (as in (3)):

$$\begin{aligned} dx &= (2\alpha)^{(1/2)} p d\omega + \{-\alpha x + \{H', x\}\} dt \\ dp &= -(2\alpha)^{(1/2)} x d\omega + \{-\alpha p + \{H', p\}\} dt \end{aligned} \quad (8)$$

Ito's formula implies that $d\{x, p\}_t = 0$ and hence $\{x, p\}_t = -1 \forall t$. However, a closer examination of the Hamiltonian H' shows that it need not be a physical Hamiltonian for most physical potentials V since it may not be bounded below. A remedy to this situation will be given in the next section in the context of quantum mechanics.

3. Dissipative Quantum Systems

We shall consider only a single quantal particle in one space dimension, the conservative version of whose motion is governed by a general Hamiltonian of the form H given in the last paragraph of the last section. If we take the route of modifying the Hamiltonian to H' , then H' , though essentially self-adjoint³ on $C_0^\infty(IR)$, is not bounded below in general.

For the stochastic description, we need to use the theory of quantum stochastic differential equations (q.s.d.e), an introduction to which may be found in [4,5,6]. The Hilbert space of the states of the total system is $\mathcal{H} = L^2(IR) \otimes \Gamma(L^2(IR_+, \mathcal{C}^2))$, where $L^2(IR)$ is the usual state space of a 1-dim quantal particle, IR_+ stands for time and Γ for bosonic second quantization. It may be useful to mention here that the symmetric Fock space $\Gamma(L^2(IR_+, \mathcal{C}^2))$ is unitarily isomorphic to $L^2(IP_1 \times IP_2)$, the space of square-integrable functionals of two independent standard Brownian motions. Here let $A_j(t)$ and $A_j^\dagger(t)$ ($j=1,2$) be the pair of annihilation and creation operators respectively, and let us consider the q.s.d.e.:

$$dU(t) = U(t) \left[\sum_{j=1}^2 \left\{ R_j^* dA_j - R_j dA_j^\dagger - \frac{1}{2} R_j^* R_j dt \right\} + iH dt \right] \quad (9)$$

with initial value $U(0) = I$, and $R_1 = \frac{1}{2}(2\alpha)^{(1/2)}(x^2 + p^2) \equiv (2\alpha)^{(1/2)}N$. (N is the self-adjoint number operator in $L^2(IR)$), $R_2 =$ the unitary multiplication operator by $\exp(\frac{1}{2}\alpha^2 x^2)$ in $L^2(IR)$, and H is the self-adjoint Hamiltonian operator for a wide class of potentials³. Note that one of the operator coefficients R_1 is unbounded though self-adjoint and the theory of such q.s.d.e. has been dealt with in [6] and references therein. It can be shown that equation (9) has a unique unitary solution in \mathcal{H} . It is worth remarking that when the friction coefficient (in this model the coefficient of coupling between the system and the stochastic process) α is set equal to 0, $U(t)$ becomes equal to $e^{iHt} \exp(i\omega_2(t))$, which is a trivial random phase-change from the usual Schrödinger evolution. Also the solution of (9) leads to an unitary evolution (or a propagator) which is not a group (when $\alpha = 0$ the solution $e^{iHt} e^{i\omega_2(t)}$ is not a group but a projective group).

Next we want to look at the evolution of the observables position and momentum in the Heisenberg picture. For a bounded observable $B \in \mathcal{B}(L^2(\mathbb{R}))$, we set $j_t(B) \equiv U(t)(B \otimes I)U(t)^*$ and find that $j_t(B)$ satisfies formally the q.s.d.e. :

$$dj_t(B) = \sum_{j=1}^2 \left\{ j_t([R_j^*, B])dA_j - j_t([R_j, B]dA_j^+) \right\} + j_t(\mathcal{L}(B))dt, \tag{10}$$

where

$$\mathcal{L}(B) = \sum_{j=1}^2 \left(R_j^* B R_j - \frac{1}{2} R_j^* R_j B - \frac{1}{2} B R_j^* R_j \right) + i[H, B] = -\alpha[N, [N, B]] + (R_2^* B R_2 - B) + i[H, B]. \tag{11}$$

If we now want to study the effect of the stochastic coupling on the system only, we have to average out the stochastic variables, i.e. take the expectation $\langle \cdot \rangle$ with respect to the vacuum in the Fock space $\Gamma(L^2(\mathbb{R}_+, \mathcal{C}^2))$. Setting $B(t) = \langle j_t(B) \rangle$, we get

$$\frac{dB(t)}{dt} = \mathcal{L}(B)(t) \text{ or equivalent } B(t) = e^{\mathcal{L}t}(B) \tag{12}$$

To study the equations of evolution of interesting (but unbounded) observables like x and p , either one writes (12) in these cases formally or one assumes that the potential V is sufficiently smooth (as in the classical situation). In the later case, one can compute $\mathcal{L}(x)$ or $\mathcal{L}(p)$ on $C_0^\infty(\mathbb{R})$. Either way one gets:

$$\frac{dx(t)}{dt} = -\alpha x + p, \quad \frac{dp}{dt} = -\alpha p + \alpha^2 x - v'(x), \tag{13}$$

the same equation as (6) or equivalently (5).

Discussion

(1) One could have worked with the unphysical Hamiltonian H' (which incidentally becomes H as $\alpha \rightarrow 0$) in which case the stochastic coupling in the quantum case could have been successfully achieved with only one copy of classical Brownian motion and there would be no need to introduce quantum Brownian motion.

(2) It is probably evident that the stochastic mechanism employed here to obtain a frictional or dissipative evolution is quite different in spirit to the conventional Langevin equation or to the independent oscillator model of Ford-Lewis-O'Connell⁷. In a suitable limit of the independent oscillator model, one gets the quantum Langevin equation:

$$\frac{dx}{dt} = p, \quad \frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + V'(x) = F(t), \tag{4}$$

where $F(t) = \sigma \dot{\omega}(t)$. For simplicity of discussion, let us assume $V = 0$ in which case (14) can be explicitly solved as $p(t) = pc^{-2\alpha t} + \sigma \int_0^t e^{2\alpha(s-t)} d\omega(s)$, and $x(t) = x(0) + \int_0^t p(s)ds$. If one now computes the commutator bracket at time t , one finds that $[x(t), p(t)] = e^{-2\alpha t}[x(0), p(0)] = ie^{-2\alpha t}$.

Thus if one believes that by coupling the system to its environment and by looking at the total evolution the conservativity should be restored, then that belief is certainly not borne out in the usual Langevin-type model.

(3) A similar exercise can be carried out if velocity-dependent forces are present, i.e. for gyroscopic system (*e.g.* in presence of a Lorentz force) as has been done in Li-Ford-O'Connell⁸ in the context of conventional Langevin-type equation. For static electric field, static and uniform magnetic field, and with no source current, one can essentially repeat the calculations of section 3 with four quantum stochastic processes and obtain the damped equation of motion for the expectation of observables with Lorentz force present. For a more general situation, the operator R_2 in section 3 has to be replaced by multiplication by $\exp(\frac{1}{2}\alpha^2|x|^2 + \varphi(x))$, where the function φ is the solution of a certain differential equation.

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