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A new perturbation solution to the scalar wave equation

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Abstract

In this study, a perturbation series solution to the scalar wave equation has been presented, solving for the total fields propagating through an arbitrary refractive index distribution. Results are presented for the case of straight step-index integrated optics waveguides for modal excitation. Equivalence of the presented solutions to those derived from the Green's function formalism of solving the scalar wave equation for some special cases has been shown.

Keywords: Perturbation solutions, optical waveguides.

1. Introduction

Electromagnetic wave propagation through source-free isotropic dielectric media is modelled in the scalar approximation by the scalar wave equation. It serves as a fairly accurate model for electromagnetic wave propagation for many classes of practical problems encountered in science and engineering.

We have presented a perturbation-series solution (PSS)[1,2], in closed series form, to the scalar wave equation for the total fields propagating through arbitrary refractive index variations. The first-order solution has been explicitly shown and basic validation of the theory has been performed for the cases of straight integrated optics (IO) waveguides. Further validation of the above developed perturbation solution has been carried out by demonstrating its equivalence to the Born approximation to the solution of the scalar wave equation.

Longitudinally varying waveguides form a very important class of optical waveguides. Previous research in the analysis of such waveguiding structures has resulted in the development of computational methods such as the beam propagation method (BPM)[3], the generalized propagation techniques (GPTs)[4], collocation methods[5], or analytical methods like path integration[6], all of which solve the paraxial approximation to the scalar wave equation. Another analytical approach for IO tapered waveguides has been the local normal mode theory[2,7], which, however, is only viable for waveguides with known modal solutions.

The present approach differs from these previous approaches to the problem in the following ways:

1. The scalar wave equation itself is solved, not its paraxial/extended Fresnel approximation.

2. There is no conceptual difficulty of non-commutation of operators in the propagator as in the case of the BPM or the GPTs, and as a result we can write out a closed series form solution, which is not possible in the above methods.

The outline of the paper is as follows. The form of solutions obtained for the total fields is given in Section 2. In Section 3, the PSS is applied to the problem of evaluation of fields propagating through IO waveguides. Further analytical validation of the PSS is presented in Section 4. Conclusions are presented in Section 5.

2. Form of solutions obtained for the total fields

The scalar wave equation (also called the Helmholtz equation) in a source-free, isotropic medium is

$$\nabla^2 \Psi + k_0^2 n^2(x,y,z)\Psi = 0 \tag{1}$$

where ∇^2 is the Laplacian operator, $\Psi(x, y, z)$ is the complex amplitude of a monochromatic wave propagating through a medium characterised by a refractive index, $n(x, y, z)$, and k_0 is the free space wave number.

The refractive index can be written as

$$n^2(x,y,z) = n_0^2 + \Delta n^2(x,y,z) \tag{2}$$

where n_0 is the ambient refractive index, and $\Delta n(x, y, z)$ represents the spatially dependent part of the refractive index.

Thus the complex amplitude of the propagating monochromatic wave at any space point, called the total field, can be considered to be the sum of two components, an incident component and a scattered component corresponding to the propagation of the incident field through the ambient medium and through the space-dependent part of the refractive index, respectively.

If $\Psi(x, y, z)$ denotes the total field, and defining $\Phi(x, y, z)$ as, $\Phi(x, y, z) \equiv \frac{\partial \Psi}{\partial z}$, we can write,

$$\begin{pmatrix} \Psi(x,y,z) \\ \Phi(x,y,z) \end{pmatrix} = U_1 \begin{pmatrix} \Psi(x,y,z=0) \\ \Phi(x,y,z=0) \end{pmatrix} \tag{3}$$

where U_1 is an operator representing the propagator for the fields and is split as

$$U_1 = U_0.P \tag{4}$$

where U_0 is the propagator for the reference z-independent problem (*i.e.*, propagation of the fields through the ambient refractive index), and is given by[2]

$$U_0 = e^{S_0 z} \tag{5}$$

where

$$S_0 = \begin{pmatrix} 0 & 1 \\ -H_0 & 0 \end{pmatrix} \tag{6}$$

with $H_0 = \nabla_T^2 + k_0^2 n_0^2$, where ∇_T^2 is the transverse Laplacian.

The operator P represents a perturbation to U_0 and has been found to be[2],

$$P = \left(e^{\int_0^z Q(z')dz'} \right)_+ \tag{7}$$

where + denotes a space ordered product, and

$$Q = U_0^{-1} S_1 U_0 \tag{8}$$

where again

$$S_1 = \begin{pmatrix} 0 & 0 \\ -H_1 & 0 \end{pmatrix} \tag{9}$$

where $H_1 = k_0^2\ \Delta n^2(x, y, z)$.

Denoting operator $\cos H_0^{1/2} z$ by A, and operator $H_0^{-1/2}\ \sin H_0^{1/2} z$ by B, operator Q is found to be[2]

$$Q = \begin{pmatrix} B\,H_1\,A & B\,H_1\,B \\ -A\,H_1 A & -A\,H_1\,B \end{pmatrix}. \tag{10}$$

When $\Delta n^2(x, y, z) \ll n_0^2$, considering a first-order evaluation of the operator P, we get

$$P = P_0 + P_1 \tag{11}$$

where $P_0 = I$, the identity operator, and,

$$P_1 = \int_0^z Q(z')dz'.$$

It is seen that all the terms composing the propagator are series of integral powers of the operator H_0. Hence, in order to evaluate eqn(1) analytically, we decompose the functions Ψ and Φ in terms of the eigenfunction basis of H_0, *i.e.*, plane waves in Cartesian coordinates.

3. Application of the PSS to integrated optics

The validity of the above-presented perturbation solution to the Helmholtz equation is demonstrated by applying it to the problem of evaluation of the total fields propagating through an integrated optics waveguiding structure.

Considering the case of planar waveguides, designating the x-axis as the transverse coordinate and the z-axis as the longitudinal coordinate along the axis of the waveguide, the field solutions at the output section $z = z_{out}$ are obtained on evaluating eqn (3) as

$$\Psi(x, z_{out}) = \Psi_0(x, z_{out}) + \Psi_1(x, z_{out}) \tag{12}$$

where $\Psi_0(x, z_{out})$ is the incident component of the total field at the output line $z = z_{out}$ and is given by

$$\Psi_0(x, z_{out}) = \int_u f(u, z_{out}) e^{j2\pi ux} du \quad (13)$$

where

1. $f(u, z) = g_1(u, z = 0) \cos(r z) + g_2(u, z = 0) \sin(r z)/r$.

2. $-k_0 n_0 < 2\pi u < k_0 n_0$, u being a spatial frequency coordinate corresponding to the transverse coordinate, and $r = \sqrt{k_0^2 n_0^2 - 4\pi^2 u^2}$.

3. $g_1(u, z = 0)$ and $g_2(u, z = 0)$ are Fourier transforms of the total field and its z-derivative, respectively, with respect to the transverse x-coordinate at the input section, $z = 0$, of the waveguide.

The field $\Psi_1(x, z_{out})$ is the scattered component of the total field, and is given by

$$\Psi_1(x, z_{out}) = \int_{z,\upsilon,u} (-\sin(q(z_{out} - z))/q)\, g(v - u; z) e^{j2\pi \upsilon x} f(u, z) du dv dz \quad (14)$$

where

1. $0 \le z \le z_{out}$, where $z = 0$ and $z = z_{out}$ are the input and output cross-sections, respectively, of the waveguide section considered.

2. $-k_0 n_0 < 2\pi v < k_0 n_0$, v being a spatial frequency coordinate corresponding to the transverse coordinate.

3. $q = \sqrt{k_0^2 n_0^2 - 4\pi^2 v^2}$.

4. $g(u, z) = \mathcal{F}(\Delta n^2(x, z): x \to u)$, $\mathcal{F}$ representing the Fourier transform operator.

The first-order solutions for the total fields are found to have nice computational properties in that they reduce to the forms of a convolution and an inverse Fourier transform along with a z-integral so that we can utilize the rapid evaluation properties of the FFT algorithm.

The above solutions for the total fields are validated for the case of straight step-index IO waveguides with fundamental mode excitation through comparison with exact modal solutions.

The plots of electric field intensity, E *vs* transverse coordinate x (whose origin is at the centre of the waveguide) are shown in Figs 1–4, that compare the PSS (solid curve) with the modal solution (dotted curve) for different refractive index profiles and various propagation distances. Here n_f and n_s are film and substrate refractive indices, respectively, d is the width of the waveguide, λ, the wavelength of light used, z_{out}, the propagation distance.

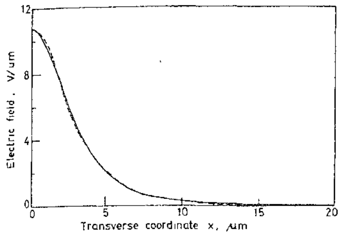

FIG 1. Plot of E vs x. $n_f = 1.503$, $n_s = 1.500$, $d = 4$ μm, $\lambda = 1.0$ μm propagation distance, $z_{out} = 10$ μm

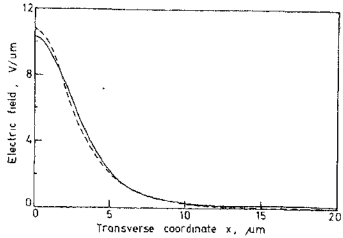

FIG. 2. Plot of E vs x. $n_f = 1.503$, $n_s = 1.500$, $d = 4$ μm, $\lambda = 1.0$ μm propagation distance, $z_{out} = 20$ μm.

It is seen from the figures that the agreement of the PSS with the exact modal solution is quite good, the correspondence being better when $(n_f - n_s)$ is smaller than when larger, which is due to the perturbation nature of the solution to the problem.

4. Further analytical validation of the PSS

The above demonstrated first-order perturbation solution has been further validated analytically in the following ways.

1. It is shown that the closed-form solution for scattered fields obtained from it for the special case of a constant ambient refractive index and scattering from an object due to plane wave incidence is the same as the solution (called the Fourier diffraction theorem (FDT)) obtained using the Born or Rytov approximations to the scalar wave equation.

2. In addition, it is shown that for the case of constant ambient refractive index, the Green's function formalism of solving the scalar wave equation is the same as the above developed perturbation solution.

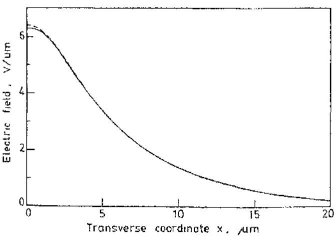

FIG. 3 Plot of E vs x. $n_f = 2.2005$, $n_s = 2.2$, $d = 6$ μm, $\lambda = 1.0$ μm propagation distance, $z_{out} = 10$ μm.

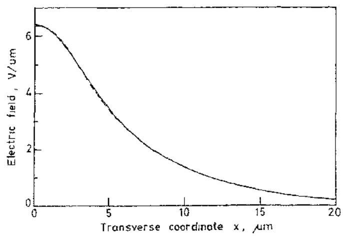

FIG 4. Plot of E vs x. $n_f = 2.2005$, $n_s = 2.2$ $d = 6$ μm, $\lambda = 1.0$ μm propagation distance, $z_{out} = 20$ μm.

The above validations are carried out for the two-dimensional scattering problem, the corresponding three-dimensional versions being their straightforward extensions.

4.1. *Obtaining the Fourier diffraction theorem from the first-order perturbation solution developed*

The scattered component of the total field, $\Psi_1(x, z)$, when only solutions in terms of plane waves propagating in the positive z-direction are considered, can be written from eqn (14) (considering $z = z_{in}$ to be the input cross-section) as

$$\Psi_1(x,z) = (j/2q)\int_{z,v,u} e^{q(z_{out}-(z-z_{in}))} g(v-u; z+z_{in}) e^{j2\pi v x} f(u,z)\,du\,dv\,dz \tag{15}$$

where the terms of the integrand and the limits of integration have been defined in the previous section.

Define the Fourier transform of the scattered field, $\Psi_1(x, z)$ with respect to the transverse coordinate, x, as

$$\Gamma(v, z) \equiv \mathcal{F}(\Psi_1(x, z) : x \to v) = \int_{-\infty}^{\infty} \Psi_1(x, z) e^{-j2\pi v x}\, dx.$$

Assuming incident illumination to be a plane wave travelling along the z-axis, *i.e.*, $\Psi(x, z_{in}) = e^{jk_0 n_0 z_{in}}, z_{in} = -L/2$, we obtain $\Gamma(v, z_{out} = L/2)$ to be,

$$\Gamma(v, L/2) = \int_{-\infty}^{\infty} N(v,w) K(v, w, L/2) dw \tag{16}$$

where

1. $N(v, w) = \mathcal{F}(\Delta n^2(x, z) : x, z \to v, w)$, where $\mathcal{F}$ represents a Fourier transform operator and (v, w) are the spatial frequency coordinates corresponding to the spatial coordinates (x, z); $-k_0 n_0 < 2\pi v < k_0 n_0$.

2. $K(v, w, L/2) = k_0^2 (j/2q) e^{jq(L/2)} F(w - B)$
where

$$F(w) = L\ \mathrm{sinc}(wL),$$

$$q = \sqrt{k_0^2 n_0^2 - 4\pi v^2},$$

and

$$B = (q - k_0 n_0)/(2\pi).$$

When the object is entirely contained in $(-L/2, L/2)$, it follows that,

$$\Gamma(v, L/2) = k_0^2 \left(\frac{j}{2q}\right) e^{jq(L/2)} N(v, B). \tag{17}$$

This relation that relates the one-dimensional Fourier transform of the forward scattered field at a spatial frequency to a point on a semi-circular arc in the Fourier transform of the object refractive index is called the Fourier diffraction theorem, and is also obtained from the Born or Rytov solutions to the scalar wave equation for scattered fields[8,9].

4.2. Obtaining the first-order perturbation solution from the first-order Born approximation

The scattered component, $\Psi_1(x, y, z)$, of the total field, $\Psi(x, y, z)$, is a solution of the equation,

$$(\nabla^2 + k_0^2)\Psi_1 = -k_0^2 \Delta n^2(x, y, z)\Psi \tag{18}$$

where the ambient refractive index is considered as unity without any loss of generality.

In two dimensions, the first-order Born approximation solution to the above equation[9] is

$$\Psi_b(x, z) = \int_{x', z'} G(x, z / x', z') H_1(x', z') \Psi_0(x', z') dx'dz' \tag{19}$$

where

1. $G(x, z/x', z')$ is the free-space Green's function corresponding to the two-dimensional scalar wave equation, and is given by

$$G(x, z / x', z') = G(x - x', z - z') = (j / 4) H_0^{(1)}(k_0 R) \tag{20}$$

where $H_0^{(1)}$ denotes the Hankel function of the zeroth order and the first kind, and $R = \sqrt{(x - x')^2 + (z - z')^2}$.

2. $H_1(x, z) = k_0^2 \Delta n^2(x, z)$.

3. $\Psi_0(x, z)$ is the total field that would have existed at (x, z) if the refractive index inhomogeniety were absent.

4. The integration is over the entire two-dimensional space.

Consider a field propagating towards increasing z. For a given z, considering the field only due to sources (inhomogeneities) in the region $z' < z$, we can write the first-order Born scattered field as

$$\Psi_b(x, z) = \int_{z'=0}^{z} \int_{x'=-\infty}^{\infty} G(x, z / x', z') H_1(x', z') \Psi_0(x', z') dx'dz' \tag{21}$$

where the lower limit for z' has been chosen without any loss of generality.

The two-dimensional free-space Green's function has the following plane wave decomposition for $z > z'$[9]

$$G(x - x', z - z') = \int_{\nu} (j / 2q) e^{j2\pi[\nu(x - x') + (q/2\pi)(z - z')]} d\nu \tag{22}$$

where

1. $-k_0 < 2\pi\nu < k_0$, if the contribution from the evanascent waves is neglected, and
2. $q = \sqrt{k_0^2 - 4\pi^2 \nu^2}$.

Defining $\Gamma_b\ (\nu,\ z) \equiv \mathcal{F}(\Psi_b(x,\ z)\ :\ x \to \nu)$, we have, on substituting the above plane wave expansion into eqn (21), and recalling from Section 3 the definition of $g(u, z)$ and suitably interpreting $f(u, z)$ as $\mathcal{F}(\Psi_0(x, z); x \to u)$, we have,

$$\Gamma_b(\nu, z) = (j/2q)\int_{z', u} e^{q(z-z')} g(\nu - u; z')dudz' \tag{23}$$

where $0 \le z' \le z$ and $-k_0 < 2\pi u < k_0$.

The above equation can be seen to be the same as the transverse Fourier transform of eqn (15) when the substitutions $z_{in} = 0$, $n_0 = 1.0$ are made in the integrand of eqn (15). Hence the equivalence of the first-order Born and perturbation solutions has been demonstrated for the present special case of constant unperturbed refractive index.

5. Conclusions

In this paper we have presented and validated a perturbation series solution to the Helmholtz equation for the total fields propagating through an arbitrary refractive index distribution. The above developed solution has been applied to the problem of solving for the total fields propagating through an IO waveguiding structure. The field solutions developed show convenient computational properties, permitting evaluation using FFTs. Validating results have been presented for the case of straight step-index IO waveguides.

Further it has been analytically shown that the perturbation solution obtained is equivalent to the Born approximation to the solution of the Helmholtz equation for the case of constant unperturbed refractive index.

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