J. Indian Inst. Sci., Jan.–Feb. 1990, 70, 1–11.

An extension of the Thwaites method for calculation of incompressible laminar boundary layers

J. DEY AND R. NARASIMHA*
Aerospace Engineering Department, Indian Institute of Science, Bangalore 560 012, India.

Received on March 8, 1989; Revised on September 21, 1989.

Abstract

Certain extensions and modifications of the Thwaites integral method for laminar boundary layer calculation are proposed here in order to better handle large favourable pressure gradients and to provide certain additional parameters not considered by Thwaites.

Key words: Incompressible laminar boundary layers, integral methods, highly accelerating flows.

1. Introduction

Although the laminar boundary layer is relatively easy to handle by modern numerical methods, many design codes in engineering applications still employ integral methods because of the associated computational speed and economy. In a recent work on a transition zone model[1], we have found that currently used integral methods for laminar boundary layers are in general not satisfactory in highly accelerating flows. For example, Thwaites[2] has proposed a simple relation for the estimation of the momentum thickness Θ for any arbitrary free-stream velocity $U(x)$, and provided tabulated values of the shape parameter H and the skin friction parameter $T(= = U\Theta C_f/2\nu$, where C_f is the skin friction coefficient, and ν the kinematic viscosity) for various values of the pressure-gradient parameter $L(= \Theta^2 U'/\nu;\ U' = \mathrm{d}U/\mathrm{d}x)$. However, the tables are limited to the range $L < 0.25$, whereas higher values are now of great interest. Highly favourable pressure-gradient flows, even with a tendency to relaminarise, are also encountered near the leading edge of turbine blades, as revealed in the cascade tests of Hodson[3]; in the transition experiments of Narasimha *et al*[4], for example, L reached values as high as 0.4. It seems natural therefore to devise, if possible, an extension of Thwaites's method to higher values of L in order to exploit the general attractiveness of the method.

It appears that in the development of integral methods in the past, more attention has generally been given to adverse (rather than favourable) pressure-gradient flows. For

*and National Aeronautical Laboratory, Bangalore 560 017.

example, while Thwaites did examine the solution of the Falkner-Skan equation for $L=0.12$, he considered that the 'practical value of such distributions (corresponding to $L>0.1$) is not very great'. In order to extend the range of L to 0.25, he relied chiefly on sucked boundary layers rather than favourable pressure-gradient flows. Curle & Skan[5] have suggested modifications to the Thwaites method in the region near separation; Ojha[6] and Iida & Fujimoto[7] studied flows only for the Falkner-Skan pressure-gradient parameter $\beta \leqslant 1$ (the constant β being defined, following Evans[8], by the relation $U'=\text{constant} \times U^{2(\beta-1)/\beta}$). It has often been considered that the Pohlhausen[9] method is adequate for accelerating flows (*e.g.*, Ojha[6]).

Furthermore, the boundary-layer thickness, δ, was not considered in the proposal of Thwaites, but is an essential parameter in the transition zone model[1]. Although it can, in principle, be estimated from the (inverse) velocity profile proposed by Thwaites[2], this profile not only renders it difficult to handle integrals of the type

$$\int_0^\delta u[\ldots.]\,\mathrm{d}y \tag{1}$$

encountered in the transition zone model[1], but also gives misleading values of δ. For example, the value of δ/Θ obtained from the Thwaites profile at $L=0$ is lower than that given by Pohlhausen's[9] method by 31%.

These facts have led us to extend and modify the Thwaites method in order to better handle large pressure-gradient flows and to provide the additional boundary-layer parameters required for transition-zone models. For the present we consider only incompressible flows, leaving an extension to include compressibility effects to a later study.

2. The present proposal

Although the emphasis of the present study is on large favourable pressure gradients, retarded flows are also considered, in order that the proposal made here is complete and shows a smooth variation of boundary-layer parameters over the entire range of L of interest. Modifications to the Thwaites proposals for retarded flows are however minimal; in particular, we have found no reason to alter his separation criterion ($L=-0.082$). The basis for the present proposals is provided by solutions of the Falkner-Skan equations and for Howarth's[10] flow, in the light of the proposals of Thwaites and Curle & Skan[5]. The Falkner-Skan solutions used here are due to Smith[11] and Evans[8], with the parameter β ranging from -0.199 ($L=-0.06814$, corresponding to separation) to -1 ($L=0.3852$, highly accelerated) through $\beta=0$. As the separation criterion of Thwaites is retained here, it is thus found possible to devise a method whose range of validity is $-0.082 \leqslant L \leqslant 0.4$.

2.1. *Estimation of the momentum thickness*

Thwaites[2] notes that the right hand side in the momentum integral equation

$$(U/\nu)\frac{\mathrm{d}}{\mathrm{d}x}(\Theta^2)=2T-2L(H+2) \tag{2}$$

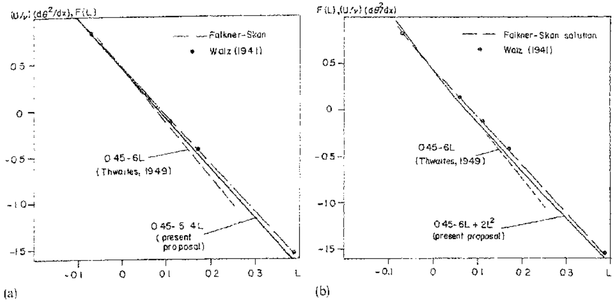

FIG. 1. Variation of the quantity $(U/\nu)\,d\Theta^2/dx$ and $F(L)$ with L for some solutions and proposals.

can be approximated as a linear function solely of L,

$$F(L) = 0.45 - 6L. \tag{3a}$$

Thwaites also proposed the alternative relation

$$F(L) = 0.455 - 6.16L + 1.37L^2, \tag{3b}$$

but felt that (3a) was adequate.

It is seen from fig. 1a that (U/ν) $(d\Theta^2/dx)$ for the Falkner-Skan solutions deviates considerably from (3a) at higher values of L; the proposal of Walz[12], who considered the Falkner-Skan solutions for $-0.0682 \leqslant L \leqslant 0.384$, is also shown in this figure. These large deviations clearly indicate the inadequacy of the Thwaites method at large values of L; in fact, an extrapolation of Thwaites's proposal to $L = 0.4$ will lead to misleading values of Θ.

For the Falkner-Skan solutions over the range $-0.0681 \leqslant L \leqslant 0.385$, the expression

$$F(L) \approx 0.45 - 5.12L, \tag{4a}$$

is a good fit. If we include the other data considered by Thwaites, the relation

$$F(L) \approx 0.45 - 5.4L \tag{4b}$$

provides an alternative to (3a) that is valid for large values of L as well (fig. 1a). Recently, Govindarajan & Narasimha[13] have shown that Granville's[14] analysis also leads to (4b) if one considers a quadratic velocity profile instead of the linear profile considered by him.

Alternatively, an approximate correction to (3a) may be devised. Figure 1b shows that the expression

$$F(L) = 0.45 - 6L + 2L^2 \tag{4c}$$

is adequate, the last term making up for the deficiency of (3a) at higher favourable pressure gradients. It is important to note here that the use of (4c) extends the range of the Thwaites method for estimation of Θ to $L=0.4$. It is however difficult to integrate the momentum equation with (4c). Fortunately, a neat approximate solution can be worked out if we consider the last term in (4c) as a small perturbation ΔF on the Thwaites expression,

$$F(L)=0.45-6L+\Delta F; \quad \Delta F=2L^2. \tag{4d}$$

Denoting the momentum thickness for $\Delta F=0$ by Θ_0, one obtains the required correction for Θ as follows. Let

$$\Theta^2=\Theta_0^2+\Theta_1^2, \tag{5a}$$

where Θ_0 is given by the Thwaites relation. Substituting (5a) into (2), and using (4d), we can write

$$\begin{aligned}(U/\nu)\mathrm{d}(\Theta_0^2+\Theta_1^2)/\mathrm{d}x=0.45-6(U'/\nu)(\Theta_0^2+\Theta_1^2)\\ +2(U'/\nu)^2(\Theta_0^4+\Theta_1^4+2\Theta_0^2\Theta_1^2).\end{aligned} \tag{5b}$$

We note that L is in general quite small itself, and from (4d) it should suffice to retain terms of $O(L^2)$; from (2) we expect $U'\Theta_1^2/\nu$ to be of the same order. We can therefore neglect higher powers and products of L^2 and $(\Theta_1/\Theta_0)^2$, *i.e.*, the last two terms of (5b), getting

$$(U/\nu)\mathrm{d}(\Theta_0^2+\Theta_1^2)/\mathrm{d}x=0.45-6(U'/\nu)(\Theta_0^2+\Theta_1^2)+2(U'/\nu)^2\Theta_0^4. \tag{5c}$$

Subtracting the Thwaites equation from the above we get

$$(U/\nu)\frac{\mathrm{d}}{\mathrm{d}x}(\Theta_1^2)=-6(U'/\nu)\Theta_1^2+L_0^2; \quad L_0=U'\Theta_0^2/\nu. \tag{6a}$$

This immediately gives

$$U^6\Theta_1^2=\int_0^x (U^5U'^2\Theta_0^4/\nu)\,\mathrm{d}x'. \tag{6b}$$

Thus, an improved approximation to Θ in highly favourable pressure gradients may be obtained by using the expression

$$\Theta^2=(0.45\nu/U^6)\left\{\int_0^x U^5\,\mathrm{d}x'+0.9\int_0^x\left[(U'^2/U^7)\left(\int_0^{x'}U^5\,\mathrm{d}x''\right)^2\right]\mathrm{d}x'\right\}. \tag{7}$$

The first term on the right represents the Thwaites value. The second term on the right is usually small, and provides a useful correction at high favourable pressure gradients. The relation (7) appears to be useful for the estimation of Θ over the whole range $-0.082\leqslant L\leqslant 0.4$.

2.2. *Boundary-layer thickness*

The variation of $H_\delta(=\delta/\Theta, \delta$ being defined to correspond to $0.995U)$ with L in various exact solutions along with the present proposal is shown in fig. 2; the present proposal is also given in Table I. It is interesting to note that for $-0.066\leqslant L\leqslant 0$, both the Falkner-Skan

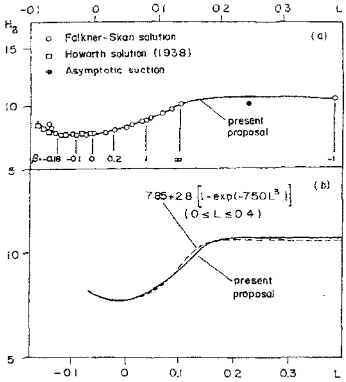

FIG. 2. Variation of H_δ with L for some solutions (a) and in the present proposal (b), which can be approximated by the correlation indicated.

and Howarth[10] solutions provide practically the same H_δ, but they differ for $L \leqslant -0.066$. However, at separation, which corresponds to $L = -0.06814$ for Falkner-Skan and $L = -0.084$ for Howarth's flow, H_δ is nearly the same (8.7 and 8.6, respectively). Further, it can be seen that for $L \geqslant 0.16$ H_δ attains almost a constant value of 10.7. The proposed variation of H_δ with L can be approximated by the expressions

$$H_\delta \approx 7.85 + 2.8[1 - \exp(-750L^3)], \quad 0 \leqslant L \leqslant 0.4, \tag{8a}$$

$$\approx 7.85 + 10.5L + 232.14L^2, \qquad -0.082 \leqslant L \leqslant 0, \tag{8b}$$

also shown in fig. 2. The correlation for retarded flows provides such an excellent fit that the data are indistinguishable from the present proposal.

2.3. *Proposed velocity profile*

The main purpose of approximate methods in the past has been to provide as accurately as possible momentum thickness and skin friction distributions. However, a simple representation of the velocity profile is desirable, as mentioned earlier, in order to handle the integral in (1). We consider for this purpose the quartic

$$u = 2\eta - 2\eta^3 + \eta^4 + P^*\eta(1-\eta)^3, \tag{9}$$

where $\eta = y/\delta$. This profile is similar to that of Pohlhausen[9], whose parameter L_p has been replaced here by $6P^*$, but with the important difference that P^* in (9) is *not* necessarily proportional to $\delta^2 U'/\nu$, as the Pohlhausen pressure-gradient parameter is. Instead, P^* is

Table I
Proposed functions for the estimation of various laminar parameters

L	T	H	H_δ	P^*	L	T	H	H_δ	P^*
−0.082	0	3.78	8.6	−2.50	0.01	0.235	2.56	7.85	−0.36
−0.08	0.02	3.72	8.55	−2.50	0.02	0.25	2.52	7.9	−0.16
−0.078	0.03	3.66	8.5	−2.50	0.03	0.265	2.48	8.0	0.04
−0.075	0.045	3.54	8.4	−2.50	0.04	0.28	2.44	8.1	0.24
−0.07	0.06	3.4	8.28	−2.50	0.05	0.295	2.4	8.2	0.44
−0.065	0.075	3.25	8.16	−2.44	0.06	0.31	2.36	8.4	0.62
−0.06	0.095	3.16	8.05	−2.32	0.07	0.325	2.32	8.6	0.80
−0.055	0.115	3.05	7.95	−2.14	0.08	0.34	2.29	8.8	0.98
−0.05	0.125	2.98	7.9	−1.96	0.09	0.35	2.26	9.0	1.16
−0.045	0.135	2.92	7.85	−1.81	0.1	0.365	2.23	9.2	1.34
−0.04	0.145	2.86	7.82	−1.66	0.11	0.375	2.20	9.4	1.52
−0.035	0.155	2.82	7.79	−1.51	0.12	0.385	2.18	9.6	1.68
−0.03	0.165	2.78	7.77	−1.36	0.13	0.40	2.15	9.85	1.84
−0.025	0.175	2.74	7.75	−1.21	0.14	0.41	2.13	10.1	1.98
−0.02	0.185	2.72	7.75	−1.06	0.15	0.42	2.11	10.3	2.11
−0.015	0.195	2.69	7.75	−0.96	0.16	0.435	2.09	10.5	2.21
−0.01	0.20	2.66	7.75	−0.80	0.17	0.445	2.07	10.6	2.30
−0.005	0.21	2.63	7.8	−0.68	0.18	0.455	2.05	10.7	2.38
0.0	0.22	2.6	7.8	−0.56	0.19	0.465	2.03	10.7	2.44
					0.2	0.475	2.01	10.7	2.50
					0.25	0.525	1.94	10.7	2.50
					0.3	0.575	1.86	10.7	2.50
					0.35	0.625	1.8	10.7	2.50
					0.4	0.675	1.74	10.7	2.50

considered here as a velocity profile factor, so selected that for each β (or the corresponding L) the profile (9) gives a good representation of the corresponding Falkner-Skan solution: examples are shown in fig. 3 (from a large number of Falkner-Skan solutions considered elsewhere[15]). It is important to emphasise that, unlike in the Pohlhausen method, (9) is *not* utilised here to construct relations for other laminar flow parameters. As a result, the constraints associated with the Pohlhausen method are *not* considered applicable here. For example, the maximum value of L_P in Pohlhausen's method is considered to be 12, as beyond this there is an overshoot in the assumed profile. Here, however, it is considered useful to go up to $P^* = \pm 2.5$, as both overshoot ($u = 1.009$ around $\eta = 0.7$) and undershoot ($u = -0.008$ at $\eta = 0.05$) in the velocity profile (9) are insignificant (as can be seen from fig. 4, for example) for the calculations we have in mind. It is seen in fig. 3 that (9) generally provides excellent approximations to these solutions; the largest deviation is at $\beta = -1$ ($L = 0.385$), but even here (9) is not inadequate. (The asymptotic suction profile is also seen to provide a good representation of the Falkner-Skan solution for $\beta = \infty$.) Furthermore, although the velocity profile (9) with $P^* = -2.5$ is entirely adequate for the present purpose, it provides a slightly better representation of the Howarth[10] separation profile than of the Falkner-Skan.

The values of P^* proposed here are listed in Table I and shown graphically in fig. 5. It may be noted that $P^* = -0.56$ when $L = 0$. The following expressions are also proposed for

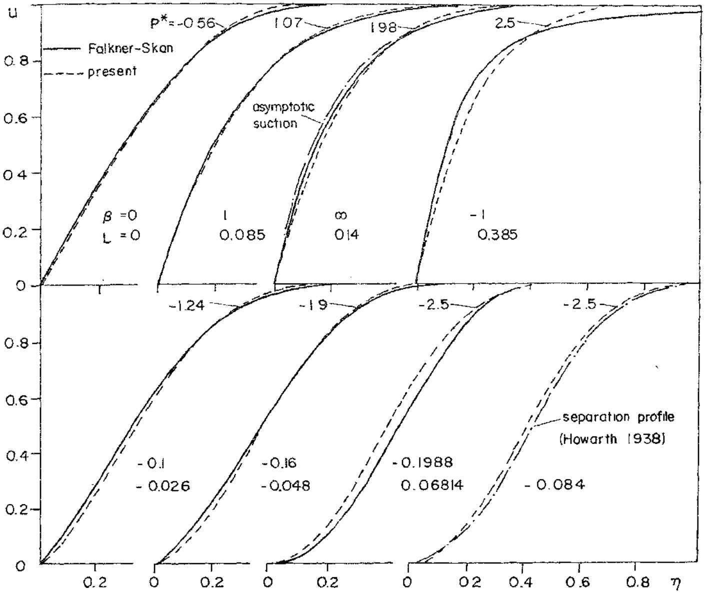

FIG. 3. Approximate representation of the Falkner-Skan profiles for various β and Howarth's separation profile by the proposed velocity profile (9) with the values of P^* indicated here.

the variation of P^* with L.

$$\begin{aligned} P^* &\approx 2.5, & 0.2 \leqslant L \leqslant 0.4, \\ &\approx -3.97 - 18.9L + 22.9L^{1/2}, & 0.08 \leqslant L \leqslant 0.2, \\ &\approx -0.56 + 19.5L, & 0 \leqslant L \leqslant 0.08, \\ &\approx -0.56 + 28.5L, & -0.066 \leqslant L \leqslant 0, \\ &\approx -2.5, & -0.082 \leqslant L \leqslant -0.066. \end{aligned} \tag{10}$$

2.4. *Shape factor*

Shown in fig. 6 is the variation of H^{-1} with L for the Falkner-Skan solutions and Howarth's flow, together with the present and Thwaites's proposals. The present proposal, which gives

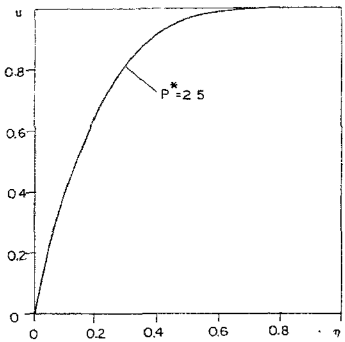

FIG. 4. Boundary-layer velocity profile (9) with $P^* = 2.5$. The velocity overshoot is 1.009 around $\eta = 0.7$.

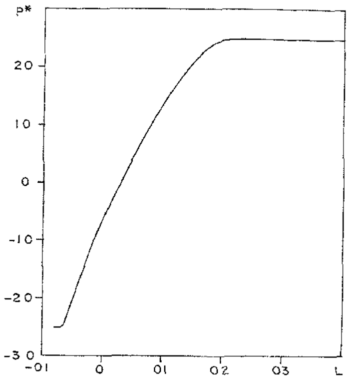

FIG. 5. Proposed variation of P^* with L.

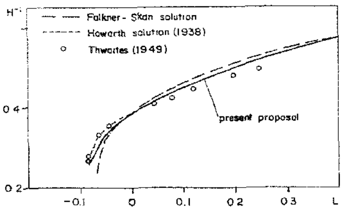

FIG. 6. Variation of H^{-1} with L for some solutions and proposals. Only a few points from Thwaites are shown for clarity.

greater weight to the highly accelerated Falkner-Skan flows, is also given in Table I and can be approximated by the expressions

$$H^{-1} \approx 0.385 + 0.37L + 0.073L^{1/2}, \qquad 0 \leqslant L \leqslant 0.4, \tag{11a}$$

$$\approx 0.385 + 0.44L - 13.26L^2, \qquad -0.082 \leqslant L \leqslant 0. \tag{11b}$$

2.5. *Skin-friction coefficient*

From a comparison of the proposal of Thwaites on T with the Falkner-Skan and Howarth[10] solutions (fig. 7a), it is seen that there are appreciable deviations at both extremes of the range of L. Giving due weight to the various data shown in fig. 7a, the

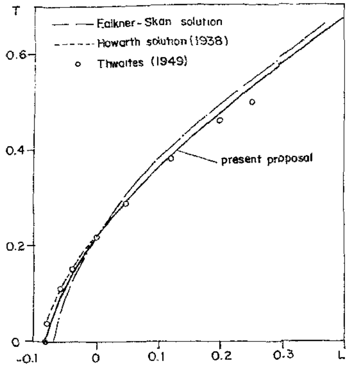

FIG. 7a. Variation of T with L for some solutions and proposals. Only a few points from Thwaites are shown for clarity.

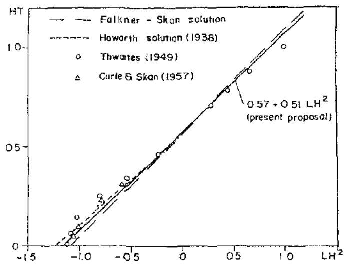

FIG. 7b. Variation of the product HT with the quantity LH^2 for some solutions and proposals. Only a few points from Thwaites and Curle & Skan are shown for clarity.

variation of T with L proposed here is shown in this figure (and also given in Table I). Further, as may be seen from fig. 7b, the product HT varies almost linearly with the quantity LH^2 for the Falkner-Skan solutions; the corresponding expression is

$$HT \approx 0.57 + 0.52LH^2, \quad -1.1 \leqslant LH^2 \leqslant 1.18, \quad \text{or} \quad -0.06814 \leqslant L \leqslant 0.4. \tag{12}$$

Evans[8] has noted the linear variation of $T_1(= = U\delta^* C_f/2\nu)$ with $L_1(=\delta^{*2}U'/\nu)$. In fig. 7b the proposals of Thwaites and of Curle & Skan[5] (only a few points are shown for the sake of clarity), as well as the Howarth solution, are also shown. Note that $HT=0$ corresponds to separation. The Thwaites proposal exhibits linear variation of HT with LH^2 for $-1 \leqslant LH^2 \leqslant 1$, but approaches $HT=0$ rapidly. Figure 7b shows that although the Thwaites proposal deviates appreciably from the Falkner-Skan solutions in the range $-0.5 \leqslant LH^2 \leqslant -1$, the suggested value of $LH^2(=-1.12)$ at separation $(L=-0.082)$ lies close to that for the Falkner-Skan solution ($LH^2=-1.107$; $L=-0.06814$). On the other hand, the Howarth solution gives $LH^2=-1.23$, and Curle & Skan[5] propose $LH^2=-1.13$, at separation. It is interesting to note that unlike the value of L found at separation in the different solutions, that of LH^2 at separation differs very little. This suggests that near separation a single-parameter method cannot be satisfactory but a two-parameter method might be adequate: this idea will be pursued elsewhere. Figure 7b shows that the correlation

$$HT \approx 0.57 + 0.51LH^2 \tag{13}$$

is a good approximation to the various data considered here, and can be used to estimate C_f for $-0.082 \leqslant L \leqslant 0.4$.

3. Conclusion

To sum up, certain extensions and modifications of the Thwaites method have been proposed here to handle large favourable pressure gradients and to provide certain additional boundary-layer parameters required in transition-zone modelling. The present proposal extends the range of the Thwaites method to $L = 0.4$.

Acknowledgement

The authors wish to express their appreciation to Prof. S. Soundranayagam of the Mechanical Engineering Department and Prof. A. Prabhu of the Aerospace Engineering Department for their support. The financial support of DST through the project 'Transition and turbulence in shear flows' is gratefully acknowledged.

Nomenclature

C_f	:	skin-friction coefficient;
$F(L)$	:	the quantity $[2T-2L(H+2)]$ in (2);
H	:	shape factor, $= \delta^*/\Theta$;
H_δ	:	δ/Θ;
L	:	a pressure-gradient parameter, $= (\Theta^2/\nu)\,\mathrm{d}U/\mathrm{d}x$;
L_0	:	value of L based on Θ_0;
L_1	:	a pressure-gradient parameter, $= (\delta^{*2}/\nu)\,\mathrm{d}U/\mathrm{d}x$;
L_p	:	a pressure-gradient parameter, $= (\delta^2/\nu)\,\mathrm{d}U/\mathrm{d}x$;
P^*	:	velocity profile factor in (9);
T	:	the quantity $U\Theta C_f/2\nu$;
T_1	:	the quantity $U\delta^* C_f/2\nu$;
U	:	free-stream velocity;
x	:	streamwise coordinate;
y	:	coordinate normal to x;
δ	:	boundary-layer thickness;
δ^*	:	boundary-layer displacement thickness;
ν	:	kinematic viscosity;
η	:	y/δ;
β	:	Falkner-Skan pressure-gradient parameter, defined by the relation $\mathrm{d}U/\mathrm{d}x = \text{constant} \times U^{2(\beta-1)/\beta}$;
Θ	:	boundary-layer momentum thickness;
Θ_0, Θ_1	:	respectively first and perturbed values of Θ, as in (5).

References

1. DEY, J. AND NARASIMHA, R. — An integral method for the calculation of 2-D transitional boundary layers, to appear in *J. Aircraft*

2. THWAITES, B. — Approximate calculation of the laminar boundary layer, *Aeronaut. Q.*, 1949, **1**, 245–280.

3. HODSON, H. P. — Boundary-layer transition and separation near the leading edge of high-speed turbine blade, *ASME J Engng Gas Turbine Power*, 1985, **107**, 127–134.

4. NARASIMHA, R., DEVASIA, K. J., GURURANI, G. AND BADRINARAYANAN, M. A. — Transitional intermittency in boundary layers subjected to pressure gradient, *Exp. Fluids*, 1984, **2**, 171–176.

5. CURLE, N. AND SKAN, S. W. — Approximate methods for predicting separtion properties of laminar boundary layers, *Aeronaut. Q.*, 1957, **8**, 257–268.

6. OJHA, S. K. — On approximate calculation of laminar boundary layer using polynomials, *Z. Angew Math Mech.*, 1965, **45**, 277–288.

7. IIDA, SEI-ICHI AND FUJIMOTO, A. — A fast approximate solution of the laminar boundary-layer equations, *ASME J. Fluids Engng*, 1986, **108**, 201–207.

8. EVANS, H. L. — *Laminar boundary-layer theory*, 1968, Addison-Wesley.

9. POHLHAUSEN, K. — Zur naherungsweisen integration der differentialgleichung der laminaren reibungsschicht, *Z. Angew. Math. Mech.*, 1921, **1**, 252–268.

10. HOWARTH, L. — On the solution of the laminar boundary layer equations, *Proc. R. Soc. Lond. A*, 1938, **164**, 547–579.

11. SMITH, A. M. O. — *Improved solutions of the Falkner and Skan boundary-layer equation*, Sherman M. Fairchild Paper no. FF-10, Inst. Aero. Sciences, 1954.

12. WALZ, A. — In *Laminar boundary layers*, Rosenhead, L. (ed.), 1963, pp. 301–303, Clarendon Press.

13. GOVINDARAJAN, R. AND NARASIMHA, R. — *A new correlation for prediction of transition onset*, Tech. Memo, DU88 (under preparation), National Aeronautical Laboratory, Bangalore, India.

14. GRANVILLE, P. S. — *The calculation of the viscous drag of bodies of revolution*, Report 849, David Taylor Model Basin, Washington, 1953.

15. DEY, J. AND NARASIMHA, R. — *An extension of Thwaites's method for calculation of incompressible laminar boundary layers*, Report 88 FM 4, Aerospace Engineering Department, Indian Institute of Science, Bangalore, 1988.