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A recursive observer design in multi-output systems

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Abstract

In this paper, it is shown that the asymptotic state-observer problem for a linear time-invariant system of order n having q output can be solved through the solution of the same problem for a similar system of order n-q-r (with $1 \leqslant r \leqslant q_1$) having q_1 (where $q_1 \leqslant q$) number of output. The procedure for determining the parameters of the asymptotic state-observer (matrices D and G) allows considerable simplification of the computations and can be repeated in a recursive manner. The result leads to a new algorithm for designing asymptotic state-observers for multivariable linear time-invariant systems.

Key words: System output, system state vector, multivariable linear time-invariant systems

1. Introduction

The output or a linear time-invariant system may be used to construct an estimate of the system-state vector. The device which reconstructs the state vector is called an observer. The observer itself is a time-invariant linear system driven by the input and output of the system it observes. Kalman and Bucy[1] dealt with the problem of state estimation for a linear, finite-dimensional dynamic plant when all measurements are corrupted by white noise. Bryson and Johansen[2] have shown that when the measurements are noise-free the optimal estimator will be a modification of the Kalman-Bucy filter. Simon[3] and Wonham[4] have recognized the duality between the pole-assignment problem and the problem of building an asymptotic-state observer. Luenberger[5] has proposed an excellent method for constructing an asymptotic-state estimator for a single-output system. His observer design for a system with Q output can be reduced to the design of q separate observers for a single-output subsystem. This result is a consequence of a special canonical form. Almost all of the published solutions resort to canonical forms and are not convenient to work with in the multiple input-output cases. Since the system is often described in terms of variables that are of direct interest, a transformation to canonical form is inconvenient. The present solution does not resort to the use of canonical forms for the multiple observer design.

2. Statement of the observer problem

Assume that a multivariable linear time-invariant dynamical plant with q output

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Hx \tag{2}$$

drives an observer

$$\dot{z} = Dz + Bu + Gy \tag{3}$$

$$D = A - GH \tag{4}$$

with

$x = x(t) = n \times 1$ state vector;
$u = u(t) = \vartheta \times 1$ input vector;
$y = y(t) = q \times 1$ output vector;
$z = z(t) = n \times 1$ reconstructed state vector.

where A, B and H are constant matrices of appropriate dimensions. Now it is required to find a linear observer law $z = Gy$, where $z = z\,(t)$ is an $n \times 1$ observer signal vector and G is an $n \times q$ observer matrix in such a way that the $n \times n$ observer system matrix D is assigned arbitrary dynamics (fig. 1).

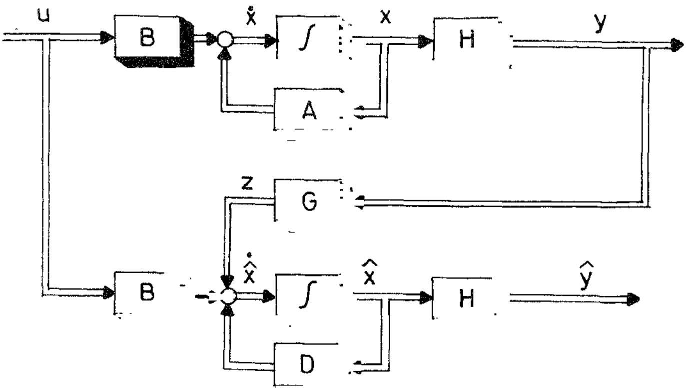

Fig. 1. Observer in general representation.

3. Method of the design procedure

The method of this procedure is based on a step-by-step transformation of the time-invariant linear mathematical model (eqns 1 and 2) of the production engineering process obtained by means of appropriate process analysis statements and given in a state-space representation. Calculation and an inverse transformation are subsequently performed. In this connection, a unique and simple calculation of the observer matrix is aimed at, in such a way that the system observer matrix D shows the dynamical behaviour demanded.

The procedure presented is recursive and can be used for the observer calculation of any system/output combination (n/q).

Based on eqn (4) the observer matrix can be determined as:

$$G=[A-D]H^{-1}. \tag{5}$$

However, this procedure is to be avoided as the output matrix H cannot be inverted immediately.

In order to perform the observer calculation in any case it is assumed that the output matrix H is of full rank and can be structured to

$$H=[H_1 \quad H_2] \tag{6}$$

where $H_1=q\times q$ non-singular matrix.

Now such a transformation matrix is required for allowing H^{-1} of eqn (5) to be substituted by H_1^{-1}.

$$T_1=\begin{bmatrix} I_q & H_1^{-1}H_2 \\ 0 & I_{n-q} \end{bmatrix} \tag{7}$$

where I_q is identity matrix of order q.

Using $x=T_1\rho$ eqns (1), (2) and (4) will be transformed into

$$\dot{\rho}=\hat{A}\rho+\hat{B}u \tag{8}$$

$$y=\hat{H}\rho \tag{9}$$

where

$$\hat{A}=T_1^{-1}AT_1=\begin{bmatrix} C_1 & E \\ C_2 & V \end{bmatrix} \tag{10}$$

$$\hat{H}=HT_1=[H_1 \quad 0] \tag{11}$$

$$\hat{B}=T_1^{-1}B \tag{12}$$

$$\hat{D}=T_1^{-1}AT_1-T_1^{-1}GHT_1 \tag{13}$$

$$\hat{D}=\hat{A}-\hat{G}\hat{H} \tag{14}$$

where $\hat{A}$ is the $n\times n$ matrix, C_1 the $q\times q$ matrix and in the theorem of Bhandarkar and Fahmy[6], (V,E) is an observable pair if and only if (A,H) is an observable pair.

Thus eqn (14) can be rewritten as

$$\hat{G}[H_1 0] = \begin{bmatrix} C_1 & E \\ C_2 & V \end{bmatrix} - \begin{bmatrix} \Delta_1 & \Delta_3 \\ \Delta_2 & \Delta_4 \end{bmatrix}. \tag{15}$$

Equation (15) is valid if and only if $\Delta_3 = E$, $\Delta_4 = V$.

$$\hat{G} = \left[\frac{C_1 - \Delta_1}{C_2 - \Delta_2}\right] H_1^{-1} \tag{16}$$

$$G = T_1 \left[\frac{C_1 - \Delta_1}{C_2 - \Delta_2}\right] H_1^{-1}. \tag{17}$$

The determination of matrices Δ_1 and Δ_2 is based on the following theorem and is done in such a way that matrix D has the required dynamical behaviour.

Theorem

A $2q \times 2q$ constant matrix $\hat{D}$

$$\hat{D} = \begin{bmatrix} \Delta_1 & E \\ \Delta_2 & V \end{bmatrix} \tag{18}$$

where E is the non-singular matrix of order q, and V the arbitrary matrix of order $q \times q$, can be assigned to arbitrary eigenvalues by a suitable computation of Δ_1 and Δ_2.

Proof

Let

$$M\hat{D} = JM \tag{19}$$

with

$$M = \begin{bmatrix} M_1 & M_3 \\ M_2 & M_4 \end{bmatrix} \quad J = \begin{bmatrix} J_1 & R \\ 0 & J_2 \end{bmatrix} \tag{20}$$

where J_1 and J_2 contain the arbitrarily specified eigenvalues and R is yet to be specified to guarantee non-singularity of matrix M. Expanding eqn (19), the following becomes valid:

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = M^{-1} J \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \tag{21}$$

where

$$M_1 = RE^{-1}; \tag{22}$$

$$M_2 = [J_2 - V]E^{-1}; \tag{23}$$

$$\Delta_1 = M_1^{-1}[J_1 M_1 + RM_2]; \tag{24}$$

$$\Delta_2 = J_2 M_2 - M_2 \Delta_1. \tag{25}$$

The determination of Δ_1 and Δ_2 requires that E, M_4, and R are non-singular matrices, and thus the determinant of M is non-zero.

$$M=\begin{bmatrix} R & M_3 \\ J_2-V & M_4 \end{bmatrix}\begin{bmatrix} E^{-1} & 0 \\ 0 & I \end{bmatrix}. \tag{26}$$

Non-singularity of M implies the similarity of matrices $\hat{D}$ and J and hence the proof of the theorem. By choosing M_4 such that its inverse exists, it can be shown that non-singularity of M is guaranteed if the determinant of $(RM_4-M_3J_2+M_3V)$ is not zero.

For example, by letting $M_3=0$, $M_4=I$, M^{-1} exists as long as det $[R]\neq 0$.

4. Computation procedure

Depending on the n/q combination, three special cases have to be distinguished in performing the observer calculation. These calculations are aimed at separating matrix E as an invertible matrix in any case. Thus the methodical procedure will be supported. In this connection, special cases II and III will be returned to special case I in a recursive manner.

4.1. *Special case I*

This case assumes that the system/output combination is $n=2q$ and matrix E in eqn (10) is invertible.

Example 1

Given

Linear time-invariant dynamical plant

$$\dot{x}=\left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ \hline 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 2 \end{array}\right]x+Bu$$

$$y=\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]x$$

Objective

a) Observer matrix G;
b) System observer matrix D.

Step 1

Determination of the transformation matrix T_1 using eqn (7).

Step 2

System transformation to eqns (8) and (9).

$$\dot{\varphi}=\begin{array}{c} \quad C_1 \qquad E \\ \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ \hline 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 2 \end{array}\right] \\ \quad C_2 \qquad V \end{array} \varphi + Bu$$

$$y=\begin{array}{c} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \\ H_1 \quad H_2 \end{array} \varphi$$

Step 3

Dynamical determination system observer matrix D using eigenvalues.

$$J_1 = \begin{array}{l} \lambda_1 = -1+1i \\ \lambda_2 = -1+1i \end{array} \qquad J=\left[\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ \hline 0 & 0 & -2 & 1 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

$$J_2 = \begin{array}{l} \lambda_3 = -2+1i \\ \lambda_4 = -2-1i \end{array}$$

Step 4

Calculate matrices M_1, M_2, Δ_1, Δ_2 using eqns (22)–(25), and $R=I$.

$$M_1=\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \quad M_2=\begin{bmatrix} 2 & -2 \\ -6 & 2 \end{bmatrix}$$

$$\Delta_1=\begin{bmatrix} -2 & -2 \\ 3 & -6 \end{bmatrix} \quad \Delta_2=\begin{bmatrix} 0 & -2 \\ -8 & -2 \end{bmatrix}$$

Step 5

Calculate matrices G and D using eqns (17) and (4).

$$[C_1-\Delta_1]=\begin{bmatrix} 3 & 4 \\ -3 & 7 \end{bmatrix} \quad [C_2-\Delta_2]=\begin{bmatrix} 1 & 4 \\ 10 & 3 \end{bmatrix}$$

$$G=\begin{bmatrix} 3 & 4 \\ -3 & 7 \\ 1 & 4 \\ 10 & 3 \end{bmatrix} \quad D=\begin{bmatrix} -2 & -2 & 1 & 1 \\ 3 & -6 & 2 & 1 \\ 0 & -2 & 0 & 1 \\ -8 & -2 & 1 & 2 \end{bmatrix}$$

Step 6

Proof: det $(\lambda_1 - D)=0$, *i.e.*, the system observer matrix shows the dynamical behaviour demanded.

4.2. *Special case II*

This case assumes that the system/output combination is $n<2q$ and matrix E in eqn (10) is not invertible.

Objective

The non-singularity and hence inversion of matrix E can be achieved by means of transforming T_2 and structuring matrix E into the matrices E_1 and E_2 by suitably selecting matrix P, with $\varphi = T_2\beta$ and

$$T_2 = \begin{bmatrix} P_q & 0 \\ 0 & I_m \end{bmatrix} \tag{27}$$

where P is the $q \times q$ permutation matrix, and m, $n-q$. System eqns (8) and (9) will be transformed into

$$\dot{\beta} = \tilde{A}\beta + \tilde{B}u \tag{28}$$

$$y = \tilde{H}\beta \tag{29}$$

with

$$\tilde{A} = T_2^{-1}\hat{A}T_2 = \begin{bmatrix} \tilde{C}_1 & \tilde{E} \\ \tilde{C}_2 & V \end{bmatrix} \tag{30}$$

$$\tilde{E} = P_q^{-1}E = \begin{bmatrix} \tilde{E}_1 \\ \tilde{E}_2 \end{bmatrix} \tag{31}$$

where $\tilde{E}_2$ is the non-singular $m \times m$ matrix. The observer system matrix $\tilde{D}$ is obtained using eqn (18)

$$\tilde{D} = \begin{bmatrix} \Delta_1 & \tilde{E} \\ \Delta_2 & V \end{bmatrix} \tag{32}$$

with eqn (31) and

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} J_0 & 0 \\ 0 & \tilde{\Delta}_1 \\ 0 & \tilde{\Delta}_2 \end{bmatrix} \tag{33}$$

eqn (32) is extended to

$$\tilde{D} = \begin{array}{r} q \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} J_0 & 0 & \tilde{E}_1 \\ 0 & \tilde{\Delta}_1 & \tilde{E}_2 \\ 0 & \tilde{\Delta}_2 & V \end{bmatrix} \begin{array}{l} \\ \rightarrow 2m \\ \\ \end{array} \tag{34}$$

$2m$

Special case I ($\tilde{D}_1$)

J_0 is a $(q-m)\times(q-m)$ matrix with $(n-2m)$ specified eigenvalues. The $m \times m$ matrices $\tilde{\Delta}_1$, $\tilde{\Delta}_2$ are computed in such a way that matrix $\tilde{D}_1$ is assigned to the remaining $2m$ specified eigenvalues in view of the theorem given in §3. The calculation of the observer matrix G is done by an inverse transformation using eqn (13).

If the matrix E structuring reveals an immediate inversion of E_2, transformation T_2 can be omitted ($E_2 \neq 0$, $T_2 = I$).

Example 2

Given

Linear time-invariant dynamical plant

$$\dot{x} = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \\ \hline 2 & 0 & 1 \end{array}\right] x + Bu$$

$$y = \left[\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] x$$

Objective

a) Observer matrix G;
b) System observer matrix D.

Step 1

The submatrix H_1 cannot be inverted immediately, *i.e.*, the permutation of the system is necessary.

$$P = P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \dot{\eta} = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 0 \\ \hline 1 & 2 & 1 \end{array}\right] \eta + B_p u$$

$$y = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \eta$$

Step 2

Determination of the transformation matrix T_1 using eqn (7).

Step 3

System transformation to eqns (8) and (9).

$$\dot{\varphi} = \overset{\quad C_1 \qquad E}{\underset{\quad C_2 \qquad V}{\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 0 \\ \hline 1 & 2 & 1 \end{array}\right]}} \varphi + Bu$$

$$y=\underbrace{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}_{H_1}\left|\underbrace{\begin{matrix}0\\0\end{matrix}}_{H_2}\right]\varphi$$

Step 4

System transformation with matrix T_2 using eqn (27). By inspection $E=\begin{bmatrix}E_1\\E_2\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$. It is to be seen that the submatrix E_2 cannot be inverted. Therefore, a transformation with matrix T_2 is necessary.

$$T_2=T_2^{-1}=\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\quad \dot{\beta}=\begin{array}{c}\begin{array}{cc}\tilde{C}_1 & \tilde{E}\end{array}\\ \left[\begin{array}{cc|c}1 & 2 & 0\\0 & 1 & 2\\\hline 2 & 1 & 1\end{array}\right]\\ \begin{array}{cc}\tilde{C}_2 & V\end{array}\end{array}\beta+\tilde{B}u$$

$$y=\left[\begin{array}{cc|c}0 & 1 & 0\\1 & 0 & 0\end{array}\right]\beta$$

Step 5

Dynamical determination system observer matrix D using eigenvalues

$$J=\left[\begin{array}{c|c}J_1 & R\\\hline 0 & J_2\end{array}\right]=\left[\begin{array}{c|cc}J_0 & 0 & 0\\\hline 0 & J_1 & R\\0 & 0 & J_2\end{array}\right]\quad \begin{array}{l}J_0=-1\\J_1=-2\\J_2=-3\end{array}$$

Step 6

Calculate matrices M_1, M_2, $\tilde{\Delta}_1$, $\tilde{\Delta}_2$ using eqns (22)–(25) and $R=I$.

$$M_1=0,5,\ M_2=-2,\ \tilde{\Delta}_1=-6,\ \tilde{\Delta}_2=-6.$$

Step 7

Calculate matrices G and D using eqns (17) and (4).

$$[\tilde{C}_1-\Delta_1]=\begin{bmatrix}2 & 2\\0 & 7\end{bmatrix}\quad [\tilde{C}_2\Delta_2]=[2 \quad 7]$$

$$G=\begin{bmatrix}7 & 2\\2 & 2\\7 & 0\end{bmatrix}\qquad D=\begin{bmatrix}1 & 0 & -6\\0 & -1 & 0\\2 & 0 & -6\end{bmatrix}$$

Step 8

Proof: det $(\lambda_I-D)=0$, *i.e.*, the system observer matrix shows the dynamical behaviour demanded.

4.3 *Special case III*

This case assumes that the system/output/combination is $n>2q$ and E in eqn (10) can not be inverted (rank r of matrix E is $1 \leqslant r \leqslant q$).

Objective

Provided matrix E_2 cannot be inverted by transformation T_2, the non-singularity of E_2 and its ability to be inverted has to be realized by transforming T_3 and structuring matrix E into matrices E_1 through E_4 by selecting matrix P in T_3 in a suitable manner. Transformation T_4 enables approaching $E_4=0$. Thus, matrix g_2 can be determined and hence the observer problem can be solved using eqns (48)–(50). With $\beta=T_3\delta$ and

$$T_3=\begin{bmatrix} I_q & 0 \\ 0 & P_m \end{bmatrix}, \tag{35}$$

where P is $m \times m$ permutation matrix such that

$$E=\tilde{E}P_m=\begin{bmatrix} \hat{E}_1 & \hat{E}_3 \\ \hat{E}_2 & \hat{E}_4 \end{bmatrix} \tag{36}$$

where $\hat{E}_2$ is the non-singular matrix of order r. Using $\delta=T_4\gamma$ and T_4

$$T_4=\begin{bmatrix} I_q & 0 & 0 \\ 0 & I_r & -\hat{E}_2^{-1}\hat{E}_4 \\ 0 & 0 & I_{m-r} \end{bmatrix} \tag{37}$$

the transformation gives

$$\dot{\gamma}=\bar{A}\gamma+\bar{B}u \tag{38}$$

$$y=\bar{H}\gamma \tag{39}$$

with

$$\bar{A}=T_4^{-1}\hat{A}T_4=\begin{bmatrix} \bar{C}_{1.1,1}\bar{C}_{1\ 1,2}\bar{E}_1 & \bar{E}_3 \\ \bar{C}_{1.2,1}\bar{C}_{1.2,2}\bar{E}_2 & 0 \\ \bar{C}_{2.3,1}\bar{C}_{2\ 3,2}\bar{V}_1 & \bar{V}_3 \\ \bar{C}_{2.4,1}\bar{C}_{2.4,2}\bar{V}_2 & \bar{V}_4 \end{bmatrix}. \tag{40}$$

The observer system matrix $\tilde{D}$ will be obtained using eqn (18) with

$$=\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{bmatrix}=\begin{bmatrix} J_0 & 0 \\ 0 & \bar{\Delta}_1 \\ 0 & \bar{\Delta}_2 \\ 0 & \bar{\Delta} \end{bmatrix} \tag{41}$$

as

$$\bar{D}=\begin{bmatrix} J_0 & 0 & \bar{E}_1 & \bar{E}_3 \\ 0 & \bar{\Delta}_1 & \bar{E}_2 & 0 \\ 0 & \bar{\Delta}_2 & \bar{V}_1 & \bar{V}_3 \\ 0 & \bar{\Delta} & \bar{V}_2 & \bar{V}_4 \end{bmatrix} \tag{42}$$

Special case II Special case I

The determination of matrices $\bar{\Delta}_1$, $\bar{\Delta}_2$ and $\bar{\Delta}$ is done by solving the matrix equation

$$\bar{D}_1 R_0 = R_0 \bar{D}_4 \tag{43}$$

where R_0 is the non-singular matrix. With

$$\bar{D}_1=\begin{bmatrix} \bar{\Delta}_1 & \bar{E}_2 & 0 \\ \bar{\Delta}_2 & \bar{V}_1 & \bar{V}_3 \\ \bar{\Delta} & \bar{V}_2 & \bar{V}_4 \end{bmatrix}, \tag{44}$$

$$\bar{D}_4=\begin{bmatrix} \bar{M}_1 & \bar{E}_2 & 0 \\ \bar{M}_2 & \bar{V}_5 & \bar{V}_3 \\ 0 & 0 & \bar{D}_3 \end{bmatrix}, \tag{45}$$

and

$$\bar{D}_4=\begin{bmatrix} \bar{D}_2 & \bar{R} \\ 0 & \bar{D}_3 \end{bmatrix} \quad \text{with } \bar{R}=\begin{bmatrix} 0 \\ \bar{V}_3 \end{bmatrix}, \tag{46}$$

respectively

$$R_0=\begin{bmatrix} I_r & 0 & 0 \\ 0 & I_r & 0 \\ g_1 & g_2 & I_{n-q-r} \end{bmatrix}. \tag{47}$$

In this connection, matrix $\bar{D}_4$ is equivalent to matrix J in eqn (20) and g_1 and g_2 are matrices temporarily unknown. However, they can be determined in such a way that the eigenvalues of $\bar{D}_1$ and $\bar{D}_2$ are identical. Then the calculation of matrix g_2 is equivalent to the observer problem solution

$$\bar{D}_3 = \bar{V}_4 - g_2 \bar{V}_3 \tag{48}$$

for the reduced system

$$\dot{\alpha} = \bar{V}_4 \alpha; \tag{49}$$

$$y = \bar{V}_3 \alpha \tag{50}$$

where $\bar{V}_4$ is a matrix of order $n-q-r$ and y involves r output. If the rank of $\bar{V}_3 = r_1$ $(r_1 \leqslant r)$ the effective number of output will be r_1.

Based on eqn (43) the following relations for determining matrices $\bar{\Delta}_1$, $\bar{\Delta}_2$ can be given:

$$\bar{V}_5 = \bar{V}_1 + \bar{V}_3 g_2 \tag{51}$$

$$g_1 = [\bar{V}_4 g_2 + \bar{V}_2 - g_2 \bar{V}_5]\bar{E}_2^{-1} \tag{52}$$

$$\bar{\Delta}_1 = \bar{M}_1 \tag{53}$$

$$\bar{\Delta}_2 = \bar{M}_2 - \bar{V}_3 g_1 \tag{54}$$

$$\bar{\Delta} = g_1 \bar{M}_1 + g_2 \bar{M}_2 - \bar{V}_4 g_1 . \tag{55}$$

Matrix J_0 is of order $(q-r)\times(q-r)$ and involves $(q-r)$-dominating eigenvalues.

Matrices $\bar{D}_2$ and $\bar{D}_3$ are of the orders $2r\times 2r$ and $(n-q-r)\times(n-q-r)$, respectively, and involve $2r$ eigenvalues and $(n-q-r)$ eigenvalue remainder terms, respectively. The transformation with T_3 can be omitted ($T_3=1$) if the structuring of matrix E reveals that $E_2\neq 0$.

Example 3

Given

Linear time-invariant dynamical plant

$$\dot{x} = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{array}\right] x + Bu$$

$$y = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] x$$

Objective

a) Observer matrix G;
b) System observer matrix D.

Step 1

Determination of the transformation matrix T_1 using eqn (7). $T_1 = T_1^{-1} = I$, Rank $E = 1$

$$E = \begin{bmatrix} E_1 & E_3 \\ E_2 & E_4 \end{bmatrix} = \begin{array}{c} \quad E_1 \;\; E_3 \\ \left[\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] \\ \quad E_2 \;\; E_4 \end{array} .$$

With $E_2\neq 0$ is obtained $T_2 = T_3 = I$. With $E_2^{-1}E_4 = [0 \quad 0]$ is obtained $T_4 = T_4^{-1} = I$.

Step 2

System transformation to eqns (38) and (39).

$$\dot{\gamma} = \begin{array}{c} \begin{array}{cc} \bar{C}_1 & \bar{E} \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{cc} \bar{C}_2 & \bar{V} \end{array} \end{array} \gamma + \bar{B}u$$

$$y = \begin{array}{c} \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cc} \bar{H}_1 & \bar{H}_2 \end{array} \end{array} \gamma$$

Step 3

Dynamical determination system observer matrix D using eigenvalues.

$$J = \begin{bmatrix} J_0 & 0 & 0 \\ 0 & J_1 & R \\ 0 & 0 & J_2 \end{bmatrix} \qquad \begin{array}{c} J_0 = \lambda_1 = -1 \\ R = I \end{array}$$

$$J_1 = \begin{array}{c} \lambda_2 = -2 \\ \lambda_3 = -3 \end{array} \qquad J_2 = \begin{array}{c} \lambda_4 = -4 \\ \lambda_5 = -5 \end{array}$$

Step 4

Calculate matrix g_2 using eqn (48).

$$\dot{\alpha} = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 1 \end{array}\right] \alpha; \quad y = [1|0]\alpha.$$

The system calculation is done by system/output combination $n = 2q$.

$$\lambda_4 = -4, \quad \lambda_5 = -5, \quad R = 1$$

$$M_1 = 1,\ M_2 = -6,\ \Delta_1 = -10, \quad \Delta_2 = -30$$

$$g_2 = \begin{bmatrix} 10 \\ 31 \end{bmatrix} \quad \bar{D}_3 = \begin{bmatrix} -10 & 1 \\ -30 & 1 \end{bmatrix}$$

Step 5

Determine matrices $\bar{V}_5$ and g_1 using eqns (51) and (52).

Step 6

Determine matrices $\bar{\Delta}_1, \bar{\Delta}_2$ using eqns (45)–(46).

$$\bar{D}_2 = \begin{bmatrix} \bar{M}_1 & \bar{E}_2 \\ \bar{M}_2 & \bar{V}_5 \end{bmatrix} = \begin{bmatrix} \bar{M}_1 & 1 \\ \bar{M}_2 & 11 \end{bmatrix} \quad \begin{matrix} \lambda_2 = -2 \\ \lambda_3 = -3 \end{matrix}$$

$M_1 = 1,\ M_2 = -14,\ \bar{M}_1 = -16,\ \bar{M}_2 = -182$

$$\bar{\Delta}_1 = -16,\ \bar{\Delta}_2 = -104 \quad \bar{\Delta} = \begin{bmatrix} -272 \\ -464 \end{bmatrix}.$$

Step 7

Calculate matrices G and D using eqns (41) and (42).

$$G = \begin{bmatrix} 2 & 0 \\ 0 & 16 \\ 1 & 105 \\ 1 & 272 \\ 1 & 465 \end{bmatrix} \quad D = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & -16 & 1 & 0 & 0 \\ 0 & -104 & 1 & 1 & 0 \\ 0 & -272 & 1 & 0 & 1 \\ 0 & -464 & 0 & 1 & 1 \end{bmatrix}$$

Step 8

Proof: $\det(\lambda_1 - D) = 0$, *i.e.*, the system observer matrix shows the dynamical behaviour demanded.

5. Conclusion

The design procedure developed represents a novel reconstruction theorem for the deterministic design of complete state observers for multivariable control systems not requiring any transformation into canonical forms, as well as for system developments into single systems. Furthermore, all eigenvalue forms can be used without any exception.

The methodical basis of the procedure is a transformation of the time-invariant linear process model given in state-space representation, the actual design procedure, as well as a subsequent inverse transformation. Under certain conditions this design cycle will have to be done repeatedly.

The observer design for a system of complete order is realized by the observer design for a system of reduced order in a recursive manner. Respective observer laws can be represented.

It is important to note that it is not necessary to check, for complete observability, the system given by (1) through the investigation of the rank of the corresponding $n \times nq$ matrix.

If (A, H) is a completely observable pair, the recursive simplification will terminate in one of the two special cases discussed in §§4.1. and 4.2. All the eigenvalues can be arbitrarily specified. All the transformations within the procedure are very simple.

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