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# Fluid flow induced by a travelling thermal wave in a saturated porous medium

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**Abstract**

Axisymmetric free convection due to a travelling thermal wave imposed on the circumference of a long vertical column of a fluid-saturated porous medium with circular cross-section is studied analytically. Solutions for both the velocity and temperature fields are obtained using the long-wave approximation The medium is assumed to be highly porous and the fluid saturating the medium Boussinesq incompressible. The interaction of the first-order fluctuations gives rise to second-order mean in the velocity and temperature fields for which analytical solutions are obtained.

**Key words:** Free convection, porous medium, vertical column, thermal wave, mean flow

## 1. Introduction

Free-convection flow and heat transfer in enclosures filled with porous materials is one of the contemporary subjects of study owing to its intrinsic importance in various fields of geological and geophysical interest. Consequently for such problems of flow and heat transfer, several investigators have presented analytical, numerical and experimental results for many of the fundamental geometries[1–7].

Fluid flow induced by a moving source of heat in the form of a travelling thermal wave was investigated both analytically and experimentally by Whitehead[8]. By postulating series expansions in the square of the aspect ratio (assumed small) for both the temperature and flow fields, Whitehead obtained an analytical solution for the mean flow produced by a moving source. Theoretical predictions regarding the ratio of the mean flow velocity to the source speed were found to be in good agreement with experimental observations in mercury which therefore justified the validity of the asymptotic expansions *a posteriori*.

Following Whitehead[8], Nanda and Purushothaman[9] made a linearised analysis of the free-convection flow induced by a travelling thermal wave on the circumference of a long vertical circular cylindrical pipe filled with a thermally conducting viscous fluid and obtained solutions for the velocity and temperature fields using the long-wave approximation.

In the present work, we investigate yet another problem of fundamental interest, namely, the unsteady flow induced by a travelling thermal wave imposed on the circumference of a long vertical cylindrical column of a fluid-saturated porous medium with a circular cross-section. The medium is assumed to be homogeneous and isotropic and of porosity close to unity; for instance, the xylem of a plant containing the pith cells. As the medium is highly porous, non-Darcian phenomenon is predominantly felt near the boundary[10], and therefore we consider the generalised equation of Darcy's law taking into account both the viscous and inertia effects. Assuming the wave to be sinusoidal, we carry out the analysis for the case of both incoming and outgoing waves, and following Whitehead[8], we seek a perturbation solution using the long-wave approximation. Owing to nonlinear interactions of the lower-order effects a secondary mean flow is encountered for which an analytical solution is obtained. The interest in this topic is motivated in part by the growing emphasis on the possible convective transport processes in highly porous media like the pappus of dandelion and fibres when they are exposed to periodic fluctuations in the temperature.

## 2. Mathematical formulation

We consider the free convection flow through a porous medium in a long vertical cylinder of circular cross-section due to a travelling thermal wave imposed on its boundary. We assume the boundary to be impermeable and the wave to be sinusoidal of amplitude $\Delta t$, wave length $2\pi/k$ and frequency, $\omega$. The fluid in Boussinesq incompressible with the density temperature relation,

$$\rho = \rho_0(1-\beta T), \tag{1}$$

where $\rho$ is the fluid density, $\rho_0$ its value in the reference state, $\beta$ the thermal expansion coefficient and $T$ the temperature.

A cylindrical polar coordinate system $(r, \varphi, z)$ is chosen with the $z$-axis as axis of symmetry and in the direction opposite to that of the gravity vector. On account of axial symmetry neither $\varphi$ nor the $\varphi$-component of velocity appears in the analysis. Then, neglecting the compressibility effects of the fluid, the equations for the conservation of mass, momentum and energy in the medium in the absence of dispersion effects are:

$$\frac{\partial}{\partial r}(ur)+\frac{\partial}{\partial z}(wr)=0, \tag{2}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{u}{r}\right)+\frac{\partial^2 u}{\partial z^2}\right)-\nu\frac{u}{K}, \tag{3a}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right)-g(1-\beta T)-\nu\frac{w}{K}, \tag{3b}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}=\alpha\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right), \tag{4}$$

where $p$ is the pressure, $\nu$ the kinematic viscosity, $K$ the medium permeability, $\alpha$ the effective thermal diffusivity, $t$ the time and $u$ and $w$ are the velocity components in the increasing directions of $r$ and $z$, respectively. It is further assumed that both the medium and the saturating fluid are in thermal equilibrium.

The boundary conditions necessary for the completion of the mathematical formulation are

$$u=0, w=0,\ T=\Delta T\sin(kz+\omega t)\ \text{on}\ r=a, u, w,\ T\ \text{finite at}\ r=0. \tag{5}$$

We take advantage of the continuity equation (2) to define the stream function $\psi$ such that

$$u=r^{-1}\delta\psi/\delta z, w=-r^{-1}\partial\psi/\partial r, \tag{6}$$

and eliminate the pressure terms in the momentum equations through cross differentiation. Following Whitehead[8], we introduce the non-dimensional variables $r', z', t', \psi', T'$:

$$r'=r/a, z'=zk, t'=(\nu k^2)t, \psi'=\psi/ak\nu^2,\ T'=T/\Delta T \tag{7}$$

and substitute into (3) and (4) to obtain for the conservation of momentum and energy (after dropping the primes),

$$\left[\lambda^2\left(\frac{\partial}{\partial t}+\frac{1}{r}\left(\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}-\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\right)-\frac{1}{r^2}\frac{\partial\psi}{\partial z}+\frac{1}{K'}\right)-\nabla_1^2+\frac{1}{r^2}\right]$$
$$\times\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{\lambda^2}{r}\frac{\partial^2\psi}{\partial z^2}\right]=-G\frac{\partial T}{\partial r}, \tag{8}$$

$$\lambda^2 P\left[\frac{\partial T}{\partial t}+\frac{1}{r}\frac{\partial(T,\psi)}{\partial(r,z)}\right]=\nabla_1^2 T, \tag{9}$$

where

$$\nabla_1^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\lambda^2\frac{\partial^2}{\partial z^2}, \tag{10}$$

$K'$ is the permeability parameter $(Kk^2)$, $\lambda$ the aspect ratio $(ak)$, $P$ the Prandtl number $(\nu/\alpha)$ and $G$ the Grashof number $[=(\beta a^2 g/k\nu^2)\Delta T]$. In this formulation, the boundary conditions reduce to

$$\frac{\partial\psi}{\partial r}=0,\ \frac{\partial\psi}{\partial z}=0,\ T=\sin(z+Vt)\ \text{on}\ r=1, \tag{11a}$$

$$\frac{1}{r}\frac{\partial\psi}{\partial z}, \frac{1}{r}\frac{\partial\psi}{\partial r},\ T\ \text{finite at}\ r=0, \tag{11b}$$

where $V(=\omega/\nu k^2)$ is the non-dimensional form of the velocity of propagation. Positive and negative $V$s represent, respectively, an incoming and an outgoing wave. While the boundary condition (11b) ensures the regularity of the flow along the axis of the pipe, the continuity requirements insist that the total mass flux across any normal section must be a constant. This then implies

$$\int_0^{2\pi}\int_0^1 rw\,\mathrm{d}r\,\mathrm{d}\varphi = \text{constant},$$

yielding one additional condition, namely,

$$\psi(1,z,t) = \text{constant}, \tag{12}$$

which we shall denote by $Q$.

## 3. Results and discussion

In view of the boundary conditions and the assumption of the validity of long-wave approximation ($\lambda < 1$), we seek solutions for $\psi$ and $T$ as suggested by Whitehead[8] in the form

$$\psi = \psi_0 + \lambda^2\psi_1 + \lambda^4\psi_2 + \ldots,$$
$$T = \theta_0 + \lambda^2\theta_1 + \lambda^4\theta_2 + \ldots. \tag{13}$$

Substituting (13) into (8) and (9) and collecting terms of equal powers in $\lambda$, we obtain the equations for the solutions of the functions $\psi_i$ and $\theta_i (i = 0, 1, 2, \ldots)$. The appropriate boundary conditions are obtained from (11) and (12) with the help of (13).

### 3.1. *Zeroth-order solution*

The functions $\psi_0$ and $\theta_0$ are found from the solutions of the equations

$$L\psi_0 = G\frac{\partial\theta_0}{\partial r};\quad \frac{\partial^2\theta_0}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_0}{\partial r} = 0, \tag{14}$$

where $L$ is the differential operator defined by

$$L = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right]\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\right\}. \tag{15}$$

The appropriate boundary conditions are

$$\frac{\partial\psi_0}{\partial r} = 0,\ \frac{\partial\psi_0}{\partial z} = 0,\ \theta_0 = \sin(z + Vt) \text{ on } r = 1, \tag{16a}$$

$$\frac{1}{r}\frac{\partial\psi_0}{\partial z}, \frac{1}{r}\frac{\partial\psi_0}{\partial r},\ \theta_0 \text{ finite at } r = 0, \tag{16b}$$

$$\psi_0(1,z,t) = Q. \tag{16c}$$

System (14)–(16) yields,

$$\psi_0 = Q(2r^2 - r^4), \theta_0 = \sin(z + Vt). \tag{17}$$

As expected, the zeroth-order solutions do not involve the permeability parameter $K'$. Further, the expression for $\psi_0$, being independent of $z$ and $t$, corresponds to Poiseuille flow due to the constant mass flux in the axial direction and it exists in the zeroth order only.

### 3.2. *First-order solution*

Substituting (13) into (8) and (9) and collecting the coefficients of $\lambda^2$, we obtain the equations for the determination of $\psi_1$ and $\theta_1$:

$$L\psi_1 = G\frac{\partial\theta_1}{\partial r} - \frac{8Q}{K'}r, \tag{18}$$

$$\frac{\partial^2\theta_1}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_1}{\partial r} = P\left[\frac{\partial\theta_0}{\partial t} + \frac{1}{r} + \frac{\partial(\psi_0, \theta_0)}{\partial(z, r)}\right] - \frac{\partial^2\theta_0}{\partial z^2}. \tag{19}$$

The associated boundary conditions are

$$\frac{\partial\psi_1}{\partial r} = 0,\ \frac{\partial\psi_1}{\partial z} = 0,\ \theta_1 = 0 \text{ on } r = 1, \tag{20a}$$

$$\frac{1}{r}\frac{\partial\psi_1}{\partial z},\ \frac{1}{r}\frac{\partial\psi_1}{\partial r},\ \theta_1 \text{ finite at } r = 0, \tag{20b}$$

$$\psi_1(1, z, t) = 0. \tag{20c}$$

We note that equation (20c) is a direct consequence of the fact that there is no mass flux at higher orders. From (18)–(20) we have

$$\psi_1 = -\frac{Q}{24K'}(r^6 - 2r^4 + r^2) + \frac{G}{1152}[P\{Q(r^8 - 12r^6 + 21r^4 - 10r^2) + 3V(r^6 - 2r^4 + r^2)\}\cos(z + Vt) + 3(r^6 - 2r^4 + r^2)\sin(z + Vt)] \tag{21}$$

and

$$\theta_1 = P\left[\frac{Q}{4}(r^4 - 4r^2 + 3) - \frac{V}{4}(1 - r^2)\right]\cos(z + Vt) - \frac{1}{4}(1 - r^2)\sin(z + Vt). \tag{22}$$

The expression for $\theta_1$ consists of two parts: the first relating to free convection and the second to pure diffusion only. On the other hand, the first term in the expression for $\psi_1$ represents the mean mass flux which is essentially due to the porous medium whereas the second term is due to the thermal forcing.

The vertical velocity fluctuations, caused by the buoyancy force, induce fluctuations in the radial velocity also and are found to be not in phase with the forcing mechanism. These phase differences, which are functions of $r$ only, are necessary to support the mean flow in the fluid. As the interaction of the first-order fluctuations in the stream function and temperature gives rise to the second-order mean in the velocity and temperature fields, the coefficients of $\lambda^4$ in (13) can be split into two parts, one dealing with the second-order mean and the other, the unsteady second harmonic solution[11,12]

Thus if $\psi_{2m}$ and $\theta_{2m}$ denote the second-order mean in the stream function and temperature, respectively, then

$$L\psi_{2m}=G\frac{\partial\theta_{2m}}{\partial r}-\frac{Q}{3K'^2}(3r^3-2r), \tag{23}$$

$$\frac{\partial^2\theta_{2m}}{\partial r^2}+\frac{1}{r}\frac{\partial\theta_{2m}}{\partial r}=-\frac{GP^2}{2304}[Q(8r^6-72r^4+84r^2-20)+3V(6r^4-8r^2+2)], \tag{24}$$

$$\psi_{2m}=0,\ \frac{\partial\psi_{2m}}{\partial r}=0,\ \theta_{2m}=0 \text{ on } r=1,\ \frac{1}{r}\frac{\partial\psi_{2m}}{\partial r},\ \theta_{2m} \text{ finite at } r=0. \tag{25}$$

From (23)–(25) we obtain

$$\begin{aligned}\psi_{2m}= &-\frac{Q}{1152K'^2}(r^8-4r^6+5r^4-2r^2)\\ &-\frac{G^2P^2}{4423680}\left[\frac{Q}{5}(r^{12}-30r^{10}+175r^8-500r^6+590r^4-236r^2)\right.\\ &\left.+\frac{V}{2}(3r^{10}-20r^8+60r^6-72r^4+29r^2)\right],\end{aligned} \tag{26}$$

and

$$\theta_{2m}=(GP^2/18432)[Q.H(r)+V.J(r)], \tag{27}$$

where

$$H(r)=-(r^8-16r^6+42r^4-40r^2+13),$$

$$J(r)=4(1-r^2)^3.$$

The mean axial velocity is now given by

$$w_{2m}=\frac{1}{288}[QK'^{-2}X(r)+G^2P^2\{Q.Y(r)+V.Z(r)\}], \tag{29}$$

where,

$$\begin{aligned}X(r)&=2r^6-6r^4+5r^2-1,\\ Y(r)&=(3r^{10}-75r^8+350r^6-750r^4+590r^2-118)/19200,\\ Z(r)&=(15r^8-80r^6+180r^4-144r^2+29)/15360.\end{aligned} \tag{30}$$

In the limit $K'\to\infty$, we recover the results obtained by Nanda and Purushothaman[9]. Graphs of the functions $X$, $Y$ and $Z$ are plotted in fig. 1 from which one can observe that the functions $X$ and $Y$ are both negative near the axis of the pipe and positive in the core region near the boundary, and that they are not affected by the phase velocity. On the other hand, the function $Z$ is strongly dependent on the direction of $V$ and it plays a dominant role in the assessment of the magnitude of $w_{2m}$. When the wave is incoming the magnitude of $w_{2m}$ is suppressed by $Z$, whereas it helps in increasing the magnitude of $w_{2m}$ when the wave is outgoing, so that the mean flow is more rapid when the wave is outgoing than in the opposite case. However, $w_{2m}$ is negative near the axis of the pipe and positive in the core region near the boundary so that it tends to zero somewhere midway between the axis and the boundary and on the side opposite to that of the axis. When the wave is incoming, back

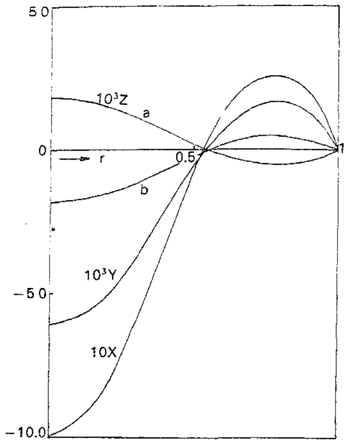


FIG. 1 Graphs of the functions $X$, $Y$, $Z$; (a) incoming wave: $10^3 Z$, (b) outgoing wave: $10^3 Z$.

flow sets in this core region near the boundary, whereas the mean flow near the boundary is in the same direction as that of the travelling wave when the wave is outgoing. Physically this is meaningful, since the heated fluid is displaced upwards by the ambient fluid from approximately its own level due to density variations in this buoyancy-driven thermal convection. One of the striking features of the flow is that for smaller values of $K'$, $w_{2m}$ is large. In fact, lower the value of $K'$ is, higher is the mean velocity $w_{2m}$. This is physically possible, because, when the permeability of the porous medium decreases, the medium becomes less porous and hence the flow resistance decreases. Lastly we note that the mean flow will be absent if the thermal wave is stationary and the imposed flow rate is zero. In the absence of mass flow rate, the mean axial velocity is independent of the parameter $K'$ and is directly proportional to the square of the amplitude of the thermal forcing which is the same as obtained by Nanda and Purushothaman[9].

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## References


1. BURNS, P J., CHOW, L. C. AND TIEN, C. L. — Convection in a vertical slot filled with porous insulation, *Int. J. Heat Mass Transfer*, 1977, **20**, 919–926.
2. BURNS, P. J AND TIEN, C. L. — Natural convection in porous media bounded by concentric spheres and horizontal cylinders, *Int. J. Heat Mass Transfer*, 1979, **22**, 929–939.
3. SIMPKINS, P. G. AND BLYTHE, P. A — Convection in a porous layer, *Int. J. Heat Mass Transfer*, 1980, **23**, 881–887.
4. VASSEUR, P., HUNG NGUYEN, T., ROBILLARD, L. AND TONG THI, V. K — Natural convection between horizontal concentric cylinders filled with a porous layer with internal heat generation, *Int. J. Heat Mass Transfer*, 1984, **27**, 337–349.
5. BAU, H. H. AND TORRANCE, K. E. — Low Rayleigh number thermal convection in a vertical cylinder filled with porous materials and heated from below, *ASME Trans. J. Heat Transfer*, 1982, **104**, 166–172.
6. PRASAD, V AND KULACKI, F. A. — Natural convection in porous media bounded by short concentric vertical cylinders, *ASME Trans. J. Heat Transfer*, 1985, **107**, 147–154.
7. PHILIP, J. R — Axisymmetric free convection at small Rayleigh number in porous cavities, *Int. J. Heat Mass Transfer*, 1982, **25**, 1689–1699.
8. WHITEHEAD, J. A. — Observations of rapid mean flow produced in mercury by a moving heater, *Geophys Fluid Dynamics*, 1972, **3**, 161–180.
9. NANDA, R. S. AND PURUSHOTHAMAN, R. — The rotating flame problem for a circular pipe, *Int. Dedication Seminar on Recent Advances in Maths and Applications*, Varanasi, Dec. 1976
10. RAMACHARYULU, N CH. P. — Annular effect in porous media, *Indian J. Pure Appl. Math.*, 1979, **10**, 761–762.
11. ROSENHEAD, L. — *Laminar boundary layers*, 1963, p. 383, Oxford University Press.
12. TELIONIS, D. P. — *Unsteady viscous flows*, 1981, p. 194, Springer Verlag.