

An external crack in a semi-infinite cylinder

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Abstract

The problem of an external crack in a semi-infinite circular cylinder has been investigated under a special type of loading of the curved boundary of the cylinder. The problem has been solved through the approach of dual series relations involving Bessel functions and their reduction to a system of Fredholm integral equations. The resulting Fredholm equations have been solved numerically and the results have been presented in the form of graphs for various values of the parameters involved.

Key words: External crack, stress-intensity factor, dual series relations, Fredholm integral equations.

1. Introduction

The use of integral equation methods in mixed boundary-value problems of elasticity theory forms a wide topic in applied mathematics. A special class of axisymmetric problems, in the cylindrical coordinate system, is known to result in the problem of solving a dual integral equation or a dual series relation (see Sneddon¹), involving Bessel functions of the first kind. Various authors^{2–8} have tackled a variety of axisymmetric problems of elasticity theory and viscous flow theory, by using the method of reduction to dual integral equations and allied ideas. In the area of fracture mechanics, the problems of internal and external cracks in half-spaces have been tackled by Srivastava and Singh⁹ and Dhawan¹⁰, respectively, and it is observed from the works of Sneddon and Tait¹¹, Chakrabarti¹², Sneddon and Srivastav¹³ and Srivastav¹⁴ that there exists a certain amount of mathematical complexity to handle similar axisymmetric problems associated with cylinders of finite radius.

In the present paper, we have demonstrated the application of dual series relations to an axisymmetric mixed boundary-value problem of elasticity theory associated with an external crack in a semi-infinite cylinder whose flat end is stress free and the curved boundary is constrained in a particular manner. Using standard notations^{11,12,15} the boundary

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conditions of the problem are given by:

$$\sigma_{zz} = 0 = \sigma_{rz}, \quad \text{on } z = -h, \quad \text{for } 0 \leq r \leq a \quad (1.1)$$

$$u_r = 0 = \sigma_{rz}, \quad \text{on } r = a, \quad \text{for } -h \leq z < \infty \quad (1.2)$$

$$\sigma_{zz}(r, 0^-) = \sigma_{zz}(r, 0^+) = -p(r), \quad \text{for } 1 \leq r \leq a \quad (1.3)$$

and

$$\sigma_{rz}(r, 0^-) = \sigma_{rz}(r, 0^+) = 0, \quad \text{for } 1 \leq r \leq a \quad (1.4)$$

where u_r , u_θ and u_z represent the components of displacement and σ_{rr} , σ_{zz} , etc., represent the components of stress in the cylindrical polar coordinate system (r, θ, z) with its origin lying on the plane of the crack situated at a distance $h(>0)$ from the flat end of the semi-infinite circular cylinder of radius a , the crack itself occupying the region $1 < r < a$ of the plane $z = 0$.

The above problem will be solved here by using an approach similar to that employed by Sneddon and Tait¹¹ and Chakrabarti¹² involving the application of dual-series relations to such boundary-value problems where the problems of an internal penny-shaped crack in an infinite and a semi-infinite circular cylinder, respectively, have been tackled. By following the procedure of Sneddon and Srivastav¹³ and Srivastav¹⁴, we have reduced the present problem to that of solving a system of Fredholm integral equations. After showing the connection of the solution of these equations with the quantities of practical interest to workers of fracture mechanics, we have solved these equations numerically and have presented the solution in the form of graphs for various values of parameters.

2. Reduction to a system of Fredholm integral equations

We start with the following representations of the biharmonic axisymmetric stress function $\psi(r, z)$ (see Chakrabarti¹² and Love¹⁵) in the two regions $z > 0$ and $-h < z < 0$ ($0 < r < a$), respectively:

$$\psi(r, z) = \sum_{n=1}^{\infty} (A_n + B_n z) J_0(\xi_n r) \exp(-\xi_n z), \quad (\text{for } z > 0, 0 < r < a) \quad (2.1)$$

and

$$\begin{aligned} \psi(r, z) = \sum_{n=1}^{\infty} [C_n \cosh \xi_n(z+h) + D_n \sinh \xi_n(z+h) + E_n(z+h) \cosh \xi_n(z+h) \\ + F_n(z+h) \sinh \xi_n(z+h)] J_0(\xi_n r) \quad (\text{for } -h < z < 0, 0 < r < a) \end{aligned} \quad (2.2)$$

where $A_n - F_n$ are arbitrary constants to be determined with the help of boundary conditions (1.1)–(1.4) and ξ_n s are positive zeros of $J_1(\xi a)$, $J_\gamma(x)$ being the Bessel function of the first kind of order γ . The choice of the ξ_n s automatically satisfies the conditions (1.2), and the rest of the conditions in (1.1)–(1.4) lead to the following system of dual series relations (see Sneddon and Tait¹¹ and Chakrabarti¹²) for the two sets of unknowns A_n and G_n :

$$\sum_{n=1}^{\infty} (M_n A_n + N_n G_n) \xi_n^2 J_1(\xi_n r) = 0, \quad 0 < r < 1 \quad (2.3)$$

$$\sum_{n=1}^{\infty} (P_n A_n + Q_n G_n) \xi_n^2 J_0(\xi_n r) = 0, 0 < r < 1 \quad (2.4)$$

$$\sum_{n=1}^{\infty} [A_n + (1 - 2\eta) G_n] \xi_n^3 J_0(\xi_n r) = -p(r), 1 < r < a \quad (2.5)$$

$$\sum_{n=1}^{\infty} (A_n - 2\eta G_n) \xi_n^3 J_1(\xi_n r) = 0, 1 < r < a \quad (2.6)$$

where η is Poisson's ratio of the material of the cylinder,

$$\begin{aligned} G_n &= B_n \xi_n^{-1} \\ y_n &= \xi_n h \end{aligned} \quad (2.7)$$

$$\left. \begin{aligned} M_n P_n &= \frac{-(2 - 2\eta)(1 \mp y_n \operatorname{cosech}^2 y_n + \coth y_n)}{y_n^2 \operatorname{cosech}^2 y_n - 1} \\ N_n &= -2\eta M_n - K_n, \\ Q_n &= K_n + (1 - 2\eta) P_n, \\ K_n &= \frac{(2 - 2\eta) y_n^2 \operatorname{cosech}^2 y_n}{y_n^2 \operatorname{cosech}^2 y_n - 1}. \end{aligned} \right\} \quad (2.8)$$

The procedure by which one arrives at the system of dual series relations (2.3)–(2.6) is laborious, but automatic and straightforward, if one makes use of the two representations (2.1) and (2.2) for the stress function in the two regions, expressions for the displacements and stresses in terms of ψ and looks at the boundary conditions (1.1)–(1.4).

We next reduce the above system of dual series relations into a system of Fredholm integral equations by an analysis that is similar to that of Sneddon and Srivastav¹³ and Srivastav¹⁴.

Setting

$$\begin{aligned} [A_n + (1 - 2\eta) G_n] \xi_n^2 &= T_n = R_n + R_n^{(1)}, \\ [A_n - 2\eta G_n] \xi_n^3 &= S_n \end{aligned} \quad (2.9)$$

and choosing

$$R_n^{(1)} = -\frac{2}{a^2 \xi_n^2 J^2(a \xi_n)} \int_1^a u J_0(\xi_n u) p(u) du \quad (2.10)$$

so that

$$\sum_{n=1}^{\infty} R_n^{(1)} \xi_n J_0(\xi_n r) = 0, \quad \text{for } 0 < r < 1 \quad (2.11)$$

and

$$\sum_{n=1}^{\infty} R_n^{(1)} \xi_n J_0(\xi_n r) = -p(r), \quad \text{for } 1 < r < a. \quad (2.12)$$

We observe that the system (2.3)–(2.6) is equivalent to the following system of dual relations

for the unknowns R_n and S_n :

$$\begin{aligned} & \sum_{n=1}^{\infty} \{ (2\eta P_n + Q_n) R_n + \xi_n^{-1} [(1 - 2\eta) P_n - Q_n] S_n \} J_0(\xi_n r) \\ & = - \sum_{n=1}^{\infty} (2\eta P_n + Q_n) R_n^{(1)} J_0(\xi_n r), (0 < r < 1) \\ & \sum_{n=1}^{\infty} \{ (2\eta M_n + N_n) R_n + \xi_n^{-1} [(1 - 2\eta) M_n - N_n] S_n \} J_1(\xi_n r) \\ & = - \sum_{n=1}^{\infty} (2\eta M_n + N_n) R_n^{(1)} J_1(\xi_n r), (0 < r < 1) \\ & \sum_{n=1}^{\infty} R_n \xi_n J_0(\xi_n r) = 0 = \sum_{n=1}^{\infty} S_n J_1(\xi_n r), (1 < r < a). \end{aligned} \quad (2.13)$$

The first two equations in the above system can further be reduced to

$$(4 - 4\eta) \sum_{n=1}^{\infty} R_n J_0(\xi_n r) - \sum_{n=1}^{\infty} X_n R_n J_0(\xi_n r) - \sum_{n=1}^{\infty} \xi_n^{-1} K_n S_n J_0(\xi_n r) = \phi_1(r), (0 < r < 1) \quad (2.14)$$

$$\begin{aligned} & - \sum_{n=1}^{\infty} K_n R_n J_1(\xi_n r) + (4 - 4\eta) \sum_{n=1}^{\infty} \xi_n^{-1} S_n J_1(\xi_n r) - \sum_{n=1}^{\infty} \xi_n^{-1} Y_n S_n J_1(\xi_n r) \\ & = \phi_2(r), (0 < r < 1) \end{aligned} \quad (2.15)$$

by making the following substitutions

$$\begin{aligned} X_n &= 4 - 4\eta - (2\eta P_n + Q_n) \\ Y_n &= 4 - 4\eta - [(1 - 2\eta) M_n - N_n] \end{aligned} \quad (2.16)$$

both of which tend to zero as $y_n \rightarrow \infty$ [cf. equations (2.8)], and writing $\phi_1(r)$ and $\phi_2(r)$, for the terms on the right of these equations, which take the forms [by the relation (2.10)]

$$\begin{aligned} \phi_1(r) &= \frac{2}{a^2} \int_1^a t p(t) \left[\sum_{n=1}^{\infty} \frac{(4 - 4\eta - X_n) J_0(\xi_n t) J_0(\xi_n r)}{\xi_n J_2^2(a \xi_n)} \right] dt \\ \text{and} \\ \phi_2(r) &= - \frac{2}{a^2} \int_1^a t p(t) \left[\sum_{n=1}^{\infty} \frac{K_n J_0(\xi_n t) J_1(\xi_n r)}{\xi_n J_2^2(a \xi_n)} \right] dt. \end{aligned} \quad (2.17)$$

Assuming the unknowns R_n and S_n to have the following forms, in terms of two unknown functions $g_1(t)$ and $g_2(t)$:

$$\begin{aligned} R_n &= \frac{2}{a^2 \xi_n J_2^2(a \xi_n)} \int_0^1 g_1(t) \cos \xi_n t \, dt \\ \text{and} \\ S_n &= \frac{2}{a^2 J_2^2(a \xi_n)} \int_0^1 \frac{g_2(t)}{t} \sin \xi_n t \, dt \end{aligned} \quad (2.18)$$

and following the procedure adopted by Sneddon and Srivastav¹³, Srivastav¹⁴ and

Chakrabarti¹⁶, we see that the last two of the equations (2.13) are automatically satisfied and the first two which have been rewritten as the equations (2.14) and (2.15) get transformed into the following system of Fredholm integral equations:

$$g_1(r) + \int_0^1 [L_1(r, t)g_1(t) + L_2(r, t)g_2(t)] dt = f_1(r) \tag{2.19}$$

and

$$g_2(r) + \int_0^1 [M_1(r, t)g_1(t) + M_2(r, t)g_2(t)] dt = f_2(r) \tag{2.20}$$

for $0 < r < 1$, where

$$L_1(r, t) = -\frac{4}{\pi a} + \frac{4}{a\pi^2} \int_0^\infty K_1(x) \left[\frac{1}{I_1(x)} \cosh\left(\frac{tx}{a}\right) \cosh\left(\frac{rx}{a}\right) - \frac{2}{x} \right] dx - \frac{1}{\pi a^2(1-\eta)} \sum_{n=1}^\infty \frac{X_n \cos(\xi_n t) \cos(\xi_n r)}{\xi_n J_2^2(a\xi_n)} \tag{2.21}$$

$$L_2(r, t) = -\frac{1}{\pi a^2(1-\eta)} \frac{1}{t} \sum_{n=1}^\infty \frac{K_n \sin(\xi_n t) \cos(\xi_n r)}{\xi_n J_2^2(a\xi_n)} \tag{2.22}$$

$$M_1(r, t) = -\frac{1}{\pi a^2(1-\eta)} \left[-\frac{1}{2} r^{1/2} \sum_{n=1}^\infty \frac{K_n \cos(\xi_n t) \cos(\xi_n r)}{\xi_n^{3/2} J_2^2(a\xi_n)} + \frac{1}{2r^{1/2}} \sum_{n=1}^\infty \frac{K_n \cos(\xi_n t) \sin(\xi_n r)}{\xi_n^{5/2} J_2^2(a\xi_n)} + r^{3/2} \sum_{n=1}^\infty \frac{K_n \cos(\xi_n t) \sin(\xi_n r)}{\xi_n^{1/2} J_2^2(a\xi_n)} \right] \tag{2.23}$$

$$M_2(r, t) = -\frac{1}{\pi a^2(1-\eta)} \frac{1}{t} \left[-\frac{1}{2} r^{1/2} \sum_{n=1}^\infty \frac{Y_n \sin(\xi_n t) \cos(\xi_n r)}{\xi_n^{3/2} J_2^2(a\xi_n)} + \frac{1}{2r^{1/2}} \sum_{n=1}^\infty \frac{Y_n \sin(\xi_n t) \sin(\xi_n r)}{\xi_n^{5/2} J_2^2(a\xi_n)} + r^{3/2} \sum_{n=1}^\infty \frac{Y_n \sin(\xi_n t) \sin(\xi_n r)}{\xi_n^{1/2} J_2^2(a\xi_n)} - \frac{2}{\pi t^{1/2}} [r^2 K_{1, 1/2, 1/2, 1/2}(r, t; a) + \frac{1}{2} r K_{1, 3/2, 1/2, -1/2}(r, t; a)] \right] \tag{2.24}$$

$$f_1(r) = \frac{1}{\pi a^2(1-\eta)} \int_1^a tp(t) \left[\sum_{n=1}^\infty \frac{(4-4\eta-X_n) J_0(\xi_n t) \cos(\xi_n r)}{\xi_n J_2^2(a\xi_n)} \right] dt \tag{2.25}$$

and

$$f_2(r) = -\frac{1}{\pi a^2(1-\eta)} \int_1^a tp(t) \left[r^{3/2} \sum_{n=1}^\infty \frac{K_n J_0(\xi_n t) \sin(\xi_n r)}{\xi_n^{1/2} J_2^2(a\xi_n)} + \frac{1}{2r^{1/2}} \sum_{n=1}^\infty \frac{K_n J_0(\xi_n t) \sin(\xi_n r)}{\xi_n^{5/2} J_2^2(a\xi_n)} - \frac{1}{2} r^{1/2} \sum_{n=1}^\infty \frac{K_n J_0(\xi_n t) \cos(\xi_n r)}{\xi_n^{3/2} J_2^2(a\xi_n)} \right] dt. \tag{2.26}$$

All notations appearing in the above equations have the same meanings as those utilised in the works of Sneddon and Srivastav¹³ and Srivastav¹⁴ and all series appearing there can be easily tested for their uniform convergence for $r, t \in (0, 1)$.

The equations (2.19) and (2.20) are the desired Fredholm integral equations for the mixed boundary-value problem posed through the conditions (1.1)–(1.4).

3. Quantities of physical interest

Using the expressions (2.10) and (2.18) as well as the expressions for displacements and stresses in terms of ψ (see Love¹⁵) and some standard results available in Sneddon's book¹, we obtain, after some lengthy but straightforward manipulations, the following results:

$$\sigma_{zz}(r, 0^+) = \frac{g_1(1)}{(1-r^2)^{1/2}} - \int_r^1 \frac{g_1'(t) dt}{(t^2-r^2)^{1/2}} \quad (0 < r < 1) \quad (3.1)$$

$$\sigma_{rz}(r, 0^+) = -\frac{d}{dr} \int_r^1 \frac{g_2(t) dt}{t(t^2-r^2)^{1/2}} \quad (0 < r < 1), \quad (3.2)$$

and

$$\begin{aligned} u_z(r, 0^+) = & \frac{-(1+\eta)}{E} \left[\frac{-4(1-\eta)}{a^2} \sum_{n=1}^{\infty} \frac{J_0(\xi_n r)}{\xi_n J_2^2(a\xi_n)} \int_1^a t p(t) J_0(\xi_n t) dt \right. \\ & + 2(1-\eta) \int_0^1 \frac{g_1(t) dt}{(r^2-t^2)^{1/2}} - \frac{4(1-\eta)}{a^2} \int_0^1 g_1(t) (a^2-t^2)^{1/2} dt \\ & \left. - \frac{4(1-\eta)}{a\pi} \int_0^1 g_1(t) \left\{ \int_0^{\infty} \frac{K_1(y)}{y I_1(y)} \cosh\left(\frac{ty}{a}\right) \left[2I_1(y) - y I_0\left(\frac{ry}{a}\right) \right] dy \right\} dt \right] \\ & \text{for } 1 < r < a, \end{aligned} \quad (3.3)$$

where, and even before $I_\nu(x)$ and $K_\nu(x)$ represent the modified Bessel functions of the first and second kind, respectively.

The most important quantities of practical interest such as the stress-intensity factors (K_1, K_2) at the tip of the external crack and the work done (W) in opening the crack are then given by the following relations:

$$\begin{aligned} K_1 = \lim_{r \rightarrow 1} [& \{2(1-r)\}^{1/2} \{\sigma_{zz}(r, 0); 0 < r < 1\}] \\ = & g_1(1), \end{aligned} \quad (3.4)$$

$$\begin{aligned} K_2 = \lim_{r \rightarrow 1} [& \{2(1-r)\}^{1/2} \{\sigma_{rz}(r, 0); 0 < r < 1\}] \\ = & g_2(1), \end{aligned} \quad (3.5)$$

and

$$W = 2\pi \int_1^a r p(r) u_z(r, 0) dr. \quad (3.6)$$

In order to facilitate the computation of the above quantities of practical interest, we have taken up a particular case of the crack being opened by a constant pressure p_0 and have presented in the next section, the numerical method of solution of the system of Fredholm

equations (2.19) and (2.20). Using these numerical solutions the stress-intensity factors K_1 and K_2 are presented in the form of graphs.

4. Numerical solution of integral equations: Case of constant pressure (p_0)

In order to solve the system of integral equations (2.19) and (2.20), numerically, we have reduced the system to a system of algebraic equations by a technique similar to that described by Srivastav and Narain¹⁷. The integrals involved in the kernels $L_{1,2}$ and $M_{1,2}$ have been evaluated by ten-point Gauss quadrature formula whilst the series appearing therein have been summed up by taking 10 terms. Then the integrals in the equations (2.19) and (2.20) have been replaced by Simpson's 11-point formula of the type:

$$\int_0^1 L_1(r, t) g_1(t) dt = \sum_{j=1}^{11} w_j L_1(r, t_j) g(t_j). \quad (4.1)$$

We have thus obtained a system of 22 algebraic equations for the 22 unknown quantities $g_1(r_i)$, $g_2(r_i)$, $i = 1, 2, \dots, 11$, where we have taken $r_i = (.1)(i - 1)$. The inversion of the resulting system of algebraic equations has then been carried out for

$$a = 1.6, \quad 2.0$$

and

$$\eta = 0.05, 0.45, \quad \text{by taking a number of values of } h. \quad (4.2)$$

Using the numerical solution thus obtained, of the integral equations (2.19) and (2.20), and using the relations (3.4) and (3.5), the variations in the stress-intensity factors K_1 and K_2 as functions of h , a and η are presented in the form of graphs (fig. 1-4).

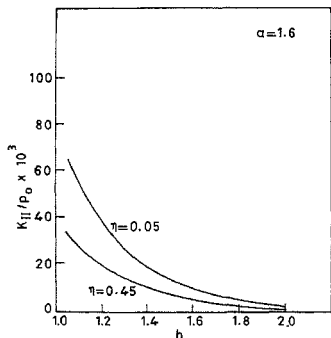


FIG. 1. Shear stress-intensity factor vs depth of the crack.

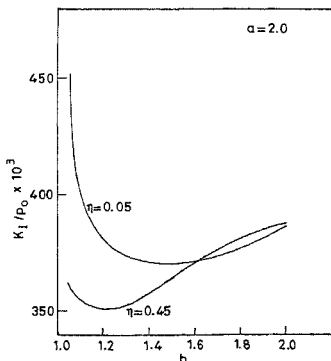


FIG. 2. Normal stress-intensity factor vs depth of the crack.

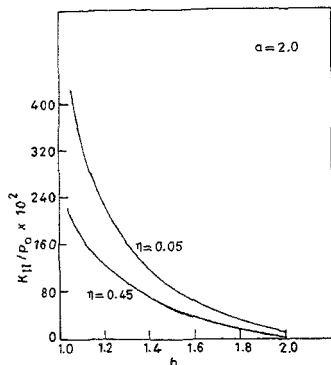


FIG. 3. Shear stress-intensity factor vs depth of the crack.

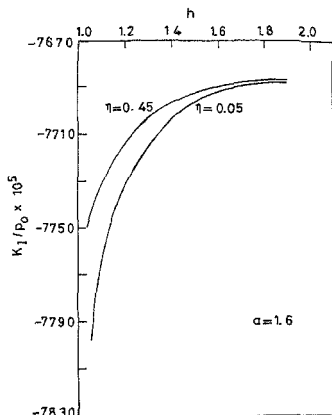


FIG. 4. Normal stress-intensity factor vs depth of the crack.

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