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Short Communication

Note on the torsion of a transversely isotropic half-space with variable modulus of elasticity

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Abstract

The object of the present note is to investigate the statical Reissner – Sagoci problem for a transversely isotropic half-space, in which the elastic coefficients are functions of depth. The solution of the problem has been reduced to the solution of Fredholm integral equation which requires numerical treatment. The expression for the applied torque has been obtained and some numerical results have been presented.

Key words: Transversely isotropic material, modulus of elasticity, torque, Fredholm integral equation.

1. Introduction

The problem of determining torsional deformation of a semi-infinite, isotropic, homogeneous, elastic solid when a circular cylinder is welded to its plane boundary and is forced to rotate about its axis is well-known[1–3]. For a nonhomogeneous, isotropic, elastic half-space in which the shear modulus is a function of depth or radial distance or both, similar problem had been solved by a number of investigators, as cited in the paper of Dhaliwal and Singh[4]. The solution for the Reissner–Sagoci problem for a transversely isotropic, elastic half-space in which the elastic coefficients are functions of radial coordinate was obtained by Ergüven[5]. The object of the present note is to study the Reissner–Sagoci problem for an elastic half-space of transversely isotropic material in which elastic coefficients increase with depth. The assumption of increasing elastic coefficients with depth is, indeed, practical, because of the expected increased rigidity due to overburden pressure as depth increases. Of course, the dependence may not be exponential in nature, but in our discussion we have assumed such a form with a view to handle complicated governing equations with rather limited mathematical tools. Although the problem is not general in this sense, it still gives information about elastic behaviour in nonhomogeneous media. The solution of the problem has been reduced to that of a solution of a dual integral equation. The applied torque required to produce a prescribed rotation has also been found. It is observed that the results for the associated homogeneous problem may be recovered from our results by setting the nonhomogeneity parameter zero. Finally, some numerical results have been presented in graphs and table to study the effect of variations of elastic coefficients on stress and torque.

2. Formulation and general solution of the problem

In our mathematical formulation of the problem, we shall use the cylindrical coordinates (r, θ, z) such that $z=0$ is the plane boundary of the half-space $z \geqslant 0$. Here, we assume that the nonhomogeneous half-space is acted on by a shearing force due to the rotation of a rigidly cemented circular shaft of radius a on $z=0$ through some given angle.

Since the problem concerned is axisymmetric in nature, the displacement vector in cylindrical coordinates (r, θ, z) has the form $[0, u(r, z), 0]$. Hence the non-vanishing stress components are

$$\sigma_{r\theta} = c_{66}\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right), \quad \sigma_{\theta z} = c_{44}\frac{\partial u}{\partial z} \tag{1}$$

where $c_{44} = c_{44}(z)$ and $c_{66} = c_{66}(z)$ are the variable moduli of rigidity whose dependence on the depth coordinate z is assumed as

$$c_{ii} = c_{ii}^0 \exp(\alpha z/a), \quad i = 4, 6; \quad \alpha > 0. \tag{2}$$

Setting $\xi = r/a$ and $\eta = z/a$ and noting (2), we get, on substituting (1) in the equation of equilibrium, the governing differential equation for the displacement u, as

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{u}{\xi^2} + \frac{1}{\xi}\frac{\partial u}{\partial \xi} + n^2\left(\alpha\frac{\partial u}{\partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right) = 0, \tag{3}$$

where $n^2 = c_{44}^0/c_{66}^0$.

The solution of equation (3) satisfying regularity condition may be written as

$$u(\xi, \eta) = \exp(-\alpha\eta/2)\int_0^\infty A(\lambda)\exp(-\beta\eta)J_1(\lambda\xi)\,\mathrm{d}\lambda \tag{4}$$

where $4\beta^2 = \alpha^2 + 4\lambda^2/n^2$.

The stress component $\sigma_{\theta z}$ is obtained from (1) as

$$\sigma_{\theta z}(\xi, \eta) = -c_{44}^0 \exp(\alpha\eta/2)\int_0^\infty \left(\frac{\alpha}{2} + \beta\right)A(\lambda)\exp(-\beta\eta)J_1(\lambda\xi)\,\mathrm{d}\lambda. \tag{5}$$

The boundary conditions of the problem are

$$u(\xi, 0) = \omega f(\xi), \quad 0 < \xi < 1 \tag{6}$$

and

$$\sigma_{\theta z}(\xi, 0) = 0, \quad \xi > 1 \tag{7}$$

where ω is a constant and $f(\xi)$ is a continuous function of ξ $(=r/a)$. Substituting (4) and (5) in (6) and (7) and writing

$$\lambda G(\lambda) = \left(\frac{\alpha}{2} + \beta\right)A(\lambda) \tag{8}$$

we find that $G(\lambda)$ satisfies the equations

$$\int_0^\infty \frac{\lambda}{\frac{\alpha}{2}+\beta} G(\lambda) J_1(\lambda\xi)\,\mathrm{d}\lambda = \omega f(\xi), \quad 0<\xi<1 \tag{9}$$

and

$$\int_0^\infty \lambda G(\lambda) J_1(\lambda\xi)\,\mathrm{d}\lambda = 0, \quad \xi>1. \tag{10}$$

Setting

$$G(\lambda) = \int_0^1 \varphi(\zeta)\sin\lambda\zeta\,\mathrm{d}\zeta \tag{10a}$$

we find that equation (10) is satisfied identically while equation (9) leads the Fredholm's integral equation for $\varphi(x)$, as

$$n\varphi(x) + \int_0^1 \varphi(\zeta) K^*(x,\zeta)\,\mathrm{d}\zeta = [2\omega/(\pi x)]\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\xi^2 f(\xi)\,\mathrm{d}\xi}{(x^2-\xi^2)^{1/2}}, \quad 0<x<1 \tag{11}$$

where

$$K^*(x,\zeta) = -(2/\pi)\int_0^\infty [n-\lambda/(\alpha/2+\beta)]\sin\lambda\zeta\sin\lambda x\,\mathrm{d}\lambda. \tag{12}$$

Now, in order to facilitate convergence of the infinite integral in (12), we define

$$g(\lambda\delta) = [n-\lambda/(\alpha/2+\beta)] - [n^2/(n/2+\lambda\delta)]$$

with $\delta=2/\alpha$, so that $g(\lambda\delta)$ is of order $(\lambda\delta)^{-3}$ for large $\lambda\delta$. It may be seen that

$$\int_0^\infty [\sin\lambda\zeta\sin\lambda x/(n/2+\lambda\delta)]\,\mathrm{d}\lambda$$
$$= (1/2\delta)\int_0^\infty [y/(y^2+m^2)]\,[e^{-y|x-\zeta|} - e^{-y(x+\zeta)}]\,\mathrm{d}y$$

where $m=n/2\delta$.

Thus, the kernel $K^*(x,\zeta)$ may be written as

$$K^*(x,\zeta) = -(2/\pi)\Bigg[\int_0^\infty g(\lambda\delta)\sin\lambda\zeta\sin\lambda x\,\mathrm{d}\lambda + mn\int_0^\infty \{y/(y^2+m^2)\}\{e^{-y|x-\zeta|} - e^{-y(x+\zeta)}\}\,\mathrm{d}y\Bigg]. \tag{13}$$

The torque T required to rotate the solid cylinder through some particular angle is given by

$$T = -2\pi\int_0^1 \xi^2 \sigma_{\theta z}(\xi,0)\,\mathrm{d}\xi. \tag{14}$$

Since, $J_1(\xi) = -J_0'(\xi)$, substituting the value of $\sigma_{\theta z}$ from (5) in (14), it is easy to verify that

$$T = 4\pi c_{44}^0 \int_0^1 \zeta\varphi(\zeta)\,\mathrm{d}\zeta. \tag{15}$$

The torque in a transversely isotropic, homogeneous medium may be obtained by putting $\alpha=0$ in (15) while that for an isotropic medium requires $c^0_{44}=c^0_{66}$.

Let us now find the surface displacement due to the torsional effect. We find from (4), that

$$u(\xi,0)=\int_0^\infty A(\lambda)J_1(\lambda\xi)\,\mathrm{d}\lambda.$$

Using (8) and (10a) and known integrals of Bessel functions[6], we obtain

$$u(\xi,0)=n\int_0^1 L(\xi,t)\varphi(t)\,\mathrm{d}t,\quad \xi>1$$

where

$$L(\xi,t)=\int_0^\infty (\lambda^2+4m^2)^{1/2}J_1(\lambda\xi)(\sin\lambda t/\lambda)\,\mathrm{d}\lambda-2mt/\xi.$$

The surface displacement $u_H(\xi,0)$ for the associated homogeneous case is obtained by putting $\alpha=0$, as

$$u_H(\xi,0)=n\int_0^1 [t/\{\xi(\xi^2-t^2)^{1/2}\}\varphi(t)\,\mathrm{d}t,\quad \xi>1.$$

3. Numerical results

For evaluation of the integral on the right-hand side of (11), specification of $f(\xi)$ is necessary. Thus, if the circular cylinder be rigidly fixed with the boundary and be rotated through an angle ω, we may take the displacement $u(\xi,0)$ as $u(\xi,0)=\omega\xi$. Hence, from (6), $f(\xi)=\xi$ and we have the equation for $\varphi(x)$ as

$$n\varphi(x)+\int_0^1 \varphi(\zeta)K^*(x,\zeta)\,\mathrm{d}\zeta=4\omega x/\pi,\quad 0<x<1. \tag{16}$$

To study the effect of nonhomogeneity on the torsional problem for a transversely isotropic half-space, we have considered two different radial distances, *viz.*, $\xi=0.1$ and $\xi=0.5$ and have computed the stress for different values of η. Numerical computations have been done on assuming the values of c^0_{ij} as those for β-quartz given by $c^0_{11}=11.66$, $c^0_{12}=1.67$, $c^0_{44}=3.61$. It follows that $c^0_{66}=\frac{1}{2}(c^0_{11}-c^0_{12})=4.995$. These values are in unit 10^{10} N/m^2 and have been taken from Ram and Parhi[7]. With the help of equations (13) and (16), the values of torque for different nonhomogeneity parameters are computed from (15) and are presented in Table I.

Table I
Variation of torque (T) with nonhomogeneity parameter (α)

α	0	0.1	0.3	0.5	0.7	0.9
$\pi T/(4\omega c^0_{44})$	4.927	5.002	5.153	5.269	5.403	5.450

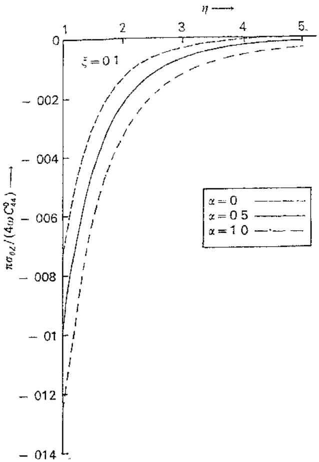

Fig. 1. Variation of stress with depth.

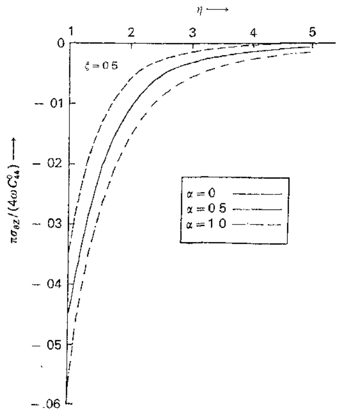

Fig. 2. Variation of stress with depth.

It is observed from Table I that torque value increases with the increase of the nonhomogeneity parameter.

Using equations (8) and (10a), the stress component $\sigma_{\theta z}$ may be evaluated from equation (5). The variations of stress component $\sigma_{\theta z}$ are shown in figs 1 and 2. The corresponding variations in the associated homogeneous medium ($\alpha = 0$) have been given in broken lines.

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