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PLANE COUETTE FLOW WITH SUCTION OR INJECTION AND HEAT TRANSFER

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Abstract

In this paper we have discussed the problem of suction and injection and of heat transfer in a plane Couette flow without imposing the condion of smallness on the suction parameter or such similar conditions on the Reynolds number to allow series solution. We have utilized some important properties of differential equations and some transformations which enable us to solve the two-point boundary value and eigenvalue problems without using the trial and error method. In fact, each integration provides us with a solution for a suction parameter and a Reynolds number. We beleive that the method outlined here can be easily adopted to a wide class of similar problems.

1. Introduction

In this paper we have discussed the problem of suction and injection and of heat transfer in a plane Couette flow without imposing the condition of smallness on the suction parameter or such similar conditions on the Reynolds number to allow series solution. We have utilized some important properties of differential equations and some transformations which enable us to solve the two-point boundary value and eigenvalue problems without using the trial and error method. In fact, each integration provides us with a solution for a suction parameter and a Reynolds number.

We believe that the method outlined here can be easily adapted to a wide class of similar problems. Besides, we have applied the suction or injection only on the fixed plate so that the usual boundary condition on the cross-flow, namely the injection at one plate is equal to the suction at the other, has not been employed.

2. BASIC EQUATIONS OF THE PROBLEM

Let the infinite plane $y = 0$ be stationary while the plane $y = a$ be moving with uniform velocity U_0 in the direction of the x-axis. We maintain these planes at constant temperatures T_0 and T_1 respectively. Moreover, uniform injection or suction with velocity $v = \pm v_0$ $(v_0 > 0)$ is applied on the plane $y = 0$, while the upper plane is non-porous. Here the plus sign refers to injection and the minus sign to suction.

Since we have taken the suction or injection to be uniform, we assume that the cross-velocity v is a function of y alone. We shall use the dimensionless variables u, v, x, y, p, θ for

$$\frac{u}{U_0}, \ \frac{v}{v_0}, \ \frac{x}{a R}, \ \frac{y}{a}, \ -\frac{p}{\rho U_0^2}, \ \frac{T - T_0}{T_1 - T_0}$$

and denote the suction parameter $v_0 a/\nu$, Reynolds number $U_0 a/\nu$, Prandtl number $\mu c_p/k$, Eckert number $U_0^2/[c_p (T_1 - T_0)]$ by λ, R, P and E respectively.

Thus the equations of the problem and the boundary conditions reduce to the following:

$$\frac{\partial u}{\partial x} + \lambda \frac{dv}{dy} = 0 \qquad [2.1]$$

$$u \frac{\partial u}{\partial x} + \lambda v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \qquad [2.2]$$

$$v \frac{dv}{dy} = -\frac{R^2}{\lambda^2} \frac{\partial p}{\partial y} + \frac{1}{\lambda} \frac{d^2 v}{dy^2} \qquad [2.3]$$

$$P \left[u \frac{\partial \theta}{\partial x} + \lambda v \frac{\partial \theta}{\partial y} \right] = \frac{1}{R^2} \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + EP \left[4 \frac{\lambda^2}{R^2} \left(\frac{dv}{dy} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] \qquad [2.4]$$

with

$$y = 0: \ u = 0, \ v = \pm 1, \ \theta = 0$$

$$y = 1: \ u = 1, \ v = 0, \ \theta = 1. \qquad [2.5]$$

Solution of the Flow Problem

3. From [2.1] we have

$$u(x, y) = -\lambda v'(y)\, x + u_0(y), \tag{3.1}$$

where $u_0(y)$ is an arbitrary function to be determined later. Equation [3.1] determines $u(x, y)$ in terms of u_0 and $v(y)$.

Since $u = 0$ at $y = 0$ and $u = 1$ at $y = 1$ for all values of x, we have the following boundary conditions to be satisfied by u_0 and v':

$$u_0(0) = 0, \quad v'(0) = 0,$$
$$u_0(1) = 1, \quad v'(1) = 0. \tag{3.2}$$

Similarly, the integration of the equation [2.3] gives us

$$p(x, y) = \frac{\lambda^2}{R^2}\left[-\frac{1}{2}v^2 + \frac{1}{\lambda}v'\right] + p_0(x), \tag{3.3}$$

where $p_0(x)$ is so far an arbitrary function to be determined later on. Equation [3.3] determines $p(x, y)$ in terms of $v(y)$ and $p_0(x)$.

Using [3.1] and [3.3] in [2.2] and concentrating on the powers of x that occur in the resulting equation, we find that we should take the following expression for $dp_0(x)/dx$:

$$-dp_0(x)/dx = A_0 + A_1 x, \tag{3.4}$$

where A_0 and A_1 are constants and then this equation breaks into the following two equations which are independent of x:

$$\lambda v(y)\, u_0'(y) - \lambda\, u_0(y)\, v'(y) = A_0 + u_0''(y), \tag{3.5}$$

$$\lambda^2 [v'(y)]^2 - \lambda^2 v(y)\, v''(y) + \lambda v'''(y) = A_1. \tag{3.6}$$

Equation [3.6] determines v for prescribed values of λ and A_1, while [3.5] then determines the value u_0 for prescribed value of A_0. Boundary conditions for [3.6] are

$$y = 0:\ v = \pm 1,\ v' = 0$$
$$y = 1:\ v = 0, \quad v' = 0, \tag{3.7}$$

while for [3.5] they are the following:

$$u_0(0) = 0,\ u_0(1) = 1. \tag{3.8}$$

We shall first concentrate on the equation [3.6] which is of third order and has to satisfy four boundary conditions [3.7]. Therefore, we shall treat this two point boundary value problem as an eigenvalue problem and determine the eigenvalue A_1 with the help of the fourth boundary condition. It is convenient to use the variable $Y = 1 - y$ so that we can start with two null conditions $v = 0$, $v' = 0$ at $Y = 0$. Further the transformation

$$v = \left[\frac{A_1^{1/4}}{\lambda}\right] V \quad \text{and} \quad Y = \left[\frac{1}{A_1^{1/4}}\right] \xi \qquad [3.9]$$

reduces the equation and boundary conditions to the following form:

$$(V')^2 - VV'' - V''' = \pm 1, \qquad [3.10]$$

with

$$\xi = 0: \qquad V = 0, \qquad V' = 0, \qquad [3.11]$$

$$\xi = \xi_0 = A_1^{1/4}: \; V = \pm \lambda / A_1^{1/4}, \; V' = 0. \qquad [3.12]$$

We have put down the $\pm$ sign on the right hand side of [3.10] in order to ensure that A_1 is positive in [3.12]. We start the integration at $\xi = 0$ with additional condition $V''(0) = m$ (say) and continue the integration till the condition $V' = 0$ is satisfied at some $\xi = \xi_0$ with $V(\xi_0) = V_0$ (say). Knowing ξ_0 and V_0, we determine the value of A_1 and λ from the boundary conditions [3.12] *viz.*,

$$A_1 = (\xi_0)^4 \text{ and } \lambda = \pm V_0 \, \xi_0. \qquad [3.13]$$

Thus we have obtained the solution of the differential equation [3.10] and the eigenvalue A_1 for the value $\pm V_0 \xi_0$ of the suction parameter λ. Different choices for the starting values of $V''(0)$ will give us solutions and eigenvalues for different suction parameters. We note that each integration with arbitrary choice of $V''(0)$ gives us a solution for some value of suction parameter and corresponding eigenvalue A_1. Since A_1 is not necessarily equal to zero, but a definite finite number for each value of suction parameter, we conclude that the suction or injection induces a pressure gradient which is dependent on x (*vide* equation [3.4]).

We shall now discuss the equation [3.5]. In order to avoid the specific assumption about the numerical value of the constant A_0 occuring in it, we make the following substitution:

$$u_0 = U + \lambda^2 v'(y) \qquad [3.14]$$

and work through the variable Y, instead of y, defined above. We thus have to solve the following two point boundary value equation:

$$(1/\xi_0) \, U'' + VU' - V'U = 0 \qquad [3.15]$$

with $$U(0)=1,\ U(1)=0, \tag{3.16}$$

provided we use

$$A_0 = -A_1\lambda\,. \tag{3.17}$$

Knowing the value of A_1, λ and ξ_0 from the integration of V-equation [3.10], we know A_0 and the coefficient of U'' in equation [3.15]. We start the solution at $Y=1$ with $U'(1)=n$ (say) and stop the integration at $Y=0$ giving us $U(0)=K$(say). Since the equation [3.15] is homogeneous linear equation in U, the solution (U/K) will satisfy the boundary condition $U(0)=1$ for the value of λ and A_0 determined above.

4. Solution of the Heat Transfer Problem

In this section we shall discuss the solution of the equation [2.4] with the boudary conditions given in [2.5]. If we substitute the value of $u(x,y)$ from equation [3.1] in [2.4] and concentrate on the powers of x that occur in the resulting equation, we find that we must take

$$\theta = \theta_0(y) + \theta_1(y)\,x + \theta_2(y)\,x^2. \tag{4.1}$$

If now, we equate the coefficients of various powers of x on the two sides of this resulting equation, we get the following three equations to be solved:

$$P[u_0\,\theta_1 + \lambda v\,\theta_0'] = (2/R^2)\,\theta_2 + \theta_0'' + EP[(4\lambda^2/R^2)(v')^2 + (u_0')^2] \tag{4.2}$$

$$P[2u_0\,\theta_2 - \lambda v'\,\theta_1 + \lambda v\,\theta_1'] = \theta_1'' + EP[-2\,\lambda v''\,u_0'] \tag{4.3}$$

$$P[-2\,\lambda v'\,\theta_2 + \lambda v\,\theta_2'] = \theta_2'' + EP\lambda^2\,(v'')^2 \tag{4.4}$$

with $$\theta_0(0) = \theta_1(0) = \theta_2(0) = 0$$

and $$\theta_0(1) = 1,\ \ \theta_1(1) = \theta_2(1) = 0. \tag{4.5}$$

We first consider the equation [4.4] in $\theta_2(y)$. We note that here we have to prescribe *apriori* the values of P and E, but the Reynolds number does not occur explicitly. Moreover λ and the corresponding values $v(y)$ and its derivatives are known to us. We write the equation in the form

$$\theta_2'' + P_1(y)\,\theta_2' + Q_1(y)\,\theta_2 = R_1(y), \tag{4.6}$$

where
$$P_1(y) = -\lambda P v$$
$$Q_1(y) = 2\lambda P v'$$
$$R_1(y) = -EP\lambda^2\,(v'')^2 \tag{4.7}$$

to be solved under two point boundary conditions

$$\theta_2(0) = 0, \ \theta_2(1) = 0. \tag{4.8}$$

Let $\theta_2 = \theta_a$ and $\theta_2 = \theta_b$ be two solutions [illegible] the [illegible] conditions

$$\theta_a(0) = 0, \ \theta_a'(0) = a\,(\text{say}). \tag{4.9}$$

$$\theta_b(0) = 0, \ \theta_b'(0) = b\,(\text{say}). \tag{4.10}$$

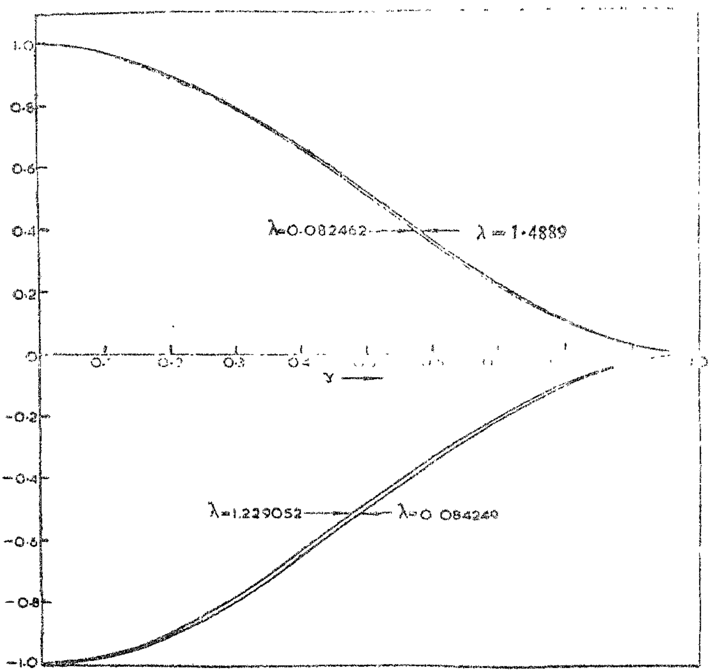

FIG. 1

Plot of v versus y

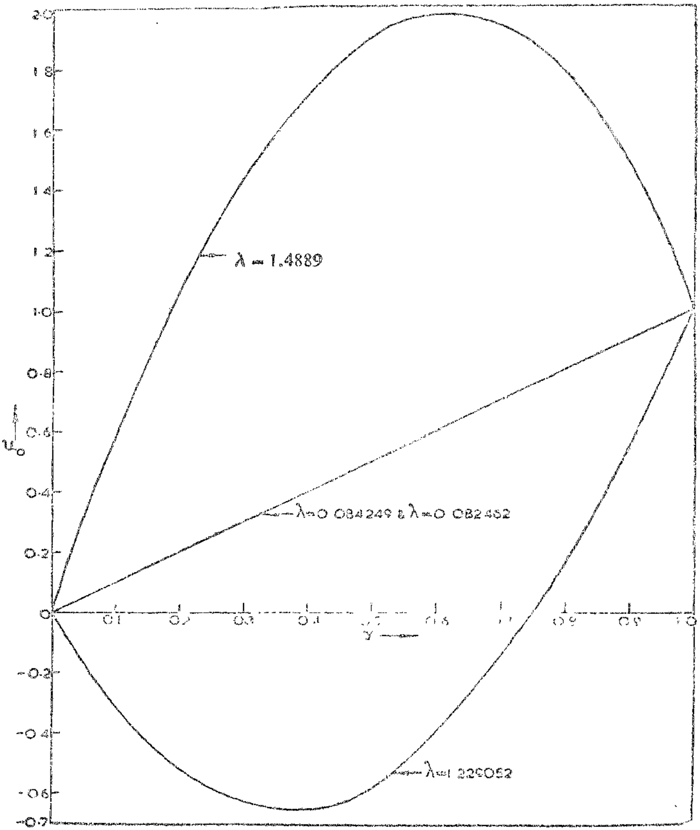

FIG. 2

Plot of u_0 versus y

Then we can easily check that

$$\theta_2(y) = \frac{\theta_b(1)}{\theta_b(1) - \theta_a(1)}\,\theta_a(y) - \frac{\theta_a(1)}{\theta_b(1) - \theta_a(1)}\,\theta_b(y) \qquad [4.11]$$

is the solution of the equation [4.4] [illegible] the [illegible] boundary conditions.

We note that this procedure allows us to solve the [illegible] boundary value problem without any trial and error.

We shall now consider the equation [4.3] in $\theta_1(y)$ which also does not contain R[illegible] number R. We write this equation in the form

$$\theta_1'' + P_2(y)\,\theta_1' + Q_2(y)\,\theta_1 = R_2(y) \qquad [4.12]$$

with boundary conditions

$$\theta_1(0) = 0, \quad \theta_1(1) = 0, \qquad [4.13]$$

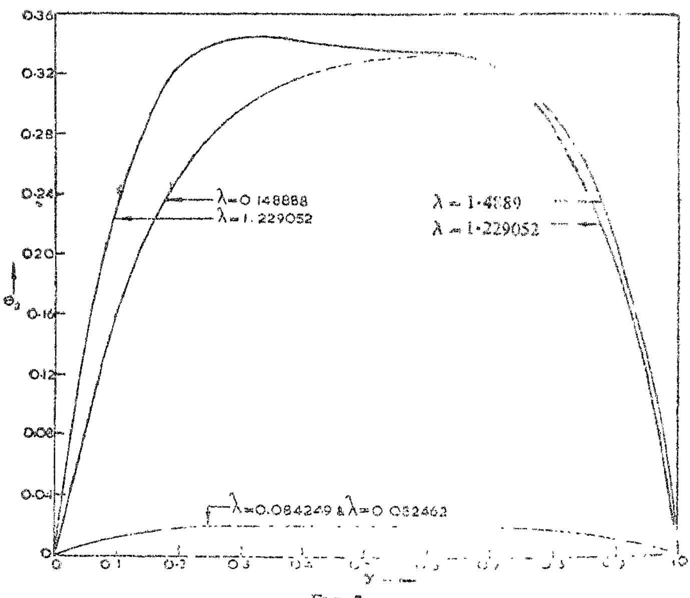

FIG. 3
Plot of θ_2 versus y

where

$$P_2(y) = -\lambda P v,$$
$$Q_2(y) = \lambda P v',$$
$$R_2(y) = 2\lambda E P v'' u_0' + 2 P u_0 \theta_2. \qquad [4.14]$$

We note that the coefficients P_2, Q_2 and R_2 are known to us from previous integrations for the chosen values of λ, P and E.

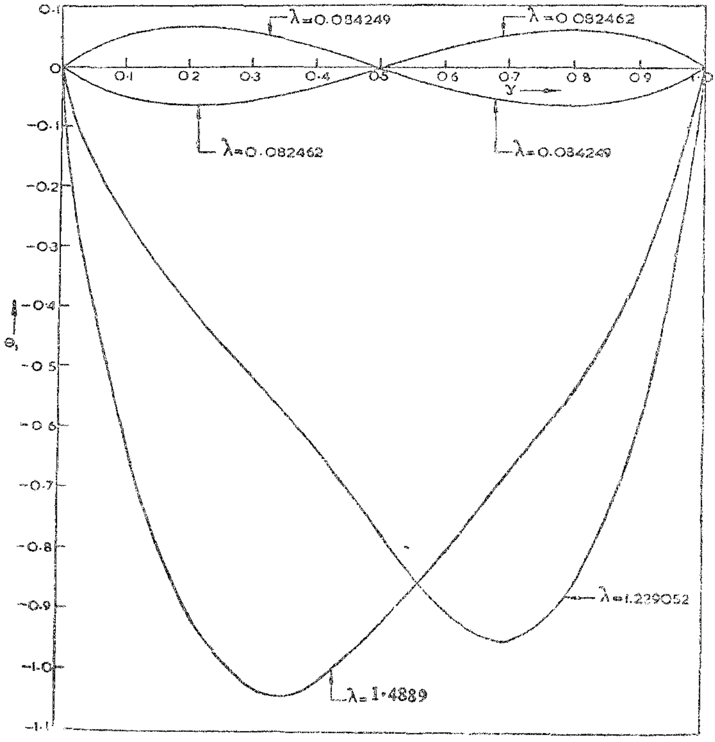

FIG. 4

Plot of θ_1 versus y

Proceeding as in the case of [illegible] equation, we find that the solution of this equation satisfying the prescribed boundary conditions is the following:

$$\theta_1(y) = \frac{\theta_B(1)}{\theta_B(1) - \theta_A(1)}\,\theta_A(y) - \frac{\theta_A(1)}{\theta_B(1) - \theta_A(1)}\,\theta_B(y), \qquad [4.15]$$

where $\theta_1 = \theta_A$ and $\theta_1 = \theta_B$ are the solutions of this equation satisfying the boundary conditions

$$\theta_A(0) = 0, \;\; \theta_A'(0) = A\,(\text{say}) \qquad [4.16]$$

$$\theta_B(0) = 0, \;\; \theta_B'(0) = B\,(\text{say}). \qquad [4.17]$$

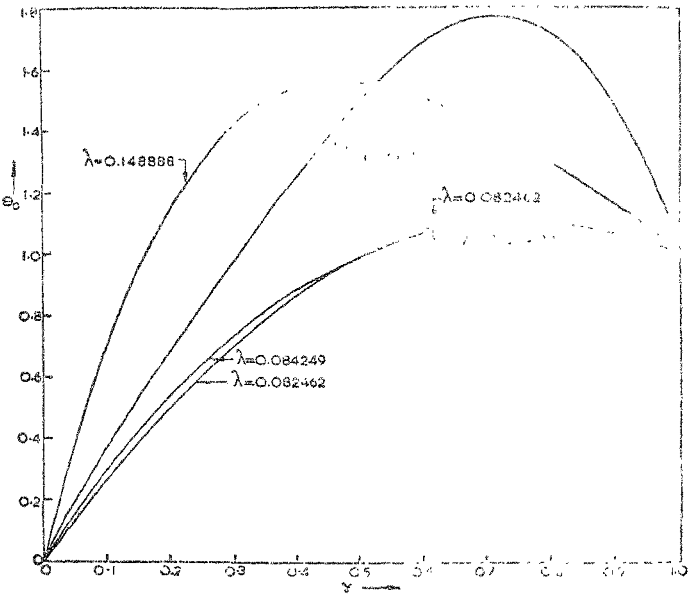

FIG. 5

Plot of θ_0 versus y

We write the equation [4 2] in the following form :

$$\theta_0'' + P_3(y)\,\theta_0' = R_3(y) \qquad [4.18]$$

with boundary conditions

$$\theta_0(0) = 0, \;\; \theta_0(1) = 1, \qquad [4.19]$$

where

$$P_3(y) = -\lambda P v$$
$$R_3(y) = -(2/R^2)\,\theta_2 + P u_0\,\theta_1 - EP[(4\lambda^2/R^2)\,(v')^2 + (u_0')^2]. \qquad [4.20]$$

We note that $P_3(y)$ and $R_3(y)$ are known to us provided we *apriori* prescribe the the value of the Reynolds number R.

Let $\theta_0 = \theta_\alpha$ and $\theta_0 = \theta_\beta$ be the solutions of this equation satisfying the boundary conditions

$$\theta_\alpha(0) = 0, \;\; \theta_\alpha'(0) = \alpha \text{ (say)}, \qquad [4.21]$$

$$\theta_\beta(0) = 0, \;\; \theta_\beta'(0) = \beta \text{ (say)}. \qquad [4.22]$$

Then we can easily check that the solution of [4.2] satisfying the prescribed boundary conditions is

$$\theta_0(y) = \frac{\theta_\beta(1) - 1}{\theta_\beta(1) - \theta_\alpha(1)}\,\theta_\alpha(y) + \frac{1 - \theta_\alpha(1)}{\theta_\beta(1) - \theta_\alpha(1)}\,\theta_\beta(y). \qquad [4.23]$$

In summary, we like to mention that the procedure prescribed above has the following advantages :

(1) The integration of the various equations does not involve any trial and error method in spite of the fact that all our equations have to satisfy two point boundary conditions.

(2) The eigenvalue A_1 occuring in the cross flow velocity is determined automatically during the process of integration of V-equation.

(3) No doubt, we do not solve the cross flow velocity equation for an *apriori* prescribed value of the suction parameter λ, but for the prescribed value of $V''(0)$ which leads to the determination of the corresponding value of λ. Thus two or three integrations with properly chosen values of $V''(0)$ will enable us to guess what value of $V''(0)$ be chosen to give approximately the solution for the prescribed value of λ.

(4) We have to make the specific choice of the Prandtl number and Eckert number in order to solve θ_1 and θ_2 equations, but we have not to prescribe the value of R till we come to the solution of θ_0-equation.

5. NUMERICAL RESULTS

We have performed the numerical calculation for the values of $m = 1, \frac{1}{2}, -1, -\frac{1}{2}$, the first two with plus sign in equation [3.10], while the last two with the negative sign in [3.10]. The [illegible] table gives the values of v'', u_0' on the boundaries and the [illegible] values of λ, A_0, A_1:

m	Sign of r.h.s. in (3.10)	λ	A_0	A_1	$v''(0)$	$v''(1)$	$u_0'(0)$	$u_0'(1)$
1	+	1.4889	−31.7828	21.3465	−1.2754	1.4437	[illegible]	[illegible]
1/2	+	0.0825	−0.0825	1.0001	−5.9650	6.0636	[illegible]	[illegible]
−1	—	1.2291	−15.3468	12.4862	1.7655	[illegible]	0.9543	[illegible]
−1/2	—	0.0842	−0.0842	1.0001	6.0365	−5.9350	−1.0679	[illegible]

Figures 1 and 2 give the plots of v and u_0, while the figures 3, 4, 5 give the plots of θ_2, θ_1 and θ_0 respectively for $R = 100$, $E = 5$, $P = 0.8$. Since the main purpose of the present paper is to establish a convenient method for solving the flow and heat transfer problems, we have undertaken only a limited number of numerical cases. This method is easily [illegible] to other geometries.

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